Multidesigns of Complete Graphs for Graph-Triples of Order 6

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Abstract

We call $T = (G_1, G_2, G_3)$ a *graph-triple of order t* if the G_i are pairwise non-isomorphic graphs on t non-isolated vertices whose edges can be combined to form K_t. If $m \geq t$, we say T *divides* K_m if $E(K_m)$ can be partitioned into copies of the graphs in T with each G_i used at least once, and we call such a partition a T-*multidecomposition*. For each graph-triple T of order 6 for which it was not previously known, we determine all K_m, $m \geq 6$, that admit a T-multidecomposition. Moreover, we determine maximum multipackings and minimum multicoverings when K_m does not admit a multidecomposition.

1 Introduction

The graph decomposition problem, in which the edges of a graph are decomposed into copies of a fixed subgraph, has been widely studied (see [BHRS80], [BS77], and [Kot65]). In [AD03], A. Abueida and M. Daven approach this problem from the perspective of *graph-pairs*. Specifically, they decompose the edges of K_t for $t = 4, 5$ into nonisomorphic graphs G_1 and G_2, and then determine complete graphs K_m with $m \geq t$ whose edges can

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be partitioned into copies of G_1 and G_2 using at least one copy of each graph. In [ADR05], Abueida, Daven and K. Roblee prove similar results for λK_m.

Our work here is a continuation of the work done on multidesigns of graph-triples in [ADD+06]. A *graph-triple of order t* is some $T = \{G_1, G_2, G_3\}$ where G_1, G_2, and G_3 are pairwise non-isomorphic subgraphs of K_t without isolated vertices whose edges partition $E(K_t)$. A *T-multidesign* of K_m with $m \geq t$ can have three forms. A *T-multidecomposition* is a partition of $E(K_m)$ into copies of the graphs of T where each G_i is used at least once. In this case, we also say that T *divides* K_m or that T *factors* K_m. In the case that a T-multidecomposition does not exist, a *maximum T-multipacking* is a partitioning of a subset of $E(K_m)$ into copies of graphs in T, where each G_i is used at least once, such that the number of edges outside the partition (called the *leave*) is minimum. A *minimum T-multicovering* is a collection of copies of graphs in T, where each G_i is used at least once, such that all edges of K_m are used once or twice and where the number of edges used twice (called the *padding*) is minimum. In [ADD+06], the authors constructed all 131 graph-triples of order 6, which are listed in Appendix B. They chose 37 of these graph-triples and determined multidesigns for all K_m with $m \geq 6$. In this paper, we determine multidesigns of K_m, $m \geq 6$, for the remaining graph-triples of order 6.

We list the graphs that are part of graph-triples of order 6 in Appendix A. We use the notation of [ADD+06], denoting the i^{th} graph on 6 vertices with j edges and no isolated vertices with the notation H_i^j. The graphs are obtained from [HP73], where we remove graphs that cannot be part of a graph-triple of order 6. Note that the vertices in Appendix A are labeled a through f. If $v_k \in V(K_m)$ for $k \in \{a, b, c, d, e, f\}$, we will denote by $[v_a, v_b, v_c, v_d, v_e, v_f]$ the subgraph of K_m isomorphic to H_i^j in which each v_k plays the role of k. This will not be ambiguous as long as we specify H_i^j.

We write $V(G)$ to denote the vertex set of G and $\deg(v)$ to denote the degree of $v \in V(G)$. Further, $\Delta(G) = \max\{\deg(v) : v \in G\}$. We write $G_1 \cup G_2$ to denote any graph whose edge set is partitioned by $E(G_1)$ and $E(G_2)$, and we define kG_1 to be any graph whose edges can be partitioned into k copies of G_1. Note that $G_1 \cup G_2$ and kG_1 are not unique up to isomorphism. For graphs G_1 and G_2 with disjoint vertex sets, we define $G_1 + G_2$ to be the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We let $V(K_m) = \mathbb{Z}_m$, and for $r \leq m$, we consider $\mathbb{Z}_r \subseteq \mathbb{Z}_m$ in the natural way. Note that $\mathbb{Z}_r$ induces a subgraph of K_m isomorphic to K_r.

A technique that is frequently used in this paper to find multidesigns is to find a way to write $K_m \cong \bigcup_{i=1}^{n} S_i$, where each S_i is a subgraph of K_m. For each $1 \leq i \leq n$, we find a T_i-multidecomposition of S_i for some

$T_i \subseteq T$. We can then combine these multidecompositions to form a T-multidecomposition of K_n, as long as each graph in T is used in at least one of the T_i-multidecompositions. This gives us the following.

Lemma 1.1. Let $m \geq 6$, and let $K_m \cong \bigcup_{i=1}^{n} S_i$, where each S_i is a subgraph of K_m. Suppose also that T is a graph-triple of order 6 and that $T_i \subseteq T$ for $1 \leq i \leq n$ such that $\bigcup_{i=1}^{n} T_i = T$. If T_i divides S_i for all i, then T divides K_m.

In particular, this can be used on the graph $G_{r,m} = K_m - K_r$, where $V(G_{r,m}) = \mathbb{Z}_m$, and we let the vertices from which the edges of K_r are removed be $\mathbb{Z}_r$. If $m \geq 6$, we have $K_m \cong K_6 \cup G_{6,m}$. We can factor K_6 into any graph-triple of order 6, and so T divides K_6. Thus, if T' divides $G_{6,m}$ for any $T' \subseteq T$, then T divides K_m.

For other terminology used but not defined herein, see [BM79], [LR97].

2 Main Multidecomposition Result

The results in [ADD+06] determine multidesigns for all K_m, $m \geq 6$, such that at least one of the graphs in the graph-triple has either three or five edges. This means that each of the remaining graph-triples of order 6 are of the form (G_1, G_2, H_k^4), where $\{G_1, G_2\} = \{H_i^7, H_j^4\}$ or $\{H_i^6, H_j^5\}$. We assume G_1 and G_2 satisfy the above throughout this paper. As suggested by Lemma 1.1, and since each triple we study includes H_k^4 for some k, it will be helpful to find H_k^4-decompositions of certain subgraphs of K_m, $m \geq 6$.

Lemma 2.1. We have the following H_i^4-decompositions for $i \in \{1, 2, 3\}$.

1. H_1^4 divides $K_{2,4}$, $K_{2,6}$, $K_{3,4}$, $K_{4,5}$, and $K_4 + K_4$.
2. H_2^4 divides $K_{3,4}$, $K_{4,4}$, $K_{4,5}$, and $K_{6,6}$.
3. H_3^4 divides $K_{2,4}$, $K_{3,4}$, $K_{4,5}$, and $K_{6,6}$.

Proof. For each of the following, we denote the partite sets of $K_{m,n}$ by $\{a, b, c, \ldots\}$ and $\mathbb{Z}_n$. We let $V(K_4 + K_4) = \mathbb{Z}_8$.

For (1), we have

$$\begin{aligned}
K_{2,4} &\cong [0, a, 1, 2, b, 3] \cup [0, b, 1, 2, a, 3] \\
K_{2,6} &\cong [0, a, 1, 2, b, 3] \cup [4, a, 5, 0, b, 1] \cup [2, a, 3, 4, b, 5] \\
K_{3,4} &\cong [0, a, 1, 2, b, 3] \cup [0, c, 1, 2, a, 3] \cup [0, b, 1, 2, c, 3] \\
K_{4,5} &\cong [a, 0, b, c, 1, d] \cup [a, 2, b, c, 3, d] \cup [a, 4, b, c, 0, d] \cup [a, 1, b, c, 2, d] \\
&\quad \cup [a, 3, b, c, 4, d] \\
K_4 + K_4 &\cong [0, 1, 2, 4, 5, 6] \cup [0, 3, 1, 4, 7, 5] \cup [0, 2, 3, 4, 6, 7]
\end{aligned}$$

For (2), we have

$$K_{3,4} \cong [0,a,1,b,2,c] \cup [a,2,b,3,0,c] \cup [a,3,c,1,0,b]$$
$$K_{4,4} \cong [0,a,1,b,d,2] \cup [b,2,c,3,d,1] \cup [3,d,0,b,c,1] \cup [b,3,a,2,c,0]$$
$$K_{4,5} \cong [0,a,1,b,4,c] \cup [b,2,c,3,d,1] \cup [3,d,4,a,0,b] \cup [c,0,d,2,b,4] \cup [b,3,a,2,c,1]$$
$$K_{6,6} \cong [0,a,1,b,2,e] \cup [b,2,c,3,f,1] \cup [3,d,4,e,2,a] \cup [e,5,f,0,a,3] \cup [0,b,3,e,5,d] \cup [e,0,c,1,f,3] \cup [1,d,2,f,5,c] \cup [f,4,a,5,d,0] \cup [5,b,4,c,1,e]$$

For (3), we have

$$K_{2,4} \cong [0,a,1,b,2,3] \cup [0,b,1,a,2,3]$$
$$K_{3,4} \cong [0,a,1,b,3,2] \cup [0,b,1,c,3,2] \cup [0,c,1,a,3,2]$$
$$K_{4,5} \cong [a,0,b,4,d,c] \cup [a,1,b,0,d,c] \cup [a,2,b,1,d,c] \cup [a,3,b,2,d,c] \cup [a,4,b,3,d,c]$$
$$K_{6,6} \cong [0,a,1,2,b,3] \cup [0,b,1,2,a,3] \cup [0,c,1,2,d,3] \cup [0,d,1,2,c,3] \cup [0,e,1,2,f,3] \cup [0,f,1,2,e,4] \cup [a,4,b,3,f,c] \cup [a,5,b,4,d,e] \cup [c,5,d,4,e,f]$$

□

We use this result to find other graphs that each H_i^4 divides.

Lemma 2.2. Each of H_i^4, $i \in \{1,2,3\}$, divides $K_{4,4}$, $K_{6,6}$, $K_{8,8}$, and K_8.

Proof. Since $K_{4,4} \cong 2K_{2,4}$, $K_{6,6} \cong 3K_{2,6}$, and $K_{8,8} \cong 8K_{2,4} \cong 4K_{4,4}$, Lemma 2.1 implies that H_i^4 divides $K_{4,4}$, $K_{6,6}$, and $K_{8,8}$. For K_8, we look at $i = 1,2,3$ separately. For $i = 1$, we have $K_8 \cong (K_4 + K_4) \cup K_{4,4}$. Since H_1^4 divides $K_4 + K_4$ and $K_{4,4}$ by Lemma 2.1(1), it follows that H_1^4 divides K_8. For $i = 2$, we begin with $[1,0,4,5,6,7]$, $[0,2,4,6,1,3]$, $[0,3,4,7,5,6]$, and $[3,2,1,4,5,7]$. The remaining edges form $K_{3,4}$, which H_2^4 divides by Lemma 2.1(2). Thus, H_2^4 divides K_8. Finally, for $i = 3$, we begin with $[0,4,1,5,6,2]$, $[5,4,6,0,2,7]$, $[0,1,2,6,7,3]$, and $[0,3,2,5,7,4]$. The remaining edges form $K_{3,4}$. As in the $i = 2$ case, we get H_3^4 divides K_8. □

The following results determine all K_m with $m \geq 6$, $m \neq 7,8$ such that T divides K_m. We determine multidesigns for K_7 and K_8 in the next section (Lemmas 3.2, 3.3, 3.4, and 3.5). The proofs here rely on many results proved in Section 3. These include multidecompositions for K_m, where $m = 9$, 10, 11, 13, and 15 (Lemmas 3.6, 3.8, 3.10, 3.11, and 3.12).

We consider the cases of m even and m odd separately.

Lemma 2.3. Let $T = (G_1, G_2, H_i^4)$ be a graph-triple of order 6. Then T divides K_m for all $m \geq 6$, m even, and $m \neq 8$.

Proof. We first consider the case $m \equiv 0 \pmod 6$. Let $m = 6k$ with $k \geq 1$. The case $k = 1$ is trivial. If $k \geq 2$, we have $K_m \cong kK_6 \cup \binom{k}{2} K_{6,6}$. Trivially, T divides K_6. Lemma 2.1 implies that H_i^4 divides $K_{6,6}$ for all $i \in \{1, 2, 3\}$. Lemma 1.1 then implies that T divides K_m.

We next take on the case $m \equiv 2 \pmod 6$. Since $m \neq 8$, $m = 6k + 2$ for some $k \geq 2$. If $k = 2$, we have

$$K_{14} \cong K_6 \cup K_8 \cup K_{6,8} \cong K_6 \cup K_8 \cup 4K_{3,4}$$

Trivially, T divides K_6, and each H_i^4 divides both K_8 by Lemma 2.2 and $K_{3,4}$ by Lemma 2.1. Thus, T divides K_{14}. For the case $k \geq 3$, we have

$$\begin{aligned} K_m &\cong K_{14} \cup K_{6(k-2)} \cup K_{6(k-2),6} \cup K_{6(k-2),8} \\ &\cong K_{14} \cup (k-2)K_6 \cup (k-2)K_{6,6} \cup (2k-4)K_{3,4} \end{aligned}$$

Note that T trivially divides K_6 and divides K_{14} by the $k = 2$ case. Moreover, each H_i^4 divides both $K_{6,6}$ by Lemma 2.2 and $K_{3,4}$ by Lemma 2.1. Thus, T divides K_m.

The last case is when $m \equiv 4 \pmod 6$. We then have $m = 6k + 4$ for some $k \geq 1$. The case $k = 1$ follows from Lemma 3.8. For $k \geq 2$, we have

$$K_m \cong K_{10} \cup K_{6(k-1)} \cup K_{10,6(k-1)} \cong K_{10} \cup (k-1)K_6 \cup (k-1)K_{6,6} \cup (k-1)K_{4,6}$$

Trivially, T divides K_6. In addition, T divides K_{10} by Lemma 3.8. By Lemmas 2.2 and 2.1, H_i^4 divides both $K_{6,6}$ and $K_{4,6} \cong 2K_{3,4}$. Therefore, T divides K_m. □

Lemma 2.4. Let $T = (G_1, G_2, H_i^4)$ be a graph-triple of order 6. Then T divides K_m for all $m \geq 9$ with m odd.

Proof. Since m is odd, we can write $m = 8k + r$, where $r \in \{1, 3, 5, 7\}$. We then have $K_m \cong K_{8+r} \cup K_{8(k-1)} \cup (k-1)K_{8+r,8}$. We have that T divides K_{8+r} by Lemmas 3.6, 3.10, 3.11, and 3.12. When $k \geq 2$, Lemma 2.2 implies that H_i^4 divides $K_{8(k-1)} \cong (k-1)K_8$. It then suffices to prove that H_i^4 divides $K_{8+r,8}$.

Note that $K_{9,8} \cong 6K_{3,4}$, $K_{11,8} \cong 4K_{3,4} \cup 2K_{4,5}$, $K_{13,8} \cong 2K_{3,4} \cup 4K_{4,5}$, and $K_{15,8} \cong 6K_{4,5}$. Thus, the edges of each $K_{8+r,8}$ can be decomposed into copies of $K_{3,4}$ and $K_{4,5}$. By Lemma 2.1, Each H_i^4 with $i \in \{1, 2, 3\}$ divides both $K_{3,4}$ and $K_{4,5}$. Thus, each H_i^4 divides $K_{8+r,8}$. It follows that T divides K_m. □

Lemmas 2.3 and 2.4 can be summarized as follows.

Theorem 2.5. Let $T = (G_1, G_2, H_i^4)$ be a graph-triple of order 6, where $i \in \{1, 2, 3\}$, and let $m \in \mathbb{Z}$ with $m \geq 6$, $m \neq 7, 8$. Then T divides K_m.

3 The Remaining Multidesigns

In this section, we determine multidesigns of K_7, K_8, K_9, K_{10}, K_{11}, K_{13}, and K_{15} for all graph-triples of the form $\{G_1, G_2, H_i^4\}$. We begin with K_7 and graph-triples that do not result in multidecompositions.

Lemma 3.1. Let $T = \{H_i^6, H_j^5, H_k^4\}$. If $i \in \{1, 8\}$ or if $T \in \{\{H_9^6, H_4^5, H_3^4\}, \{H_{10}^6, H_5^5, H_1^4\}\}$, then T does not divide K_7.

Proof. We begin with the case $i \in \{1, 8\}$. Assume that T divides K_7. The T-multidecomposition of K_7 must have two copies of H_i^6 and one copy each of H_j^5 and H_k^4. Furthermore, each vertex of H_1^6 and H_8^6 has degree 2. Thus, the degree sequence of $K_7 - 2H_i^6$ is either $(2,2,2,2,2,2,6)$ or $(2,2,2,2,2,4,4)$. We consider each of these cases in turn.

In the case that the degree sequence of $K_7 - 2H_i^6$ is $(2,2,2,2,2,2,6)$, observe that the only H_j^5 that appear in a graph-triple of order 6 with either H_1^6 or H_8^6 are H_1^5, H_5^5, and H_6^5. Thus, $\Delta(H_i^5) = 2$. Furthermore, only H_1^4 and H_2^4 appear in graph-triples with H_1^6 and H_8^6. Thus, the degree sequence of H_i^4 is $(1,1,1,1,2,2)$ It follows that removing H_j^5 and H_k^4 takes away at most 4 incident edges from the degree 6 vertex in $K_7 - 2H_i^6$. This leaves a vertex of degree at least 2 in $K_7 - 2H_i^6 - H_j^5 - H_k^4$, a contradiction.

Next is the case that the degree sequence of $K_7 - 2H_i^6$ is $(2,2,2,2,2,4,4)$. Since $K_7 - 2H_i^6 - H_k^4 \cong H_j^5$ and H_j^5 has only 6 vertices, $K_7 - 2H_i^6 - H_k^4$ must have an isolated vertex. Recall that the degree sequence of H_k^4 is $(1,1,1,1,2,2)$. One of the degree 2 vertices in $K_7 - 2H_i^6$ must then be a degree 2 vertex in H_k^4. It follows that at most one of the two degree 4 vertices in $K_7 - 2H_i^6$ can be a degree 2 vertex in H_k^4. One of the degree 4 vertices in $K_7 - 2H_i^6$ will then have degree 3 or 4 in $K_7 - 2H_i^6 - H_k^4$, which contradicts the fact that $\Delta(H_j^5) = 2$.

The case $T \in \{\{H_9^6, H_4^5, H_3^4\}, \{H_{10}^6, H_5^5, H_1^4\}\}$, follows from an exhaustive computer search. □

This result implies that the T-multidesigns of K_7, where $H_i^6 \in T$ for $i = 1$ or $i = 8$ are multipackings and multicoverings. Similarly, if $T = \{H_i^7, H_j^4, H_k^4\}$, when the edges of one copy each of H_i^7, H_j^4, and H_k^4 is removed from K_7, 6 edges remain, which means that no T-multidecompositions exist. We get the following.

Lemma 3.2. Let $T = (G_1, G_2, H_k^4)$ be graph-triple of order 6. Then

1. If $\{G_1, G_2\} = \{H_i^6, H_j^5\}$ with $i \in \{1, 8\}$, then
   (a) K_7 has a maximum T-multipacking with leave P_2.
   (b) K_7 has a minimum T-multicovering with padding $P_2 + P_2$

2. If $\{G_1, G_2\} = \{H_i^7, H_j^4\}$, then

   (a) K_7 has a maximum T-multipacking with leave P_3.

   (b) K_7 has a minimum T-multicovering with padding P_2.

3. If $T \in \{\{H_9^6, H_4^5, H_3^4\}, \{H_{10}^6, H_5^5, H_1^4\}\}$,

   (a) K_7 has a maximum T-multipacking with leave $P_2 + P_2$.

   (b) K_7 has a minimum T-multicovering with padding P_2.

Proof. We present the multidesigns for (1).

H_1^6, H_1^5, H_1^4 : Packing : $H_1^6 \cong [0,1,2,3,4,5]; H_1^5 \cong [0,2,4,1,3,6]$, $[1,6,2,0,3,5]; H_1^4 \cong [0,4,1,2,5,6]$ with leave 46
Cover : $H_1^6 \cong [0,1,2,3,4,5]; H_1^5 \cong [0,1,3,2,5,4]; H_1^4 \cong$ $[0,6,3,1,4,2], [1,6,5,2,0,3], [1,5,3,2,6,4]$ with padding 45, 01

H_1^6, H_1^5, H_2^4 : Packing : $H_1^6 \cong [0,1,2,3,4,5]; H_1^5 \cong [0,2,4,1,3,6]$, $[1,4,0,2,5,6,]; H_2^4 \cong [0,3,5,1,2,6]$ with leave 46
Cover: $H_1^6 \cong [0,1,2,3,4,5]; H_1^5 \cong [0,1,3,2,5,4]; H_2^4 \cong$ $[0,2,6,3,1,4], [3,0,6,4,1,5], [1,6,5,3,2,4]$ with padding 45, 01

H_1^6, H_5^5, H_2^4 : Packing: $H_1^6 \cong [0,1,2,3,4,5]; H_5^5 \cong [0,2,1,3,6,4]$, $[1,4,2,5,3,6]; H_2^4 \cong [1,5,6,2,0,3]$ with leave 06
Cover: $H_1^6 \cong [0,1,2,3,4,5]; H_5^5 \cong [0,1,2,3,5,4]; H_2^4 \cong$ $[0,2,6,4,1,3], [0,3,6,1,2,5], [0,6,5,1,2,4]$ with padding 23, 01

H_1^6, H_6^5, H_1^4 : Packing: $H_1^6 \cong [0,1,2,3,4,5]; H_6^5 \cong [0,2,1,3,4,6]$, $[1,5,0,3,2,6]; H_1^4 \cong [0,4,1,3,5,6]$ with leave 36.
Cover: $H_1^6 \cong [0,1,2,3,4,5]; H_6^5 \cong [0,2,1,5,4,6]; H_1^4 \cong$ $[0,1,3,2,5,4], [0,3,5,1,6,2], [0,4,1,3,6,5]$ with padding 01, 45

H_8^6, H_1^5, H_2^4 : Packing: $H_8^6 \cong [0,1,2,3,4,5]; H_1^5 \cong [0,2,1,3,6,4]$, $[0,3,1,2,5,6]; H_2^4 \cong [1,4,5,3,2,6]$ with leave 16
Cover: $H_8^6 \cong [0,1,2,3,4,5]; H_1^5 \cong [0,1,2,3,5,4]; H_2^4 \cong$ $[0,2,6,1,3,4], [0,6,3,1,2,5]; H_2^4 \cong [1,4,6,5,0,3]$ with padding 34, 01

H_8^6, H_6^5, H_1^4 : Packing: $H_8^6 \cong [0,1,2,3,4,5]; H_6^5 \cong [0,2,1,3,6,4]$, $[0,3,1,2,5,6]; H_1^4 \cong [1,6,3,2,5,4]$ with leave 14
Cover: $H_8^6 \cong [0,1,2,3,4,5]; H_6^5 \cong [0,1,2,5,3,4]; H_1^4 \cong$ $[0,6,3,2,1,4], [2,0,3,4,6,5], [1,6,2,3,5,4]$ with padding 34, 01

We continue with (2a). Note that $K_7 \cong K_6 \cup G_{6,7} \cong (K_6 - H_i^4) \cup (G_{6,7} \cup H_i^4)$. Since $\{H_i^6, H_j^5\}$ divides $K_6 - H_i^4$, it suffices to find an H_i^4-packing of $G_{6,7} \cup H_i^4$ with leave P_3 for each $i \in \{1, 2, 3\}$. We get

$$H_1^4 : [0,6,1,3,4,5], [0,1,2,4,6,5] \text{ with leave } 263$$
$$H_2^4 : [0,6,2,3,4,5], [5,6,4,3,0,1] \text{ with leave } 163$$
$$H_3^4 : [0,6,1,4,5,2], [3,6,4,0,1,5] \text{ with leave } 203$$

To prove that these multipackings are optimal, note that, after removing one copy of each graph in T from K_7, we are left with 6 edges that can be utilized by some combination of H_i^7, H_j^4, and H_k^4. The best we can do is add an additional copy of H_i^4 to get a leave of two edges.

We next prove (2b). We get the following minimum T-multicoverings.

$$H_1^7, H_1^4, H_2^4 : H_1^4 \cong [0,3,5,1,6,2]; H_2^4 \cong [1,5,6,0,2,4];$$
$$H_1^7 \cong [0,1,2,3,4,5], [0,2,1,3,6,4] \text{ with padding } 12$$
$$H_2^7, H_1^4, H_2^4 : H_1^4 \cong [1,5,2,3,0,6]; H_2^4 \cong [2,6,5,3,1,4];$$
$$H_2^7 \cong [0,1,2,3,4,5], [0,2,1,3,6,4] \text{ with padding } 12$$
$$H_3^7, H_1^4, H_2^4 : H_1^4 \cong [1,5,2,3,6,4]; H_2^4 \cong [1,6,5,3,2,4];$$
$$H_3^7 \cong [0,1,2,3,4,5], [0,3,4,1,2,6] \text{ with padding } 13$$
$$H_4^7, H_1^4, H_2^4 : H_1^4 \cong [1,3,5,4,0,6]; H_2^4 \cong [2,5,6,4,0,3];$$
$$H_4^7 \cong [0,1,2,3,4,5], [0,1,3,4,6,2] \text{ with padding } 01$$
$$H_4^7, H_1^4, H_3^4 : H_1^4 \cong [0,6,1,2,5,3]; H_3^4 \cong [3,6,4,0,2,5];$$
$$H_4^7 \cong [0,1,2,3,4,5], [0,3,2,6,1,4] \text{ with padding } 14$$
$$H_4^7, H_2^4, H_3^4 : H_2^4 \cong [0,6,5,3,1,2]; H_3^4 \cong [0,3,4,2,5,6];$$
$$H_4^7 \cong [0,1,2,3,4,5], [0,2,1,3,6,4] \text{ with padding } 24$$
$$H_5^7, H_1^4, H_2^4 : H_1^4 \cong [0,6,1,2,5,3]; H_2^4 \cong [3,2,0,4,5,6];$$
$$H_5^7 \cong [0,1,2,3,4,5], [0,1,4,2,6,3] \text{ with padding } 01$$
$$H_5^7, H_2^4, H_3^4 : H_2^4 \cong [1,3,5,2,0,4]; H_3^4 \cong [0,6,2,1,4,5];$$
$$H_5^7 \cong [0,1,2,3,4,5], [0,2,1,4,6,3] \text{ with padding } 12$$
$$H_6^7, H_1^4, H_2^4 : H_1^4 \cong [0,1,6,2,3,5]; H_2^4 \cong [0,6,5,2,1,3];$$
$$H_6^7 \cong [0,1,2,3,4,5], [1,0,2,3,6,4] \text{ with padding } 13$$
$$H_6^7, H_2^4, H_3^4 : H_2^4 \cong [2,5,3,4,0,1]; H_3^4 \cong [0,6,1,2,3,5];$$
$$H_6^7 \cong [0,1,2,3,4,5], [1,0,2,3,6,4] \text{ with padding } 34$$

$H^7_8, H^4_1, H^4_3 : H^4_1 \cong [2,0,3,4,6,5]; H^4_3 \cong [1,5,2,0,6,3];$
$H^7_8 \cong [0,1,2,3,4,5], [0,1,6,2,3,4]$ with padding 01
$H^7_9, H^4_1, H^4_2 : H^4_1 \cong [2,6,5,3,0,4]; H^4_2 \cong [3,5,0,6,1,4];$
$H^7_9 \cong [0,1,2,3,4,5], [0,2,1,4,6,3]$ with padding 12
$H^7_{10}, H^4_1, H^4_2 : H^4_1 \cong [0,3,5,2,1,6]; H^4_2 \cong [2,5,6,3,0,4];$
$H^7_{10} \cong [0,1,2,3,4,5], [0,2,1,3,4,6]$ with padding 34

For (3), we get

H^6_9, H^5_4, H^4_3 : Cover: $H^6_9 \cong [0,1,2,3,4,5]; H^5_4 \cong [0,3,4,1,2,5]; H^4_3 \cong$
$[1,0,2,3,4,5], [0,6,2,1,3,4], [1,6,3,0,4,5]$ with padding 10, 34
Packing: $H^6_9 \cong [0,1,2,3,4,5]; H^5_4 \cong [0,3,4,1,2,5],$
$[1,6,0,2,4,3]; H^4_3 \cong [2,6,4,1,3,5]$ with leave 05
H^6_{10}, H^5_5, H^4_1 : Cover: $H^6_{10} \cong [0,1,2,3,4,5]; H^5_5 \cong [0,1,2,4,5,3]; H^4_1 \cong$
$[0,2,1,3,6,4], [0,6,1,2,5,3], [0,4,3,2,6,5]$ with padding 45, 01
Packing: $H^6_{10} \cong [0,1,2,3,4,5]; H^5_5 \cong [0,2,1,3,6,4],$
$[1,2,0,3,4,6]; H^4_1 \cong [0,6,4,2,5,3]$ with leave 56

□

The remaining triples result in T-multidecompositions.

Lemma 3.3. Let $T = \{G_1, G_2, H^4_k\}$ be a graph-triple of order 6 such that $H^6_1, H^6_8, H^7_j \notin T$ for all j and $T \notin \{\{H^6_9, H^5_4, H^4_3\}, \{H^6_{10}, H^5_5, H^4_1\}\}$. Then T divides K_7.

Proof. We begin with $i \in \{5,6,7\}$. We have

$$K_7 \cong K_6 \cup G_{6,7} \cong (K_6 - H^6_i) \cup (G_{6,7} \cup H^6_i)$$

Since $\{H^5_j, H^4_j\}$ divides $K_6 - H^6_i$, it suffices to prove that H^6_i divides $G_{6,7} \cup H^6_i$. For $i = 5$, we have $G_{6,7} \cup H^6_5 \cong [1,0,3,5,6,2] \cup [2,3,5,1,6,4]$. For $i = 6$, we have $G_{6,7} \cup H^6_6 \cong [0,1,3,2,5,6] \cup [3,6,2,1,5,4]$. For $i = 7$, we have $G_{6,7} \cup H^6_7 \cong [4,0,1,3,2,6] \cup [1,3,4,2,5,6]$.

For the remaining cases, we list the multidecompositions for $T \neq \{H^6_9, H^5_4, H^4_3\}, \{H^6_{10}, H^5_5, H^4_1\}$.

$H_2^6, H_1^5, H_1^4 : H_2^6 \cong [0,1,2,3,4,5], [0,2,3,6,1,4]; H_1^5 \cong [5,2,1,3,0,6];$
$H_1^4 \cong [1,5,3,2,4,6]$

$H_2^6, H_1^5, H_2^4 : H_2^6 \cong [0,1,2,3,4,5], [0,2,1,4,6,3]; H_1^5 \cong [1,5,3,0,6,2];$
$H_2^4 \cong [1,6,5,2,0,4]$

$H_2^6, H_1^5, H_3^4 : H_2^6 \cong [0,1,2,3,4,5], [0,3,1,6,2,4]; H_1^5 \cong [0,6,4,5,1,2];$
$H_3^4 \cong [2,5,3,1,4,6]$

$H_2^6, H_2^5, H_2^4 : H_2^6 \cong [0,1,2,3,4,5], [0,2,1,4,6,3]; H_2^5 \cong [5,3,0,1,6,2];$
$H_2^4 \cong [2,1,5,6,0,4]$

$H_2^6, H_3^5, H_1^4 : H_2^6 \cong [0,1,2,3,4,5], [0,2,3,6,1,4]; H_3^5 \cong [6,5,2,4,0,3];$
$H_1^4 \cong [0,6,4,2,1,5]$

$H_2^6, H_3^5, H_2^4 : H_2^6 \cong [0,1,2,3,4,5], [0,2,1,4,6,3]; H_3^5 \cong [0,6,1,2,3,5];$
$H_2^4 \cong [1,5,2,6,0,4]$

$H_3^6, H_3^5, H_2^4 : H_3^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_3^5 \cong [0,6,1,2,3,5];$
$H_2^4 \cong [1,5,2,4,0,3]$

$H_3^6, H_5^5, H_1^4 : H_3^6 \cong [0,1,2,3,4,5], [0,2,3,1,6,4]; H_5^5 \cong [0,3,1,5,2,6];$
$H_1^4 \cong [1,2,4,3,5,6]$

$H_3^6, H_5^5, H_3^4 : H_3^6 \cong [0,1,2,3,4,5], [0,2,3,1,6,4]; H_5^5 \cong [0,3,1,2,5,6];$
$H_3^4 \cong [1,5,3,2,4,6]$

$H_3^6, H_6^5, H_2^4 : H_3^6 \cong [0,1,2,3,4,5], [0,2,3,1,6,4]; H_6^5 \cong [0,3,1,2,5,6];$
$H_2^4 \cong [1,5,2,4,3,6]$

$H_3^6, H_7^5, H_1^4 : H_3^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_7^5 \cong [1,6,2,4,3,5];$
$H_1^4 \cong [1,2,5,3,0,6]$

$H_4^6, H_1^5, H_1^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_1^5 \cong [2,6,1,0,3,5];$
$H_1^4 \cong [0,6,5,2,1,4]$

$H_4^6, H_1^5, H_2^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_1^5 \cong [1,2,5,3,0,6];$
$H_2^4 \cong [2,6,5,3,1,4]$

$H_4^6, H_1^5, H_3^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_1^5 \cong [2,1,6,0,3,5];$
$H_3^4 \cong [0,6,2,1,4,5]$

$H_4^6, H_2^5, H_2^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_2^5 \cong [3,0,1,2,6,5];$
$H_2^4 \cong [4,1,2,5,0,6]$

$H_4^6, H_3^5, H_1^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_3^5 \cong [1,6,0,3,5,2];$
$H_1^4 \cong [2,1,4,3,5,6]$

$H_4^6, H_3^5, H_2^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_3^5 \cong [1,2,5,3,0,6];$
$H_2^4 \cong [4,1,6,5,0,3]$

$H_4^6, H_5^5, H_2^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_5^5 \cong [1,2,0,3,5,6];$
$H_2^4 \cong [0,6,5,2,1,4]$

$H_4^6, H_6^5, H_1^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,3,1,4,6]; H_6^5 \cong [1,2,0,3,5,6];$
$H_1^4 \cong [0,4,2,5,3,6]$

$H_4^6, H_6^5, H_3^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,3,1,4,6]; H_6^5 \cong [1,2,0,4,5,6];$
$H_3^4 \cong [0,3,5,2,4,6]$

$H_4^6, H_7^5, H_1^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_7^5 \cong [2,5,1,4,0,6];$
$H_1^4 \cong [0,3,5,2,1,6]$

$H_4^6, H_7^5, H_2^4 : H_4^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_7^5 \cong [1,6,0,3,5,2];$
$H_2^4 \cong [0,6,5,3,1,4]$

$H_9^6, H_2^5, H_1^4 : H_9^6 \cong [0,1,2,3,4,5], [0,2,4,1,3,6]; H_2^5 \cong [0,6,1,4,3,5];$
$H_1^4 \cong [1,6,5,3,0,4]$

$H_9^6, H_3^5, H_2^4 : H_9^6 \cong [0,1,2,3,4,5], [0,2,4,1,3,6]; H_3^5 \cong [1,3,0,4,6,5];$
$H_2^4 \cong [1,6,0,5,3,4]$

$H_{10}^6, H_1^5, H_1^4 : H_{10}^6 \cong [0,1,2,3,4,5], [0,2,1,3,4,6]; H_1^5 \cong [3,0,4,1,6,5];$
$H_1^4 \cong [1,2,5,4,3,6]$

$H_{10}^6, H_2^5, H_1^4 : H_{10}^6 \cong [0,1,2,3,4,5], [0,2,1,3,4,6]; H_2^5 \cong [1,2,0,5,3,6];$
$H_1^4 \cong [0,4,3,2,5,6]$

$H_{10}^6, H_2^5, H_2^4 : H_{10}^6 \cong [0,1,2,3,4,5], [0,2,1,3,6,4]; H_2^5 \cong [1,2,0,5,3,6];$
$H_2^4 \cong [0,6,5,2,3,4]$

$H_{10}^6, H_2^5, H_3^4 : H_{10}^6 \cong [0,1,2,3,4,5], [0,2,1,3,4,6]; H_2^5 \cong [2,1,0,4,3,5];$
$H_3^4 \cong [1,6,3,0,4,5]$

$H_{10}^6, H_7^5, H_3^4 : H_{10}^6 \cong [0,1,2,3,4,5], [2,0,1,3,6,4]; H_7^5 \cong [2,5,0,3,1,6];$
$H_3^4 \cong [4,3,5,1,2,6]$

$H_{11}^6, H_1^5, H_2^4 : H_{11}^6 \cong [0,1,2,3,4,5], [0,3,6,1,2,4]; H_1^5 \cong [5,2,0,1,4,6];$
$H_2^4 \cong [1,3,5,4,0,6]$

$H_{11}^6, H_2^5, H_2^4 : H_{11}^6 \cong [0,1,2,3,4,5], [0,3,6,1,2,4]; H_2^5 \cong [0,2,3,4,5,6];$
$H_2^4 \cong [3,1,4,6,2,5]$

$H_{11}^6, H_3^5, H_1^4 : H_{11}^6 \cong [0,1,2,3,4,5], [0,3,6,1,2,4]; H_3^5 \cong [6,5,2,0,1,4];$
$H_1^4 \cong [0,6,4,1,3,5]$

$H^6_{11}, H^5_3, H^4_2 : H^6_{11} \cong [0,1,2,3,4,5], [0,3,6,1,2,4]; H^5_3 \cong [0,6,4,1,2,5];$
$H^4_2 \cong [1,3,5,4,0,2]$
$H^6_{11}, H^5_3, H^4_3 : H^6_{11} \cong [0,1,2,3,4,5], [0,3,6,1,2,4]; H^5_3 \cong [3,5,2,0,1,4];$
$H^4_3 \cong [0,6,4,1,3,5]$
$H^6_{11}, H^5_7, H^4_2 : H^6_{11} \cong [0,1,2,3,4,5], [0,3,6,1,2,4]; H^5_7 \cong [5,6,0,2,1,4];$
$H^4_2 \cong [1,3,5,2,0,6]$

□

Next, we turn out attention to K_8. We get the following multidecompositions.

Lemma 3.4. Let $T = \{H^6_i, H^5_j, H^4_k\}$ be a graph-triple of order 6. Then T divides K_8.

Proof. We begin with the case $T = (H^6_i, H^5_j, H^4_k)$. For each $a, b \in \mathbb{Z}_6$, let $S_{a,b}$ be the graph with vertex set $\{6, 7, a, b\}$ and edge set $\{67, 6a, 6b, 7a, 7b\}$. We then have

$$K_8 = S_{a,b} \cup K_6 \cup K_{2,4} = (S_{a,b} \cup H^5_j) \cup (K_6 - H^5_j) \cup K_{2,4}$$

where H^5_j uses the vertices in K_6. Note that $\{H^6_i, H^4_k\}$ divides $K_6 - H^5_j$, and if $i \neq 2$, then H^4_k divides $K_{2,4}$ by Lemma 2.1. Thus, in the case $k \neq 2$, Lemma 1.1 implies that we need only show that H^5_j divides $S_{a,b} \cup H^5_j$. We get

$$S_{2,3} \cup H^5_1 \cong [3,4,5,6,7,2] \cup [6,3,7,0,1,2]$$
$$S_{1,4} \cup H^5_2 \cong [7,4,0,2,1,6] \cup [1,7,6,5,4,3]$$
$$S_{2,3} \cup H^5_3 \cong [7,2,1,0,3,6] \cup [4,3,7,6,5,2]$$
$$S_{2,3} \cup H^5_5 \cong [6,7,3,4,5,2] \cup [0,1,6,3,7,2]$$
$$S_{1,2} \cup H^5_6 \cong [6,2,4,5,7,1] \cup [0,1,6,7,2,3]$$
$$S_{1,4} \cup H^5_7 \cong [6,4,2,3,1,7] \cup [0,2,4,5,6,1]$$

The remaining graph-triples are those in which either $j = 4$ or $k = 2$. We can decompose the edges of K_8 in the following ways.

$$K_8 \cong G_{6,8} \cup K_6 \cong (G_{6,8} \cup H^4_k) \cup (K_6 - H^4_k)$$
$$K_8 \cong G_{6,8} \cup K_6 \cong (G_{6,8} \cup H^5_j) \cup (K_6 - H^5_j)$$

By Lemma 1.1, we then need only show that there exists $T' \subseteq T$ such that either T' divides $G_{6,8} \cup H_k^4$, with $H_k^4 \in T'$, T' divides $G_{6,8} \cup H_j^5$ with $H_j^5 \in T'$, or T' divides $G_{6,8}$.

We get the following multidecompositions of $G_{6,8}$.

$$H_i^6, H_4^5, H_1^4 : \ H_4^5 \cong [0,6,7,2,3,1]; \ H_1^4 \cong [0,7,4,2,6,5], [1,7,5,3,6,4]$$
$$H_i^6, H_4^5, H_3^4 : \ H_4^5 \cong [0,7,6,4,5,3]; \ H_3^4 \cong [0,6,1,5,7,2], [1,7,2,3,6,4]$$

We get the following multidecompositions of $G_{6,8} \cup H_j^5$.

$$H_i^6, H_1^5, H_2^4 : \ H_1^5 \cong [6,1,0,4,3,7], [7,5,4,0,6,2]; \ H_2^4 \cong [2,3,6,5,1,7], [0,7,4,6,1,2]$$
$$H_i^6, H_6^5, H_2^4 : \ H_6^5 \cong [6,4,2,3,7,5], [6,3,1,2,7,0]; \ H_2^4 \cong [6,2,7,1,3,4], [7,6,1,4,5,0]$$
$$H_i^6, H_7^5, H_2^4 : \ H_7^5 \cong [1,2,3,4,5,6], [0,5,6,4,2,7]; \ H_2^4 \cong [4,7,6,0,2,3], [1,7,3,6,2,4]$$

We get the following multidecompositions of $G_{6,8} \cup H_k^4$.

$$H_i^6, H_2^5, H_2^4 : \ H_2^5 \cong [7,3,0,1,6,2]; \ H_2^4 \cong [6,4,7,0,2,3], [7,5,6,3,1,2], [6,7,1,0,4,5]$$
$$H_i^6, H_3^5, H_2^4 : \ H_3^5 \cong [2,1,6,0,4,7]; \ H_2^4 \cong [6,2,7,0,4,5], [7,3,6,5,0,1], [5,7,6,4,2,3]$$
$$H_i^6, H_4^5, H_2^4 : \ H_4^5 \cong [0,6,7,4,5,1]; \ H_2^4 \cong [6,2,7,0,4,5], [7,3,6,5,1,2], [6,4,1,7,0,3]$$
$$H_i^6, H_5^5, H_2^4 : \ H_5^5 \cong [2,6,4,7,5,3]; \ H_2^4 \cong [0,6,1,7,4,5], [5,6,7,3,1,2], [2,7,0,1,4,6]$$

This completes the proof. □

We continue determining multidesigns of K_8 for graph-triples of the form $T = \{H_i^7, H_j^4, H_l^4\}$. Since K_8 has 28 edges, 15 of which will be filled by the graph-triple, we find that the remaining 13 edges can not be filled with graphs of order 4 or 7. Therefore, we present optimal multipackings and multicoverings for K_8.

Lemma 3.5. Let $T = \{H_i^7, H_j^4, H_k^4\}$ be a graph-triple of order 6. Then K_8 has a maximum T-multipacking with leave P_2 and a minimum multicovering with padding P_2.

Proof. Here are maximum multipackings.

$H_1^7, H_1^4, H_2^4 : H_1^4 \cong [0,2,4,1,3,5], [0,4,1,2,6,3], [0,6,1,2,7,3], [0,7,1,4,6,5];$
$H_2^4 \cong [1,5,7,4,0,3]; H_1^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_2^7, H_1^4, H_2^4 : H_1^4 \cong [0,2,5,3,1,4], [1,6,2,3,0,4], [0,6,3,1,7,2], [0,7,4,3,5,6];$
$H_2^4 \cong [1,5,7,3,4,6]; H_2^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_3^7, H_1^4, H_2^4 : H_1^4 \cong [0,3,5,1,2,4], [0,4,1,2,6,3], [0,6,1,2,7,3], [0,7,4,1,5,6];$
$H_2^4 \cong [1,7,5,2,4,6]; H_3^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_4^7, H_1^4, H_2^4 : H_1^4 \cong [0,2,1,4,3,5], [0,3,1,2,6,4], [0,6,1,2,7,3], [0,7,1,2,5,6];$
$H_2^4 \cong [0,4,7,5,3,6]; H_4^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_4^7, H_1^4, H_3^4 : H_1^4 \cong [0,2,1,4,3,5], [0,3,1,2,6,4], [0,6,1,2,7,3], [0,4,7,2,5,6];$
$H_3^4 \cong [0,7,1,3,6,5]; H_4^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_4^7, H_2^4, H_3^4 : H_2^4 \cong [0,4,3,1,2,5], [0,3,6,4,1,2], [0,2,6,1,3,7], [0,6,7,1,3,5];$
$H_3^4 \cong [0,7,2,5,6,4]; H_4^7 \cong [0,1,2,3,4,5]$ with leave 57

$H_5^7, H_1^4, H_2^4 : H_1^4 \cong [0,2,5,3,1,4], [0,3,2,1,6,4], [0,6,2,1,7,3], [0,7,2,3,5,6];$
$H_2^4 \cong [0,4,7,5,3,6]; H_5^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_5^7, H_2^4, H_3^4 : H_2^4 \cong [0,2,3,5,1,4], [0,4,6,2,1,3], [0,3,6,1,2,7], [0,6,7,1,2,5];$
$H_3^4 \cong [0,7,3,5,6,4]; H_5^7 \cong [0,1,2,3,4,5]$ with leave 57

$H_6^7, H_1^4, H_2^4 : H_1^4 \cong [0,1,4,2,3,5], [1,6,2,3,0,4], [0,6,3,1,7,2], [0,7,3,2,5,6];$
$H_2^4 \cong [4,6,7,5,0,2]; H_6^7 \cong [0,1,2,3,4,5]$ with leave 47

$H_6^7, H_2^4, H_3^4 : H_2^4 \cong [0,2,3,5,1,4], [0,1,6,3,2,5], [1,7,0,3,2,6], [0,4,6,5,2,7];$
$H_3^4 \cong [3,7,4,0,6,5]; H_6^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_8^7, H_1^4, H_3^4 : H_1^4 \cong [0,2,5,1,3,4], [1,6,2,3,0,4], [0,6,3,1,7,2], [1,5,3,4,6,7];$
$H_3^4 \cong [0,7,3,5,6,4]; H_8^7 \cong [0,1,2,3,4,5]$ with leave 57

$H_9^7, H_1^4, H_2^4 : H_1^4 \cong [0,4,1,2,3,5], [0,3,1,2,6,4], [0,6,1,2,7,3], [0,5,6,1,7,4];$
$H_2^4 \cong [2,0,7,5,3,6]; H_9^7 \cong [0,1,2,3,4,5]$ with leave 67

$H_{10}^7, H_1^4, H_2^4 : H_1^4 \cong [0,2,5,3,1,4], [1,2,6,3,0,4], [0,6,1,2,7,3], [0,7,4,3,6,5];$
$H_2^4 \cong [1,7,5,3,4,6]; H_{10}^7 \cong [0,1,2,3,4,5]$ with leave 67

Here are the minimum multicoverings

$H_1^7, H_1^4, H_2^4 : H_1^4 \cong [0,6,2,1,7,4]; H_2^4 \cong [3,7,2,4,5,6]; H_1^7 \cong$
$[0,1,2,3,4,5], [0,2,1,3,6,4], [0,3,5,1,6,7]$ with padding 12

$H_2^7, H_1^4, H_2^4 : H_1^4 \cong [0,3,5,2,7,4]; H_2^4 \cong [3,7,5,6,1,4]; H_2^7 \cong$
$[0,1,2,3,4,5], [0,2,1,3,6,4], [5,1,7,0,6,2]$ with padding 12

$H_3^7, H_1^4, H_2^4 : H_1^4 \cong [1,6,3,4,7,5]; H_2^4 \cong [3,7,6,5,1,2]; H_3^7 \cong$
$[0,1,2,3,4,5], [0,1,3,5,2,4], [6,0,2,7,1,4]$ with padding 01

$H_4^7, H_1^4, H_2^4 : H_1^4 \cong [0,7,1,2,5,3]; H_2^4 \cong [2,7,5,6,0,3]; H_4^7 \cong$
$[0,1,2,3,4,5], [0,1,3,4,6,2], [0,4,3,1,7,6]$ with padding 01

$H_4^7, H_1^4, H_3^4 : H_1^4 \cong [0,4,6,2,5,7]; H_3^4 \cong [2,7,4,3,5,6]; H_4^7 \cong$
$[0,1,2,3,4,5], [0,1,3,4,6,2], [1,3,6,5,0,7]$ with padding 01

$H_4^7, H_2^4, H_3^4 : H_2^4 \cong [3,0,4,7,2,5]; H_3^4 \cong [1,7,2,4,6,3]; H_4^7 \cong$
$[0,1,2,3,4,5], [0,1,3,4,6,2], [0,6,3,1,5,7]$ with padding 01

$H_5^7, H_1^4, H_2^4 : H_1^4 \cong [0,7,1,2,6,5]; H_2^4 \cong [1,6,7,2,3,5]; H_5^7 \cong$
$[0,1,2,3,4,5], [0,2,5,4,1,3], [0,6,3,5,7,4]$ with padding 15

$H_5^7, H_2^4, H_3^4 : H_2^4 \cong [1,6,0,3,4,7]; H_3^4 \cong [1,7,3,2,5,6]; H_5^7 \cong$
$[0,1,2,3,4,5], [0,1,3,2,6,4], [0,2,3,6,5,7]$ with padding 01

$H_6^7, H_1^4, H_2^4 : H_1^4 \cong [0,7,2,1,6,5]; H_2^4 \cong [0,6,7,4,3,5]; H_6^7 \cong$
$[0,1,2,3,4,5], [1,0,2,3,6,4], [0,2,3,5,7,1]$ with padding 12

$H_6^7, H_2^4, H_3^4 : H_2^4 \cong [5,3,2,7,0,6]; H_3^4 \cong [3,7,4,0,1,6]; H_6^7 \cong$
$[0,1,2,3,4,5], [1,0,2,3,6,4], [0,1,2,6,5,7]$ with padding 12

$H_8^7, H_1^4, H_3^4 : H_1^4 \cong [0,4,6,5,2,7]; H_3^4 \cong [0,7,1,5,6,4]; H_8^7 \cong$
$[0,1,2,3,4,5], [0,1,3,4,6,2], [0,3,5,1,7,6]$ with padding 01

$H_9^7, H_1^4, H_2^4 : H_1^4 \cong [0,7,2,3,6,5]; H_2^4 \cong [2,6,7,5,0,3]; H_9^7 \cong$
$[0,1,2,3,4,5], [1,3,2,4,0,5], [0,6,1,3,7,4]$ with padding 25

$H_{10}^7, H_1^4, H_2^4 : H_1^4 \cong [2,1,6,3,7,5]; H_2^4 \cong [1,7,2,5,0,3]; H_{10}^7 \cong$
$[0,1,2,3,4,5], [0,2,1,3,4,6], [0,4,3,5,6,7]$ with padding 34

□

For K_9, all T-multidesigns are multidecompositions.

Lemma 3.6. Let $T = \{G_1, G_2, H_k^4\}$ be a graph-triple of order 6. Then T divides K_9.

Proof. We begin with the case $T = \{H_i^6, H_j^5, H_k^4\}$. We have $K_9 \cong K_6 \cup G_{6,9}$, and so we need only show that $\{H_j^5, H_k^4\}$ divides $G_{6,9}$ for each pair (j,k).

We have the following multidecompositions of $G_{6,9}$.

$H_i^6, H_1^5, H_1^4 : H_1^5 \cong [4,7,5,2,6,8];$

$H_1^4 \cong [1,6,7,2,8,3], [0,8,7,4,6,5], [1,7,2,3,6,0], [5,8,1,0,7,3]$

$H_i^6, H_1^5, H_2^4 : H_1^5 \cong [7,3,8,1,6,2];$

$H_2^4 \cong [4,6,8,5,0,7], [2,8,7,4,3,6], [1,7,6,5,4,8], [1,8,0,6,5,7]$

$H_i^6, H_1^5, H_3^4 : H_1^5 \cong [7,3,8,1,6,2];$

$H_3^4 \cong [4,8,5,6,7,0], [1,8,2,3,6,7], [4,6,5,0,7,8], [1,7,4,0,6,5]$

$H_i^6, H_2^5, H_1^4 : H_2^5 \cong [6,1,5,0,8,7];$

$H_1^4 \cong [8,6,2,3,7,4], [1,7,2,3,8,4], [5,6,0,1,8,2], [3,6,4,5,7,0]$

$H_i^6, H_2^5, H_2^4 : H_2^5 \cong [6,1,0,5,8,7];$

$H_2^4 \cong [2,6,8,4,3,7], [4,7,2,8,3,6], [1,7,5,6,3,8], [4,6,0,7,1,8]$

$H_i^6, H_2^5, H_3^4 : H_2^5 \cong [6,1,5,0,8,7];$

$H_3^4 \cong [2,7,3,6,8,4], [1,7,5,2,6,0], [2,8,3,0,6,4], [3,6,4,1,8,5]$

$H_i^6, H_3^5, H_1^4 : H_3^5 \cong [5,6,4,8,7,0];$

$H_1^4 \cong [1,6,7,5,8,0], [4,7,8,2,6,3], [3,8,6,1,7,2], [1,8,2,5,7,3]$

$H_i^6, H_3^5, H_2^4 : H_3^5 \cong [3,7,6,1,0,8];$

$H_2^4 \cong [2,6,8,4,5,7], [1,7,2,8,3,6], [1,8,5,6,4,7], [4,6,0,7,3,8]$

$H_i^6, H_3^5, H_3^4 : H_3^5 \cong [3,7,6,1,0,8];$

$H_3^4 \cong [2,6,4,5,7,8], [0,6,3,4,8,5], [1,7,2,5,8,4], [1,8,2,0,7,3]$

$H_i^6, H_4^5, H_1^4 : H_4^5 \cong [1,6,7,3,8,2];$

$H_1^4 \cong [6,8,4,5,7,0], [6,3,8,1,7,2], [6,4,7,5,8,0], [1,8,2,5,6,0]$

$H_i^6, H_4^5, H_2^4 : H_4^5 \cong [0,6,7,1,2,3];$

$H_2^4 \cong [0,7,8,1,2,6], [0,8,6,1,3,7], [2,8,4,6,5,7], [3,8,5,6,4,7]$

$H_i^6, H_4^5, H_3^4 : H_4^5 \cong [4,6,7,3,8,2];$

$H_3^4 \cong [0,6,1,2,7,5], [1,8,0,3,6,5], [2,8,3,1,7,4], [0,7,4,6,8,5]$

$H_i^6, H_5^5, H_1^4 : H_5^5 \cong [0,6,1,7,2,8];$

$H_1^4 \cong [0,7,3,1,6,2], [1,8,2,4,7,5], [3,6,4,5,8,7], [3,8,4,5,6,7]$

$H_i^6, H_5^5, H_2^4 : H_5^5 \cong [0,6,1,7,2,8];$

$H_2^4 \cong [0,7,6,1,2,8], [1,8,7,3,2,6], [3,6,4,8,5,7], [3,8,4,5,6,7]$

$H_i^6, H_5^5, H_3^4 : H_5^5 \cong [0,6,1,7,2,8];$

$H_3^4 \cong [0,7,3,1,6,4], [1,8,2,5,6,7], [2,6,3,5,7,4], [3,8,4,6,7,5]$

$H_i^6, H_6^5, H_1^4 : H_6^5 \cong [0,6,1,7,2,8];$

$H_1^4 \cong [0,7,2,1,6,3], [1,8,4,5,7,6], [3,7,4,5,8,6], [3,8,7,4,6,5]$

$H_i^6, H_6^5, H_2^4 : H_6^5 \cong [0,6,1,7,2,8]$;

$H_2^4 \cong [0,7,3,6,1,8],[1,6,7,4,5,8],[2,7,8,4,5,6],[3,8,6,4,5,7]$

$H_i^6, H_6^5, H_3^4 : H_6^5 \cong [0,6,1,7,2,8]$;

$H_3^4 \cong [0,7,2,1,6,8],[1,8,3,4,6,5],[3,6,5,4,7,8],[3,7,5,4,8,6]$

$H_i^6, H_7^5, H_1^4 : H_7^5 \cong [0,6,1,7,2,8]$;

$H_1^4 \cong [0,7,2,1,6,3],[1,8,7,2,6,4],[3,7,4,6,5,8],[3,8,4,5,7,6]$

$H_i^6, H_7^5, H_2^4 : H_7^5 \cong [0,6,1,7,2,8]$;

$H_2^4 \cong [0,7,2,6,1,8],[1,6,7,3,4,8],[3,6,5,8,4,7],[3,8,7,5,4,6]$

$H_i^6, H_7^5, H_3^4 : H_7^5 \cong [0,6,1,7,2,8]$;

$H_3^4 \cong [0,7,2,1,6,3],[1,8,4,2,6,5],[3,6,4,7,8,5],[4,7,5,3,8,6]$

We now consider the case $T = \{H_i^7, H_j^4, H_k^4\}$. We get the following multidecompositions of K_9.

$H_1^7, H_1^4, H_2^4 : H_1^4 \cong [1,5,3,6,7,8]; H_2^4 \cong [3,8,6,5,2,7]; H_1^7 \cong$
$[0,1,2,3,4,5],[0,2,4,1,3,6],[0,7,1,6,2,8],[0,3,7,5,8,4]$

$H_2^7, H_1^4, H_2^4 : H_1^4 \cong 1,6,8,3,7,5]; H_2^4 \cong [3,8,5,6,2,7]; H_2^7 \cong$
$[0,1,2,3,4,5],[0,2,5,3,1,4],[6,0,7,1,8,2],[0,3,6,7,4,8]$

$H_3^7, H_1^4, H_2^4 : H_1^4 \cong [1,8,3,2,6,7]; H_2^4 \cong [2,7,8,4,5,6]; H_3^7 \cong$
$[0,1,2,3,4,5],[4,1,2,5,3,0],[0,6,7,1,2,8],[6,3,4,7,5,8]$

$H_4^7, H_1^4, H_2^4 : H_1^4 \cong [4,6,7,5,3,8]; H_2^4 \cong [5,7,3,6,0,8]; H_4^7 \cong$
$[0,1,2,3,4,5],[0,2,3,4,1,6],[1,7,0,3,4,8],[6,5,7,0,2,8]$

$H_4^7, H_1^4, H_3^4 : H_1^4 \cong [2,7,5,3,6,8]; H_3^4 \cong [4,6,5,0,8,7]; H_4^7 \cong$
$[0,1,2,3,4,5],[1,7,0,3,4,8],[1,7,0,3,4,8],[2,5,7,0,3,8]$

$H_4^7, H_2^4, H_3^4 : H_2^4 \cong [2,7,6,5,0,8]; H_3^4 \cong [3,6,4,5,7,8]; H_4^7 \cong$
$[0,1,2,3,4,5],[0,2,3,4,1,6],[1,7,0,3,4,8],[2,5,7,0,3,8]$

$H_5^7, H_1^4, H_2^4 : H_1^4 \cong [5,3,6,7,4,8]; H_2^4 \cong [0,4,6,7,5,8]; H_5^7 \cong$
$[0,1,2,3,4,5],[0,2,3,4,1,6],[0,7,1,2,8,3],[2,5,6,0,8,7]$

$H_5^7, H_2^4, H_3^4 : H_2^4 \cong [0,8,6,7,3,5]; H_3^4 \cong [4,8,5,3,6,7]; H_5^7 \cong$
$[0,1,2,3,4,5],[0,2,3,4,1,6],[0,7,1,2,8,3],[2,5,6,0,4,7]$

$H_6^7, H_1^4, H_2^4 : H_1^4 \cong [2,3,5,4,7,8]; H_2^4 \cong [4,8,6,7,2,5]; H_6^7 \cong$
$[0,1,2,3,4,5],[1,0,2,3,6,4],[0,7,2,3,8,1],[1,0,7,8,5,6]$

$H_6^7, H_2^4, H_3^4 : H_2^4 \cong [2,3,5,6,7,8]; H_3^4 \cong [2,5,7,0,6,8]; H_6^7 \cong$
$[0,1,2,3,4,5],[1,0,2,3,6,4],[0,7,2,3,8,1],[1,7,0,4,8,6]$
$H_8^7, H_1^4, H_3^4 : H_1^4 \cong [1,8,3,4,7,6]; H_3^4 \cong [2,8,4,3,7,6]; H_8^7 \cong$
$[0,1,2,3,4,5],[0,4,3,1,6,2],[0,6,5,2,1,7],[0,8,7,2,5,3]$
$H_9^7, H_1^4, H_2^4 : H_1^4 \cong [0,4,6,3,5,8]; H_2^4 \cong [2,7,4,8,0,6]; H_9^7 \cong$
$[0,1,2,3,4,5],[0,2,3,4,1,6],[1,7,0,2,8,3],[0,5,6,1,8,7]$
$H_{10}^7, H_1^4, H_2^4 : H_1^4 \cong [2,6,5,3,0,8]; H_2^4 \cong [2,5,7,3,4,8]; H_{10}^7 \cong$
$[0,1,2,3,4,5],[0,2,1,3,6,7],[0,4,1,2,8,6],[1,4,3,5,8,7]$

This completes the proof. □

Now we move to the K_{10} case.

Lemma 3.7. Let $T = \{G_1, G_2, H_i^4\}$ be a graph-triple of order 6 with $i \in \{1,2,3\}$. Then there exists $T' \subseteq T$ such that T' divides $G_{6,10}$.

Proof. We begin with the case where G_1 and G_2 have 6 edges and 5 edges, respectively. In this case, we let $T' = \{H_i^5\}$. We get H_1^5-decompositions of $G_{6,10}$ as follows.

$H_1^5 \cong [0,6,9,8,1,7],[3,7,8,6,2,9],[4,6,7,9,5,8],[1,9,8,7,5,6],$
$[2,7,9,6,3,8],[0,8,6,7,4,9]$
$H_2^5 \cong [7,2,0,8,6,1],[8,2,4,6,9,3],[8,0,4,7,6,5],[9,1,0,8,7,5],$
$[8,1,3,9,7,4],[6,3,0,8,9,2]$
$H_3^5 \cong [9,1,6,7,2,8],[6,2,7,8,3,9],[7,3,8,9,4,6],[8,4,9,6,5,7],$
$[6,5,9,7,0,8],[9,0,6,8,1,7]$
$H_4^5 \cong [0,6,7,1,3,2],[2,8,9,3,5,4],[4,6,8,5,0,1],[0,7,9,1,2,5],$
$[2,7,8,3,1,4],[4,9,6,5,3,0]$
$H_5^5 \cong [0,6,8,3,9,7],[0,8,6,3,7,9],[1,6,7,4,9,8],[1,7,6,4,8,9],$
$[2,6,7,5,8,9],[2,7,6,5,9,8]$
$H_6^5 \cong [6,1,8,9,7,2],[6,3,7,9,8,4],[6,5,7,8,9,0],[7,3,6,8,9,4],$
$[7,5,6,9,8,0],[8,1,6,7,9,2]$
$H_7^5 \cong [6,7,1,9,8,0],[7,8,0,9,6,1],[8,9,5,6,7,2],[6,8,4,7,9,3],$
$[6,9,3,7,8,4],[7,9,2,6,8,5]$

Now we move on to triples of the form $T = \{H_i^7, H_j^4, H_k^4\}$. For $i = j = 1$, $k = 2$, we let $T' = \{H_1^7, H_1^4\}$ and get the following T'-multidecomposition

$H_1^7 \cong [8,9,0,6,1,7],[9,6,7,2,8,3]$
$H_1^4 \cong [6,4,7,8,5,9],[6,5,7,8,4,9],[6,8,0,2,9,7],[8,1,9,2,6,3]$

For $i = k = 2$, $j = 1$, we let $T' = \{H_2^7, H_1^4, H_2^4\}$ to get

$$H_2^7 \cong [6,1,7,8,9,0],[2,8,3,9,6,7]$$
$$H_1^4 \cong [6,4,7,8,5,9],[6,5,7,8,4,9]$$
$$H_2^4 \cong [8,6,2,9,7,0],[9,1,8,0,7,3]$$

For $i = 3$, $j = 1$, and $k = 2$, we let $T' = \{H_3^7, H_1^4, H_2^4\}$ to get

$$H_3^7 \cong [1,6,7,2,9,8],[3,8,9,4,7,6]$$
$$H_1^4 \cong [6,8,0,7,9,1],[6,9,5,7,8,2]$$
$$H_2^4 \cong [7,0,6,4,8,5],[6,5,7,3,9,0]$$

For $i = 4$, $j = 1$, and $k = 2,3$, we let $T' = \{H_4^7, H_1^4\}$ to get

$H_4^7 \cong [8,7,3,9,6,1],[6,8,7,4,2,9]$
$H_1^4 \cong [9,7,0,4,8,5],[3,7,5,0,6,2],[4,6,5,0,9,1],[0,8,3,4,9,5]$

For $i = 4$, $j = 2$, and $k = 3$, we let $T' = \{H_4^7, H_2^4\}$ to get

$H_4^7 \cong [8,7,3,9,6,1],[6,8,7,4,2,9]$
$H_2^4 \cong [7,9,0,8,6,2],[6,5,7,0,8,4],[3,8,5,9,6,0],[6,4,9,1,7,3]$

For $i = 5$, $j = 2$, and $k = 1,3$, we let $T' = \{H_5^7, H_2^4\}$ to get

$H_5^7 \cong [9,0,6,4,7,8],[8,1,7,2,9,6]$
$H_2^4 \cong [6,2,8,3,9,1],[8,4,6,3,7,2],[6,5,7,3,9,4],[8,5,9,3,7,0]$

For $i = 6$, $j = 2$, and $k = 1,3$, we let $T' = \{H_6^7, H_2^4\}$ to get

$H_6^7 \cong [5,1,8,7,9,6],[9,8,7,2,6,0]$
$H_2^4 \cong [6,4,9,5,7,0],[8,3,7,2,9,1],[8,4,7,5,9,3],[3,6,8,5,9,2]$

For $i = 8$, $j = 1$, and $k = 3$, we let $T' = \{H_8^7, H_1^4\}$ to get

$H_8^7 \cong [6,0,9,1,8,7],[6,9,7,1,2,8]$
$H_1^4 \cong [0,7,3,6,1,8],[2,6,3,8,4,9],[6,4,7,8,5,9],[6,5,7,8,3,9]$

For $i = 9$, $j = 1$, and $k = 2$, we let $T' = \{H_9^7, H_1^4\}$ to get

$$H_9^7 \cong [9,0,8,4,7,6], [2,8,9,3,7,1]$$
$$H_1^4 \cong [1,6,2,3,9,4], [3,6,4,5,7,0], [6,5,8,7,2,9], [6,9,5,3,8,4]$$

For $i = 10$, $j = 1$, and $k = 2$, we let $T' = \{H_{10}^7, H_1^4\}$ to get

$$H_{10}^7 \cong [7,0,1,8,9,6], [6,2,3,9,7,8]$$
$$H_1^4 \cong [6,4,7,8,5,9], [6,5,7,8,4,9], [6,3,8,7,2,9], [6,1,7,8,0,9]$$

This completes the proof. □

Since T divides K_6, T' divides $G_{6,10}$ for some $T' \subseteq T$, and $K_{10} \cong K_6 \cup G_{6,10}$, Lemma 1.1 gives us the following.

Lemma 3.8. Let $T = \{G_1, G_2, H_i^4\}$ be a graph-triple of order 6 with $i \in \{1,2,3\}$. Then T divides K_{10}.

For multidecompositions of K_{11}, we use the following.

Lemma 3.9. H_i^4 divides $G_{6,11}$ for $i \in \{1,2,3\}$.

Proof. We have the following H_i^4-decompositions of $G_{6,11}$.

$$H_1^4 \cong [1,6,2,7,8,9], [3,7,4,9,10,6], [5,8,0,6,7,10], [1,9,2,10,8,6], [3,10,4,6,9,7], [5,6,0,1,7,2], [3,8,4,5,9,0], [1,10,2,3,6,4], [5,7,0,1,8,2], [3,9,4,5,10,0]$$
$$H_2^4 \cong [1,6,7,2,10,4], [2,6,8,1,10,3], [3,6,9,1,7,0], [4,6,10,1,8,0], [3,7,8,2,9,5], [4,7,9,2,8,5], [5,7,10,2,6,0], [3,8,9,4,6,5], [4,8,10,5,9,0], [3,9,10,0,7,1]$$
$$H_3^4 \cong [1,6,2,7,8,3], [4,6,5,7,9,0], [1,7,2,8,10,3], [4,7,5,6,9,0], [1,8,2,9,10,3], [4,8,5,6,10,0], [1,9,2,6,7,3], [4,9,5,7,10,0], [1,10,2,8,9,3], [4,10,5,6,8,0]$$

□

Since $K_{11} \cong K_6 \cup G_{6,11}$, T divides K_6, and H_i divides $G_{6,11}$ for all $i \in \{1,2,3\}$, we have

Lemma 3.10. Let $T = \{G_1, G_2, H_k^4\}$ be a graph-triple of order 6. Then T divides K_{11}.

We now move on to K_{13}.

Lemma 3.11. Let $T = \{G_1, G_2, H_i^4\}$ with $i \in \{1,2,3\}$. Then T divides K_{13}.

Proof. Let $T = \{H_i^4, G_1, G_2\}$. We begin by noting that for each $i \in \{1,2,3\}$,

$$\begin{aligned} K_{13} &\cong 2K_6 \cup K_{6,6} \cup K_{1,12} \\ &\cong K_6 \cup (K_6 - H_i^4) \cup K_{6,6} \cup (K_{1,12} \cup H_i^4) \\ &\cong 2(K_6 - H_i^4) \cup (K_{6,6} - H_i^4) \cup (K_{1,12} \cup 3H_i^4) \end{aligned}$$

where the vertices of the copies of K_6 (as well as the partite sets of $K_{6,6}$) are $\mathbb{Z}_6$ and $\{6,7,8,9,10,11\}$. We have that T divides K_6, $\{G_1, G_2\}$ divides $K_6 - H_i^4$ and H_i^4 divides $K_{6,6}$ by Lemma 2.2. It suffices to show that H_i^4 divides either $K_{1,12} \cup H_i^4$ or $K_{1,12} \cup 3H_i^4$, where each copy of H_i^4 is taken from either a copy of K_6 or from $K_{6,6}$ if needed.

We begin with $i \in \{1,2\}$, where we decompose $K_{1,12} \cup 3H_i^4$. From the proof of Lemma 1.1(1) and (2), we can assume the copy of H_1^4 taken from $K_{6,6}$ is $[6,0,7,8,1,9]$ and the copy of H_2^4 taken from $K_{6,6}$ is $[6,3,7,4,5,11]$. We then decompose $K_{1,12} \cup 3H_i^4$ as follows.

$$\begin{aligned} H_1^4 \cong &[0,12,1,3,4,5], [2,12,3,6,7,8], [4,12,5,9,10,11], \\ &[6,12,7,0,1,2], [8,12,9,6,0,7], [10,12,11,8,1,9] \\ H_2^4 \cong &[9,12,10,11,3,7], [5,12,8,9,0,1], [6,12,7,8,4,5], \\ &[2,3,12,4,6,7], [2,12,11,5,3,6], [0,12,1,2,4,7] \end{aligned}$$

Finally, for $i = 3$, we decompose $K_{1,12} \cup H_i^4$ as follows.

$$H_3^4 \cong [0,12,1,4,5,2], [9,12,10,1,3,11], [3,12,4,0,1,5], [6,12,7,1,2,8]$$

□

Finally, we address K_{15}.

Lemma 3.12. Let $T = \{G_1, G_2, H_i^4\}$ with $i \in \{1,2,3\}$. Then T divides K_{15}.

Proof. First use the fact $G_{6,10} = K_{4,6} \cup K_4$ and $K_{4,11} = K_{4,6} \cup K_{4,5}$ to get

$$K_{15} \cong K_{11} \cup K_4 \cup K_{4,11} \cong K_{11} \cup G_{6,10} \cup K_{4,5}$$

By Theorem 3.10, T divides K_{11}. By Lemma 3.7, there exists some $T' \subseteq T$ such that T' divides $G_{6,10}$. By Lemma 2.1, H_i^4 divides $K_{4,5}$. Lemma 1.1 then implies that T divides K_{15}. □

When we combine these results with those of Theorem 2.5, we get

Theorem 3.13. Let $T = \{G_1, G_2, H_k^4\}$ be a graph-triple of order 6, where $k \in \{1, 2, 3\}$, and let $m \geq 6$.

1. If $\{G_1, G_2\} = \{H_i^6, H_j^5\}$ with $i \in \{1, 8\}$, then K_7 has a maximum T-multipacking with leave P_2 and a minimum T-multicovering with padding $P_2 + P_2$.
2. If $\{G_1, G_2\} = \{H_i^7, H_j^4\}$, then K_7 has a maximum T-multipacking with leave P_3 and a minimum T-multicovering with padding P_2.
3. If $T \in \{\{H_9^6, H_4^5, H_3^4\}, \{H_{10}^6, H_5^5, H_1^4\}\}$, then K_7 has a maximum T-multipacking with leave P_2+P_2 and a minimum T-multicovering with padding P_2.
4. If $\{G_1, G_2\} = \{H_i^7, H_j^4\}$, Then K_8 has a maximum T-multipacking with leave P_2 and a minimum multicovering with padding P_2.
5. For all graph-triples not covered by (1), (2), (3), and (4), T divides K_m.

4 Conclusion

This paper settles the T-multidesign problem of K_m into graph-triples T of order 6. However, there are several ways to extend our work.

- *Find Multidesigns for Graph-Pairs and Graph-Triples of Higher Order.* It certainly seems reasonable to attack graph-pairs and triples of order 7 or higher. However, it will become computationally more difficult to generate the graph-pairs and triples and perhaps more difficult to find arguments that generate multidesigns for large collections of the pairs and triples.
- *Multidesigns for Graphs Other than Complete Graphs.*
- *Multidesigns for K_n that have specified leaves or paddings.* Multidesigns whose leave or padding is P_2 have only one possible leave or padding up to isomorphism. However, every multidesign whose leave or padding has more than one edge has the potential of having different leaves or paddings.

References

[AD03] A. Abueida and M. Daven, *Multidesigns for graph pairs of order 4 and 5*, Graphs Combin. **19** (2003), 433–447.

[ADD+06] A. Abueida, M. Daven, W.S Diestelkamp, S.P. Edwards, and D.B. Parker, *Multidesigns for Graph-Triples of Order 6*, Congr. Numer. **183** (2006), 139–160.

[ADR05] A. Abueida, M. Daven, and K. Roblee, *Multidesigns of the λ-fold complete graph for graph-pairs of orders 4 and 5*, Australas. J. Combin. **32** (2005), 125–136.

[BHRS80] J.-C. Bermond, C. Huang, A. Rosa, and D. Sotteau, *Decompositions of complete graphs into isomorphic subgraphs with 5 vertices*, Ars Comb. **10** (1980), 211–254.

[BM79] J.A. Bondy and U.S.R Murty, *Graph Theory with Applications*, North-Holland, New York, 1979.

[BS77] J.-C. Bermond and Schönheim, *G-decomposition of K_n, where G has four vertices or less*, Discrete Math. **19** (1977), 113–120.

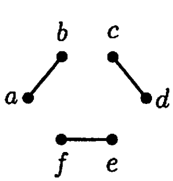
[HP73] F. Harary and E. Palmer, *Graphical Enumeration*, Academic Press, New York, 1973.

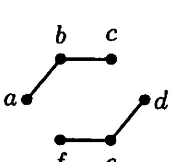
[Kot65] A. Kotzig, *On the decomposition of complete graphs into 4k-gons*, Mat. Fyz. Casop. **15** (1965), 229–233.

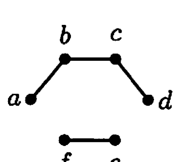
[LR97] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, Boca Raton, 1997.

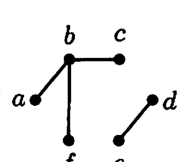
5 Appendices

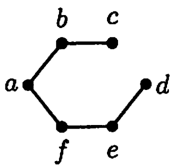
A Graphs of Order 6 that are Part of Graph-Triples

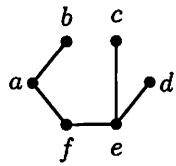
$H_1^3 \cong$ 

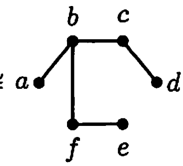
$H_1^4 \cong$ 

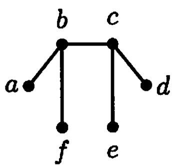
$H_2^4 \cong$ 

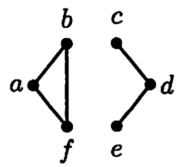
$H_3^4 \cong$ 

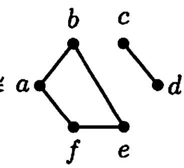
$H_1^5 \cong$ 

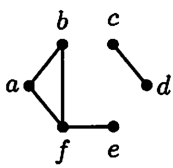
$H_2^5 \cong$ 

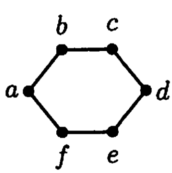
$H_3^5 \cong$ 

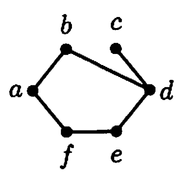
$H_4^5 \cong$ 

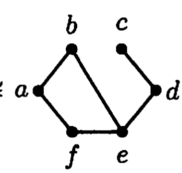
$H_5^5 \cong$ 

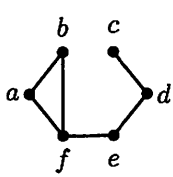
$H_6^5 \cong$ 

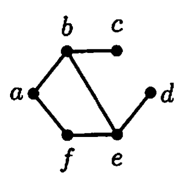
$H_7^5 \cong$ 

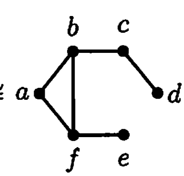
$H_1^6 \cong$ 

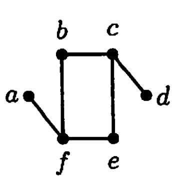
$H_2^6 \cong$ 

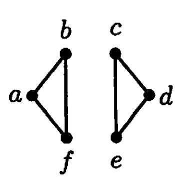
$H_3^6 \cong$ 

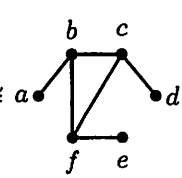
$H_4^6 \cong$ 

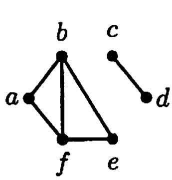
$H_5^6 \cong$ 

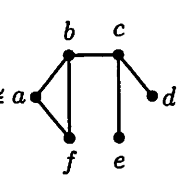
$H_6^6 \cong$ 

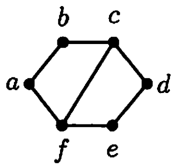
$H_7^6 \cong$ 

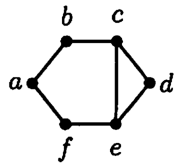
$H_8^6 \cong$ 

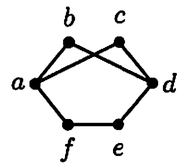
$H_9^6 \cong$ 

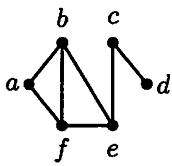
$H_{10}^6 \cong$ 

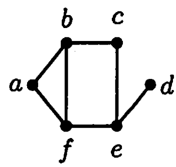
$H_{11}^6 \cong$ 

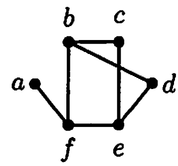
$H_1^7 \cong$ 

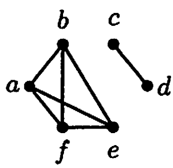
$H_2^7 \cong$ 

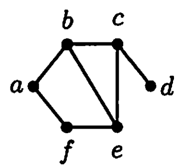
$H_3^7 \cong$ 

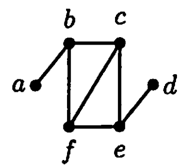
$H_4^7 \cong$ 

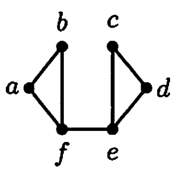
$H_5^7 \cong$ 

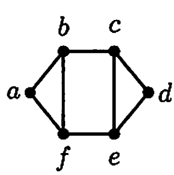
$H_6^7 \cong$ 

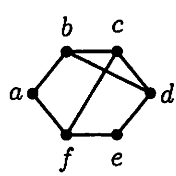
$H_7^7 \cong$ 

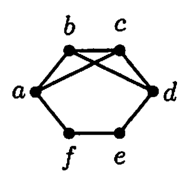
$H_8^7 \cong$ 

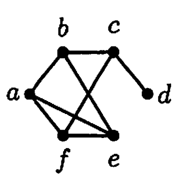
$H_9^7 \cong$ 

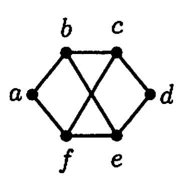
$H_{10}^7 \cong$ 

$H_1^8 \cong$ 

$H_2^8 \cong$ 

$H_3^8 \cong$ 

$H_4^8 \cong$ 

$H_5^8 \cong$ 

B The Graph-Triples of Order 6

The graph triples of order six $T = (G_1, G_2, G_3) = (H_{i_1}^{j_1}, H_{i_2}^{j_2}, H_{i_3}^{j_3})$, where j_k represents the number of edges in the graph G_k.

For $j_1 = 8,\ j_2 = 4,\ j_3 = 3$,

$$\begin{array}{llll} T = (G_1,G_2,G_3) \in & \{(H_1^8,\ H_1^4,\ H_1^3), & (H_1^8,\ H_2^4,\ H_1^3), & (H_2^8,\ H_2^4,\ H_1^3), \\ & (H_3^8,\ H_1^4,\ H_1^3), & (H_4^8,\ H_3^4,\ H_1^3)\}, & (H_5^8,\ H_1^4,\ H_1^3). \end{array}$$

For $j_1 = 7,\ j_2 = 4,\ j_3 = 4$,

$$\begin{array}{llll} T = (G_1,G_2,G_3) \in & \{(H_1^7,\ H_1^4,\ H_2^4), & (H_2^7,\ H_1^4,\ H_2^4), & (H_3^7,\ H_1^4,\ H_2^4), \\ & (H_4^7,\ H_1^4,\ H_2^4), & (H_4^7,\ H_1^4,\ H_3^4), & (H_4^7,\ H_2^4,\ H_3^4), \\ & (H_5^7,\ H_1^4,\ H_2^4), & (H_5^7,\ H_2^4,\ H_3^4), & (H_6^7,\ H_1^4,\ H_2^4), \\ & (H_6^7,\ H_2^4,\ H_3^4), & (H_8^7,\ H_1^4,\ H_3^4), & (H_9^7,\ H_1^4,\ H_2^4), \\ & (H_{10}^7,\ H_1^4,\ H_2^4)\}. & & \end{array}$$

For $j_1 = 7,\ j_2 = 5,\ j_3 = 3$,

$$\begin{array}{llll} T = (G_1,G_2,G_3) \in & \{(H_1^7,\ H_1^5,\ H_1^3), & (H_2^7,\ H_1^5,\ H_1^3), & (H_2^7,\ H_5^5,\ H_1^3), \\ & (H_3^7,\ H_1^5,\ H_1^3), & (H_3^7,\ H_6^5,\ H_1^3), & (H_4^7,\ H_2^5,\ H_1^3), \\ & (H_5^7,\ H_2^5,\ H_1^3), & (H_5^7,\ H_3^5,\ H_1^3), & (H_5^7,\ H_7^5,\ H_1^3), \\ & (H_6^7,\ H_2^5,\ H_1^3), & (H_8^7,\ H_3^5,\ H_1^3), & (H_9^7,\ H_4^5,\ H_1^3), \\ & (H_{10}^7,\ H_1^5,\ H_1^3)\}. & & \end{array}$$

For $j_1 = 6,\ j_2 = 5,\ j_3 = 4$,

$$\begin{array}{llll} T = (G_1,G_2,G_3) \in & \{(H_1^6,\ H_1^5,\ H_1^4), & (H_1^6,\ H_1^5,\ H_2^4), & (H_1^6,\ H_5^5,\ H_2^4), \\ & (H_1^6,\ H_6^5,\ H_1^4), & (H_2^6,\ H_1^5,\ H_1^4), & (H_2^6,\ H_1^5,\ H_2^4), \\ & (H_2^6,\ H_1^5,\ H_3^4), & (H_2^6,\ H_2^5,\ H_2^4), & (H_2^6,\ H_3^5,\ H_1^4), \\ & (H_2^6,\ H_3^5,\ H_2^4), & (H_2^6,\ H_5^5,\ H_2^4), & (H_2^6,\ H_6^5,\ H_2^4), \\ & (H_2^6,\ H_7^5,\ H_1^4), & (H_2^6,\ H_7^5,\ H_2^4), & (H_3^6,\ H_1^5,\ H_1^4), \\ & (H_3^6,\ H_1^5,\ H_2^4), & (H_3^6,\ H_1^5,\ H_3^4), & (H_3^6,\ H_2^5,\ H_1^4), \\ & (H_3^6,\ H_2^5,\ H_2^4), & (H_3^6,\ H_3^5,\ H_2^4), & (H_3^6,\ H_5^5,\ H_1^4), \\ & (H_3^6,\ H_5^5,\ H_3^4), & (H_3^6,\ H_6^5,\ H_2^4), & (H_3^6,\ H_7^5,\ H_1^4), \\ & (H_4^6,\ H_1^5,\ H_1^4), & (H_4^6,\ H_1^5,\ H_2^4), & (H_4^6,\ H_1^5,\ H_3^4), \\ & (H_4^6,\ H_2^5,\ H_2^4), & (H_4^6,\ H_3^5,\ H_1^4), & (H_4^6,\ H_3^5,\ H_2^4), \\ & (H_4^6,\ H_5^5,\ H_2^4), & (H_4^6,\ H_6^5,\ H_1^4), & (H_4^6,\ H_6^5,\ H_3^4), \\ & (H_4^6,\ H_7^5,\ H_1^4), & (H_4^6,\ H_7^5,\ H_2^4), & (H_5^6,\ H_1^5,\ H_2^4), \\ & (H_5^6,\ H_2^5,\ H_2^4), & (H_5^6,\ H_3^5,\ H_1^4), & (H_5^6,\ H_3^5,\ H_2^4), \\ & (H_5^6,\ H_3^5,\ H_3^4), & (H_5^6,\ H_4^5,\ H_1^4), & (H_5^6,\ H_7^5,\ H_2^4), \\ & (H_6^6,\ H_1^5,\ H_1^4), & (H_6^6,\ H_1^5,\ H_2^4), & (H_6^6,\ H_2^5,\ H_1^4), \\ & (H_6^6,\ H_2^5,\ H_2^4), & (H_6^6,\ H_2^5,\ H_3^4), & (H_6^6,\ H_3^5,\ H_1^4), \\ & (H_6^6,\ H_3^5,\ H_2^4), & (H_6^6,\ H_3^5,\ H_3^4), & (H_6^6,\ H_4^5,\ H_2^4), \\ & (H_6^6,\ H_5^5,\ H_2^4), & (H_6^6,\ H_7^5,\ H_1^4), & (H_6^6,\ H_7^5,\ H_2^4), \\ & (H_6^6,\ H_7^5,\ H_3^4), & (H_7^6,\ H_1^5,\ H_2^4), & (H_7^6,\ H_2^5,\ H_2^4), \\ & (H_7^6,\ H_3^5,\ H_2^4), & (H_7^6,\ H_4^5,\ H_2^4), & (H_7^6,\ H_6^5,\ H_1^4), \\ & (H_7^6,\ H_6^5,\ H_2^4), & (H_7^6,\ H_7^5,\ H_2^4), & (H_7^6,\ H_7^5,\ H_3^4), \end{array}$$

$$
\begin{array}{lll}
(H_8^6,\ H_1^5,\ H_2^4), & (H_8^6,\ H_6^5,\ H_1^4), & (H_9^6,\ H_2^5,\ H_1^4), \\
(H_9^6,\ H_3^5,\ H_2^4), & (H_9^6,\ H_4^5,\ H_3^4), & (H_{10}^6,\ H_1^5,\ H_1^4), \\
(H_{10}^6,\ H_2^5,\ H_1^4), & (H_{10}^6,\ H_2^5,\ H_2^4), & (H_{10}^6,\ H_2^5,\ H_3^4), \\
(H_{10}^6,\ H_5^5,\ H_1^4), & (H_{10}^6,\ H_7^5,\ H_3^4), & (H_{11}^6,\ H_1^5,\ H_2^4), \\
(H_{11}^6,\ H_2^5,\ H_2^4), & (H_{11}^6,\ H_3^5,\ H_1^4), & (H_{11}^6,\ H_3^5,\ H_2^4), \\
(H_{11}^6,\ H_3^5,\ H_3^4), & (H_{11}^6,\ H_7^5,\ H_2^4)\}. &
\end{array}
$$

For $j_1 = 6,\ j_2 = 6,\ j_3 = 3,$

$$
T = (G_1, G_2, G_3) \in \begin{array}{lll}
\{(H_1^6,\ H_8^6,\ H_1^3), & (H_2^6,\ H_3^6,\ H_1^3), & (H_2^6,\ H_4^6,\ H_1^3), \\
(H_5^6,\ H_6^6,\ H_1^3), & (H_5^6,\ H_7^6,\ H_1^3), & (H_6^6,\ H_{11}^6,\ H_1^3), \\
(H_7^6,\ H_{10}^6,\ H_1^3)\}. & &
\end{array}
$$

For $j_1 = 5,\ j_2 = 5,\ j_3 = 5$

$$
T = (G_1, G_2, G_3) \in \begin{array}{lll}
\{(H_1^5,\ H_2^5,\ H_3^5), & (H_1^5,\ H_2^5,\ H_6^5), & (H_1^5,\ H_2^5,\ H_7^5), \\
(H_1^5,\ H_3^5,\ H_5^5), & (H_1^5,\ H_3^5,\ H_7^5), & (H_1^5,\ H_5^5,\ H_7^5), \\
(H_2^5,\ H_3^5,\ H_4^5), & (H_2^5,\ H_3^5,\ H_5^5), & (H_2^5,\ H_3^5,\ H_7^5), \\
(H_2^5,\ H_5^5,\ H_6^5), & (H_2^5,\ H_6^5,\ H_7^5), & (H_3^5,\ H_4^5,\ H_7^5)\}.
\end{array}
$$