Decompositions of K_v into four kinds of graphs with eight vertices and eight edges*

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Abstract. Let K_v be the complete graph with v vertices. Let G be a finite simple graph. A G-decomposition of K_v, denoted by $(v, G, 1)$-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G. In this paper, the discussed graphs are $G_i, i = 1, 2, 3, 4$, where G_i are four kinds of graphs with eight vertices and eight edges. We obtain the existence spectrum of $(v, G_i, 1)$-GD.

Keywords: G-decomposition; G-holey design; G-incomplete holey design.

1 Introduction

Let G be a finite simple graph. Let X be a set of v vertices and $\mathcal{G} = \{X_1, X_2, \cdots, X_t\}$ be a partition of X. Denote $|X_i| = n_i$ for $1 \leq i \leq t$. Let $K_{n_1, n_2, \cdots, n_t}$ denote the complete multigraph on the set X which partition $\mathcal{G}$. For any given graph G, if the edges of $K_{n_1, n_2, \cdots, n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to G

*Research supported by NSFC Grant 11401158, NSFC Grant 11171089 and the YPSR of HUEB (No.2014KYQ04).

and is called a *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design*, denoted by G-$HD(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. A G-$IHD(h_1, h_2; w)$ is a pair $((H_1, H_2, W), \mathcal{A})$, where $\mathcal{A}$ is a collection of subgraphs in $H_1 \bigcup H_2 \bigcup W$, called *blocks*, such that each block is isomorphic to G and any two distinct vertices x, y are jointed in

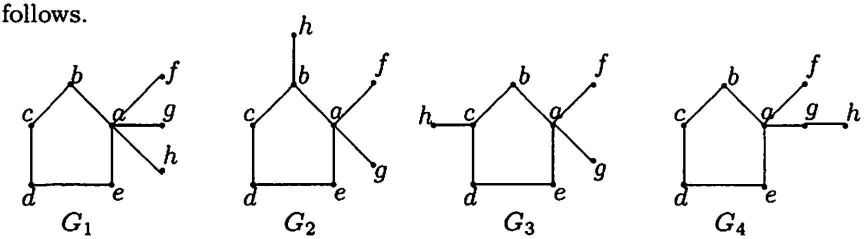
$$\begin{cases} \text{exactly } one \text{ block of } \mathcal{A} & \text{if } x, y \in H_1 \text{ or } x, y \in H_2 \text{ or } x \in H_1 \bigcup H_2, y \in W \\ \text{no block of } \mathcal{A} & \text{otherwise} \end{cases}$$

A *G-decomposition* of K_v, denoted by $(v, G, 1)$-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G. It is well known that if there exists a $(v, G, 1)$-GD, then

$$v(v-1) \equiv 0 \pmod{2e(G)} \quad \text{and} \quad v - 1 \equiv 0 \pmod{d},$$

where $e(G)$ denotes the number of edges in G and d is the greatest common divisor of the degrees of the vertices of G. For the path P_k and the star $K_{1,k}$, the existence problems of their graph designs have been solved (see [1] and [2]). For the graphs which have four vertices or less, the problem of their graph designs has already been solved (see [3]). For some graphs with less vertices and less edges, the problem of their graph designs has already been researched (see [4]-[12]).

In this paper, the discussed graphs are G_i $(i = 1, 2, 3, 4)$, where G_i are four kinds of graphs with eight vertices and eight edges, they are listed as follows.



For convenience, the graphs G_1-G_4 above are denoted by (a, b, c, d, e, f, g, h). we obtain the existence spectrum of $(v, G_i, 1)$-GD.

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers h, w and t, if there exist G-$HD(h^{2t+1})$, G-$IHD(h, h; w)$ and $(h+w, G, 1)$-GD, then $((2t+1)h+w, G, 1)$-GD exists.*

Proof. Let $X = (Z_h \times Z_{2t+1}) \bigcup W$, where $|W| = w$. Suppose there exist

$$G\text{-}HD(h^{2t+1}) = (Z_h \times Z_{2t+1}, \mathcal{A}),$$

$$G\text{-}IHD(h, h; w) = ((Z_h \times \{2i\}, Z_h \times \{2i+1\}, W), \mathcal{B}_i) \text{ for } 0 \le i \le t-1,$$

and

$$(h+w, G, 1)\text{-}GD = ((Z_h \times \{2t\}) \bigcup W, \mathcal{C}),$$

then $(X, \mathcal{A} \bigcup (\bigcup_{i=0}^{t-1} \mathcal{B}_i) \bigcup \mathcal{C})$ forms a $((2t+1)h+w, G, 1)$-GD. In fact, we have

$$|\mathcal{A}| + t|\mathcal{B}_i| + |\mathcal{C}| = \frac{\binom{2t+1}{2}h^2}{e(G)} + \frac{t(2hw+h(h-1))}{e(G)} + \frac{\binom{w+h}{2}}{e(G)} = \frac{\binom{(2t+1)h+w}{2}}{e(G)}. \quad \square$$

The necessary conditions for the existence of $(v, G_i, 1)$-GD are $v(v-1) \equiv 0 \pmod{16}$ and $v \ge 16$, i.e., $v \equiv 0, 1 \pmod{16}$ and $v \ge 16$. So by Theorem 2.1, we only need to give the constructions of HD,IHD and GD for the orders given in the following table.

(Table 2.1) For $G_i, i = 1, 2, 3, 4$

$v(mod\ 16)$	HD	IHD	GD
0	8^{2t+1}	$(8, 8; 8)$	16
1	8^{2t+1}	$(8; 8; 9)$	17

3 Holey designs

A *quasigroup* is an ordered pair $(Q, \cdot)$, which is a set Q with a binary operation $\cdot$ such that the equations $a \cdot x = b$ and $y \cdot a = b$ are uniquely solvable for every pair of elements a, b in Q. It is well known that the multiplication table of a quasigroup is equivalent with a Latin square. A quasigroup is said to be *idempotent* (or *symmetric*) if the identity $x \cdot x = x$ (or $x \cdot y = y \cdot x$) holds for all $x \in Q$ (or $x, y \in Q$). It is well known that there exists an idempotent quasigroup of order v if and only if $v \neq 2$ and there exists an idempotent symmetric quasigroup of order v if and only if v is odd.

Suppose $(I_n, \cdot)$ is an idempotent symmetric quasigroup on the set $I_n = \{1, 2, \cdots, n\}$. Now, let's construct a G-$HD(e^n)$, where $e = |E(G)|$. Let the element set be $Z_e \times I_n$, and the automorphism group of the block set be Z_e. It is enough to construct a base block for any $i, j \in I_n$ and $i < j$. In a base block of a G-$HD(e^n)$, all edges can be partitioned into three types: $\{(x, i), (x + d, j)\}$, $\{(x, i), (x + d, i \cdot j)\}$ and $\{(x, i \cdot j), (x + d, j)\}$, denoted by $d(i, j), d(i, i \cdot j)$ and $d(i \cdot j, j)$ briefly, where $x \in Z_e$, $i, j \in I_n$. We have the following Lemma.

Lemma 3.1 [7] *Let n be odd and $(I_n, \cdot)$ be an idempotent symmetric quasigroup on the set $I_n = \{1, 2, \cdots, n\}$. The block set $\mathcal{A} = \{A_{i,j} : i, j \in I_n \text{ and } i < j\}$ can be taken as a base of a G-$HD(e^n)$ if and only if the following conditions hold,*

(1) *For any given block $A_{i,j}$ in $\mathcal{A}$, the differences $d(i, i \cdot j)$ and $-d(i \cdot j, j)$ both appear or not in $A_{i,j}$;*

(2) $\{d : \exists d(i, j)\} \bigcup \{d : \exists d(i, i \cdot j)\} \bigcup \{d : \exists d(i \cdot j, j)\} = Z_e$.

Lemma 3.2 *There exists a G_k-$HD(8^{2t+1})$ for $k = 1, 2, 3, 4$ and $t > 0$.*

Proof. The number of the edges of G_k is 8. Suppose $(I_{2t+1}, \cdot)$ is an idempotent symmetric quasigroup on the set $I_{2t+1} = \{1, 2, \cdots, 2t + 1\}$. Let $X = Z_8 \times I_{2t+1}$ and

$$\mathcal{B}_1 = (0_j, 2_{i \cdot j}, 3_i, 6_{i \cdot j}, 4_i, 3_{i \cdot j}, 0_i, 7_{i \cdot j}),$$
$$\mathcal{B}_2 = (0_j, 2_{i \cdot j}, 3_i, 6_{i \cdot j}, 4_i, 0_i, 7_{i \cdot j}, 7_j),$$
$$\mathcal{B}_3 = (0_j, 4_i, 6_{i \cdot j}, 3_i, 2_{i \cdot j}, 0_i, 3_{i \cdot j}, 7_j),$$
$$\mathcal{B}_4 = (0_j, 2_{i \cdot j}, 3_i, 6_{i \cdot j}, 4_i, 0_i, 3_{i \cdot j}, 4_j),$$

where $1 \le i < j \le 2t + 1$. We can verify that each $\mathcal{B}_k$ mod $(8, -)$ gives the expected G_k-$HD(8^{2t+1})$ by Lemma 3.1 for $k = 1, 2, 3, 4$. □

4 Incomplete holey designs

Lemma 4.1 *There exist G_1-$IHD(8, 8; w)$ for $w = 8, 9$.*

Proof. Let $X = Z_8 \bigcup \bar{Z}_8 \bigcup W$, where $W = \{a_1, a_2, \cdots, a_w\}$ and G_1-$IHD(8,8;w)$=$((Z_8, \bar{Z}_8, W), \mathcal{B})$, where $|\mathcal{B}| = 7 + 2w$. The family $\mathcal{B}$ consists of the following blocks.

$w = 8$:

$(a_1, 0, 4, a_2, \bar{5}, \bar{7}, \bar{3}, \bar{2})$, $(a_2, \bar{1}, a_1, 1, 0, 7, \bar{4}, \bar{2})$, $(a_3, 2, a_1, \bar{0}, \bar{3}, 6, \bar{7}, \bar{4})$,
$(a_4, \bar{0}, \bar{2}, \bar{1}, \bar{3}, 1, \bar{6}, 6)$, $(a_5, 0, 2, a_2, 6, \bar{7}, 4, \bar{6})$, $(a_6, 4, 5, 0, 3, 1, \bar{5}, \bar{4})$,
$(a_7, 0, a_8, \bar{2}, \bar{3}, 5, \bar{0}, \bar{5})$, $(4, a_7, \bar{6}, a_2, 3, a_1, a_8, 6)$, $(\bar{0}, a_2, \bar{7}, \bar{2}, \bar{4}, a_3, \bar{1}, \bar{6})$,
$(\bar{1}, a_8, \bar{0}, a_5, \bar{5}, a_3, a_6, \bar{6})$, $(\bar{2}, a_5, 5, a_1, \bar{6}, \bar{5}, a_3, a_7)$, $(a_8, \bar{3}, \bar{5}, \bar{7}, \bar{4}, 5, 3, 2)$,
$(\bar{3}, a_2, 1, a_7, \bar{4}, a_5, \bar{7}, a_6)$, $(6, 0, a_3, 1, a_8, a_1, 2, 7)$, $(1, 4, 7, a_3, 5, a_5, 6, 3)$,
$(3, a_3, 4, a_4, 2, 7, 6, a_1)$, $(\bar{4}, a_4, 7, a_8, \bar{6}, a_1, \bar{1}, a_5)$, $(2, 5, a_4, \bar{1}, a_7, a_6, 1, 4)$,
$(5, a_6, 0, a_4, 3, 6, a_2, 7)$, $(\bar{5}, a_4, \bar{2}, a_6, \bar{0}, a_3, a_8, \bar{4})$, $(7, 2, a_5, 3, a_7, a_1, 0, 1)$,
$(\bar{7}, a_6, 7, a_5, \bar{1}, a_4, \bar{0}, a_8)$, $(\bar{6}, a_6, 6, a_7, \bar{7}, a_3, \bar{3}, \bar{5})$.

$w = 9$:

$(a_1, 2, 1, 0, 5, \bar{3}, 4, \bar{7})$, $(a_2, 0, a_1, \bar{1}, \bar{0}, 3, 6, \bar{5})$, $(a_3, 0, 2, a_2, \bar{3}, 6, \bar{2}, \bar{5})$,
$(a_4, 7, 6, a_1, 3, 0, \bar{4}, \bar{3})$, $(a_5, \bar{6}, a_1, 7, 0, 5, 1, \bar{7})$, $(a_6, \bar{1}, a_3, \bar{4}, \bar{7}, 5, 6, \bar{2})$,
$(a_7, \bar{7}, a_3, 4, 7, \bar{0}, \bar{1}, \bar{5})$, $(a_8, \bar{6}, \bar{2}, a_9, \bar{4}, 2, 7, 6)$, $(a_9, \bar{5}, \bar{0}, a_4, \bar{1}, 1, 3, 5)$,
$(0, a_7, 4, 5, a_8, a_9, 6, a_6)$, $(1, a_4, \bar{5}, \bar{4}, a_2, a_1, 4, a_7)$, $(2, a_3, 1, 3, a_6, a_4, a_9, 5)$,
$(3, a_7, \bar{6}, \bar{0}, a_8, a_3, a_5, 2)$, $(4, a_2, 7, 1, a_6, 3, a_8, 2)$, $(5, a_4, 4, 0, 3, a_2, 1, a_7)$,
$(6, a_5, \bar{3}, \bar{6}, a_9, 3, 4, 5)$, $(7, 2, a_5, 4, a_9, a_3, a_6, 3)$, $(\bar{0}, a_3, 5, 7, a_5, a_1, \bar{4}, \bar{7})$,
$(\bar{1}, a_2, \bar{6}, \bar{7}, a_8, a_5, \bar{3}, \bar{2})$, $(\bar{2}, a_4, 6, 2, a_7, a_8, a_2, \bar{7})$, $(\bar{3}, a_7, 6, 1, a_8, \bar{4}, a_9, \bar{7})$,
$(\bar{4}, a_1, \bar{2}, \bar{3}, a_6, a_7, a_5, \bar{1})$, $(\bar{5}, a_5, \bar{2}, \bar{0}, a_6, \bar{6}, a_8, a_1)$, $(\bar{6}, \bar{1}, \bar{5}, \bar{2}, \bar{4}, a_4, a_6, a_3)$,
$(\bar{7}, a_9, \bar{0}, \bar{3}, \bar{5}, a_2, a_4, \bar{1})$. □

Lemma 4.2 *There exist G_2-$IHD(8,8;w)$ for $w = 8, 9$.*

Proof. Let $X = Z_8 \bigcup \bar{Z}_8 \bigcup W$, where $W = \{a_1, a_2, \cdots, a_w\}$ and G_2-$IHD(8,8;w)$=$((Z_8, \bar{Z}_8, W), \mathcal{B})$, where $|\mathcal{B}| = 7 + 2w$. The family $\mathcal{B}$ consists of the following blocks.

$w = 8$:

$(a_1, 0, 7, a_6, 6, \bar{5}, \bar{6}, a_7)$, $(a_2, \bar{2}, \bar{6}, a_4, 7, \bar{7}, 0, \bar{1})$, $(a_3, \bar{3}, \bar{2}, a_5, \bar{4}, \bar{1}, 2, a_1)$,
$(a_4, 6, 2, a_7, \bar{3}, \bar{1}, \bar{7}, 3)$, $(a_5, 5, a_6, 4, 2, 3, \bar{7}, a_4)$, $(a_6, \bar{6}, \bar{0}, a_7, \bar{7}, \bar{4}, 2, \bar{1})$,
$(a_7, 4, a_2, 1, 6, \bar{2}, \bar{5}, a_5)$, $(a_8, \bar{0}, a_5, 0, 3, \bar{2}, \bar{5}, \bar{1})$, $(0, 2, a_4, 4, 1, 6, a_6, 7)$,
$(1, a_1, 4, 7, 5, a_4, a_5, \bar{0})$, $(2, a_2, \bar{4}, a_7, 1, a_1, a_8, \bar{3})$, $(3, a_4, 0, 5, 4, 2, a_6, \bar{0})$,
$(4, a_3, \bar{0}, \bar{3}, a_8, 6, 0, \bar{2})$, $(5, 3, a_3, 6, a_2, 2, a_7, 1)$, $(6, 7, 3, a_1, 5, a_5, a_8, a_3)$,
$(7, 1, a_3, 5, a_8, a_1, a_5, a_6)$, $(\bar{1}, \bar{4}, a_1, \bar{2}, a_6, a_2, a_7, \bar{6})$, $(\bar{2}, \bar{5}, \bar{1}, a_1, \bar{7}, a_4, \bar{0}, \bar{3})$,
$(\bar{3}, a_6, \bar{0}, a_2, \bar{6}, \bar{1}, \bar{4}, \bar{5})$, $(\bar{4}, \bar{7}, a_3, \bar{5}, a_4, \bar{2}, \bar{0}, \bar{1})$, $(\bar{5}, a_5, \bar{1}, a_8, \bar{4}, \bar{0}, \bar{7}, \bar{3})$,
$(\bar{6}, a_7, 3, a_2, \bar{5}, a_5, a_8, 7)$, $(\bar{7}, a_8, 0, a_3, \bar{6}, \bar{3}, \bar{0}, 1)$.

$w = 9$:

$(a_1,5,0,6,7,\bar{6},\bar{5},a_4)$, $(a_2,6,5,3,2,\bar{0},\bar{4},a_7)$, $(a_3,7,1,2,4,\bar{3},6,a_9)$,
$(a_4,4,a_1,\bar{2},\bar{1},1,6,0)$, $(a_5,3,a_2,\bar{5},\bar{7},5,\bar{1},4)$, $(a_6,\bar{7},a_4,\bar{2},\bar{3},3,2,a_1)$,
$(a_7,\bar{6},a_4,0,1,\bar{7},4,a_8)$, $(a_8,\bar{5},a_5,1,3,7,5,\bar{6})$, $(a_9,\bar{4},a_6,1,5,6,\bar{7},a_3)$,
$(0,a_1,\bar{0},\bar{2},a_5,3,a_3,6)$, $(1,a_2,\bar{6},\bar{3},a_1,a_3,4,5)$, $(2,a_3,\bar{0},\bar{4},a_4,a_1,7,5)$,
$(3,a_4,\bar{3},\bar{1},a_3,a_1,a_7,\bar{5})$, $(4,a_5,2,6,a_6,a_2,a_8,\bar{6})$, $(5,2,a_9,\bar{5},a_7,4,a_6,0)$,
$(6,1,a_8,\bar{0},a_5,3,4,a_9)$, $(7,0,a_8,2,a_7,4,5,a_2)$, $(\bar{0},a_6,\bar{1},\bar{4},a_7,a_9,\bar{6},7)$,
$(\bar{1},a_7,\bar{2},\bar{4},a_8,a_2,\bar{6},0)$, $(\bar{2},a_8,\bar{7},\bar{3},\bar{5},a_6,a_3,6)$, $(\bar{3},a_9,3,7,a_5,a_2,a_8,4)$
$(\bar{4},\bar{3},\bar{0},\bar{1},a_1,a_5,\bar{7},a_7)$, $(\bar{5},\bar{1},a_9,\bar{6},a_3,\bar{4},a_6,\bar{7})$, $(\bar{6},\bar{2},a_9,0,a_6,\bar{7},\bar{4},a_2)$,
$(\bar{7},\bar{0},a_4,7,a_2,\bar{2},a_3,\bar{5})$. □

Lemma 4.3 *There exist G_3-$IHD(8,8;w)$ for $w=8,9$.*

Proof. Let $X = Z_8 \bigcup \bar{Z}_8 \bigcup W$, where $W = \{a_1,a_2,\cdots,a_w\}$ and G_3-$IHD(8,8;w)$=$((Z_8,\bar{Z}_8,W),\mathcal{B})$, where $|\mathcal{B}| = 7+2w$. The family $\mathcal{B}$ consists of the following blocks.

$w=8$:

$(a_1,2,0,a_3,\bar{5},\bar{1},\bar{3},7)$, $(a_2,1,3,a_1,\bar{2},\bar{0},\bar{6},a_5)$, $(a_3,6,2,5,1,\bar{2},\bar{1},a_8)$,
$(a_4,3,5,a_7,2,\bar{0},6,a_2)$, $(a_5,2,3,a_7,\bar{0},4,0,a_3)$, $(a_6,3,4,a_1,6,5,\bar{7},a_8)$,
$(a_7,\bar{3},\bar{2},a_5,\bar{7},1,\bar{6},\bar{0})$, $(a_8,3,7,a_6,\bar{0},5,\bar{4},a_3)$, $(0,a_2,6,7,5,a_7,a_6,3)$,
$(\bar{1},\bar{0},a_1,\bar{6},\bar{5},a_5,\bar{4},0)$, $(1,7,a_2,\bar{1},a_8,a_5,a_1,3)$, $(\bar{4},\bar{0},a_3,2,a_6,\bar{2},a_1,\bar{7})$,
$(2,4,a_4,0,1,a_2,7,\bar{3})$, $(1,6,a_5,5,4,a_4,a_6,\bar{5})$, $(\bar{7},\bar{5},a_6,\bar{1},\bar{2},a_2,\bar{3},\bar{6})$,
$(0,6,a_7,7,a_8,3,4,\bar{5})$, $(4,6,a_8,\bar{3},a_2,a_7,a_6,\bar{2})$, $(5,a_4,\bar{1},\bar{7},a_1,a_3,6,a_7)$,
$(7,a_5,\bar{4},a_3,4,a_4,a_1,a_7)$, $(\bar{5},a_4,\bar{6},\bar{7},\bar{4},a_8,\bar{3},\bar{0})$, $(\bar{6},a_8,\bar{7},a_4,\bar{4},\bar{1},a_3,\bar{0})$,
$(\bar{2},a_6,\bar{3},a_5,\bar{6},a_7,a_4,a_3)$, $(\bar{3},\bar{0},\bar{5},a_2,\bar{4},\bar{6},\bar{1},\bar{2})$.

$w=9$:

$(a_1,\bar{3},\bar{6},\bar{0},\bar{5},0,5,a_9)$, $(a_2,\bar{2},a_1,\bar{7},\bar{1},5,\bar{4},4)$, $(a_3,4,a_9,\bar{5},\bar{6},1,\bar{2},\bar{0})$,
$(a_4,\bar{6},\bar{7},a_8,0,5,6,\bar{4})$, $(a_5,\bar{0},a_2,\bar{5},\bar{1},4,3,\bar{7})$, $(a_6,2,0,a_2,3,\bar{1},\bar{7},1)$,
$(a_7,6,1,a_1,3,\bar{1},\bar{6},a_4)$, $(a_8,7,a_3,2,6,5,\bar{2},\bar{3})$, $(a_9,5,2,a_1,7,\bar{2},\bar{7},4)$,
$(0,a_3,3,6,4,7,a_9,1)$, $(1,2,a_4,\bar{4},a_8,a_6,a_7,4)$, $(2,a_2,4,1,a_5,7,3,5)$,
$(3,a_4,7,1,a_9,0,a_8,6)$, $(4,7,a_5,5,a_6,3,a_8,0)$, $(5,0,a_6,\bar{5},a_7,3,6,\bar{4})$,
$(6,a_3,5,1,a_2,a_5,a_9,7)$, $(7,a_6,6,0,a_7,a_2,3,a_1)$, $(\bar{0},a_4,\bar{5},\bar{3},a_8,a_1,a_6,a_3)$,
$(\bar{1},\bar{2},a_7,2,a_9,a_3,\bar{3},4)$, $(\bar{2},\bar{6},a_8,\bar{1},\bar{0},\bar{5},a_4,2)$, $(\bar{3},a_7,\bar{0},\bar{7},a_4,a_2,a_5,a_3)$
$(\bar{4},a_1,\bar{1},\bar{6},a_5,\bar{0},a_7,a_4)$, $(\bar{5},\bar{7},\bar{2},\bar{3},\bar{4},a_8,a_5,a_6)$, $(\bar{6},a_6,\bar{3},a_9,\bar{4},a_2,a_1,\bar{0})$,
$(\bar{7},a_3,\bar{4},\bar{2},a_5,\bar{3},a_7,\bar{1})$. □

Lemma 4.4 *There exist G_4-$IHD(8,8;w)$ for $w=8,9$.*

Proof. Proof. Let $X = Z_8 \bigcup \bar{Z}_8 \bigcup W$, where $W = \{a_1,a_2,\cdots,a_w\}$ and G_4-$IHD(8,8;w)$=$((Z_8,\bar{Z}_8,W),\mathcal{B})$, where $|\mathcal{B}| = 7+2w$. The family $\mathcal{B}$ consists of the following blocks.

$w = 8$:

$(a_1, 5, 3, a_3, 0, \bar{7}, 7, a_4)$, $(a_2, \bar{5}, \bar{6}, a_1, \bar{1}, 0, \bar{0}, a_8)$, $(a_3, 7, 4, a_8, 6, \bar{3}, \bar{0}, a_7)$,
$(a_4, \bar{2}, a_3, \bar{1}, \bar{3}, 1, 6, 5)$, $(a_5, \bar{7}, a_2, 6, 1, \bar{1}, \bar{6}, \bar{4})$, $(a_6, 2, 4, a_1, \bar{4}, 5, 0, 7)$,
$(a_7, \bar{3}, \bar{4}, a_2, 2, 1, 4, a_3)$, $(a_8, \bar{4}, \bar{5}, a_1, 1, \bar{1}, 2, 3)$, $(0, 3, a_2, \bar{2}, a_5, 1, 5, a_4)$,
$(1, a_2, \bar{3}, a_5, 7, a_3, a_6, 3)$, $(2, 6, a_7, 3, a_1, a_3, 0, 4)$, $(3, a_5, \bar{0}, a_6, 6, 7, 1, 5)$,
$(4, 5, a_8, \bar{6}, a_4, a_6, a_2, 7)$, $(5, 2, 1, 4, a_5, a_3, a_2, \bar{6},)$, $(6, 0, a_4, 2, a_5, 7, 4, 3)$,
$(7, a_6, \bar{6}, \bar{7}, a_8, 2, 5, a_7)$, $(\bar{1}, a_4, 3, a_8, \bar{5}, a_7, \bar{7}, \bar{2})$, $(\bar{2}, a_7, \bar{5}, \bar{0}, a_1, \bar{1}, \bar{3}, a_6)$,
$(\bar{3}, \bar{6}, \bar{2}, a_6, \bar{7}, a_8, a_1, 6)$, $(\bar{4}, \bar{2}, a_8, 0, a_7, \bar{0}, a_3, \bar{5})$, $(\bar{5}, \bar{2}, \bar{0}, \bar{1}, a_6, \bar{7}, a_5, \bar{4})$,
$(\bar{6}, \bar{1}, \bar{4}, a_4, \bar{0}, a_7, a_3, \bar{7})$, $(\bar{7}, a_4, \bar{5}, \bar{3}, \bar{0}, \bar{4}, a_7, 7)$.

$w = 9$:

$(a_1, 4, 0, 1, 6, 7, \bar{1}, \bar{7})$, $(a_2, 1, a_1, 3, 7, \bar{3}, \bar{4}, a_3)$, $(a_3, \bar{1}, \bar{0}, \bar{4}, \bar{3}, 4, 0, a_7)$,
$(a_4, \bar{6}, a_1, \bar{3}, \bar{7}, 3, 5, 7)$, $(a_5, \bar{0}, a_3, 2, 0, \bar{1}, \bar{7}, a_9)$, $(a_6, 6, a_2, \bar{2}, \bar{4}, 1, \bar{0}, \bar{3})$,
$(a_7, 3, a_3, 6, 2, 5, \bar{5}, \bar{0})$, $(a_8, 1, a_7, \bar{2}, \bar{7}, \bar{6}, \bar{4}, a_4)$, $(0, a_4, 1, 5, a_2, a_9, a_8, \bar{5})$,
$(1, a_3, 7, 2, a_5, a_9, 4, a_4)$, $(2, a_4, \bar{2}, \bar{0}, a_2, a_1, a_9, 3)$, $(3, a_6, 7, 0, 6, a_2, 1, 2)$,
$(4, a_8, 3, 0, 5, 2, a_2, \bar{6})$, $(5, a_1, \bar{4}, \bar{5}, a_6, a_5, 3, 4)$, $(6, a_8, 2, 3, a_5, a_9, 7, 1)$,
$(7, a_7, \bar{7}, \bar{6}, a_9, a_8, a_5, \bar{5})$, $(\bar{0}, a_1, \bar{5}, \bar{6}, a_7, a_8, \bar{7}, a_3)$, $(\bar{1}, a_4, 7, 4, a_7, \bar{3}, a_2, \bar{7})$,
$(\bar{2}, \bar{5}, \bar{1}, \bar{6}, \bar{3}, a_1, a_8, 5)$, $(\bar{3}, a_6, 4, 6, a_7, \bar{5}, a_8, \bar{1})$, $(\bar{4}, \bar{6}, a_3, \bar{2}, a_5, a_7, a_9, \bar{3})$
$(\bar{5}, a_4, 6, 5, a_3, a_2, a_9, \bar{2})$, $(\bar{6}, \bar{0}, a_4, \bar{3}, a_5, \bar{2}, a_6, 0)$, $(\bar{7}, \bar{4}, \bar{1}, \bar{2}, a_6, \bar{5}, a_1, 0)$,
$(a_9, \bar{1}, a_6, 2, 5, \bar{0}, 4, a_5)$. □

5 Graph designs

Lemma 5.1 *There exist* $(v, G_1, 1)$ *for* $v = 16, 17$.

Proof. $v = 16$: $X = Z_{15} \bigcup \{\infty\}$ $(0, 1, 3, 6, 10, \infty, 9, 7)$ mod 15
$v = 17$: $X = Z_{17}$ $(0, 1, 3, 6, 10, 5, 8, 11)$ mod 17 □

Theorem 5.2 *There exist* $(v, G_1, 1)$ *if and only if* $v(v-1) \equiv 0$ *(mod* 16*) and* $v \geq 16$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.1 and Lemma 5.1. □

Lemma 5.3 *There exist* $(v, G_2, 1)$ *for* $v = 16, 17$.

Proof. $v = 16$: $X = Z_{15} \bigcup \{\infty\}$ $(0, 1, 3, 6, 10, 9, 7, \infty)$ mod 15
$v = 17$: $X = Z_{17}$ $(0, 1, 3, 6, 10, 5, 11, 9)$ mod 17 □

Theorem 5.4 *There exist* $(v, G_2, 1)$ *if and only if* $v(v-1) \equiv 0$ *(mod* 16*) and* $v \geq 16$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.2 and Lemma 5.3. □

Lemma 5.5 *There exist* $(v, G_3, 1)$ *for* $v = 16, 17$.

Proof. $v = 16$: $X = Z_{15} \bigcup \{\infty\}$ $(0, 1, 3, 6, 10, 9, 7, \infty)$ mod 15
$v = 17$: $X = Z_{17}$ $(0, 1, 3, 6, 10, 9, 11, 8)$ mod 17 □

Theorem 5.6 *There exist* $(v, G_3, 1)$ *if and only if* $v(v-1) \equiv 0$ $(mod\ 16)$ *and* $v \geq 16$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.3 and Lemma 5.5. □

Lemma 5.7 *There exist* $(v, G_4, 1)$ *for* $v = 16, 17$.

Proof. $v = 16$: $X = Z_{15} \bigcup \{\infty\}$ $(0, 1, 3, 6, 10, 7, 9, \infty)$ mod 15
$v = 17$: $X = Z_{17}$ $(0, 1, 3, 6, 10, 8, 5, 11)$ mod 17 □

Theorem 5.8 *There exist* $(v, G_4, 1)$ *if and only if* $v(v-1) \equiv 0$ $(mod\ 16)$ *and* $v \geq 16$.

Proof. By Theorem 2.1, Lemma 3.2, Lemma 4.4 and Lemma 5.7. □

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