Gap-neighbour-distinguishing colourings

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Abstract

In this work, we investigate the gap-adjacent-chromatic number, a graph colouring parameter introduced by M. A. Tahraoui, E. Duchêne, and H. Kheddouci in [5]. From an edge labelling $f : E \to \{1, \cdots, k\}$ of a graph $G = (V, E)$, the vertices of G get an induced colouring. Vertices of degree greater than one are coloured with the difference between their maximum and their minimum incident edge label, i.e., with their so-called gap, and vertices of degree one get their incident edge label as colour. The gap-adjacent-chromatic number of G is the minimum k, for which a labelling f of G exists that induces a proper vertex colouring.

The main purpose of this work is to state easy colouring approaches for bipartite graphs and to estimate the gap-adjacent-chromatic number for arbitrary graphs in terms of the chromatic number.

1 Introduction

Let a simple, finite, and non-oriented graph $G = (V, E)$ without isolated vertices be given and a labelling $f : E \to \{1, \cdots, k\}$ of its edges which is not necessarily a proper edge colouring. We define the function $l : V \to \{0, \cdots, k\}$ with

$$l(v) = \begin{cases} f(e), & \text{if } \deg_G(v) = 1 \text{ and } v \in e \\ \max\limits_{e \ni v}\{f(e)\} - \min\limits_{e \ni v}\{f(e)\}, & \text{otherwise.} \end{cases}$$

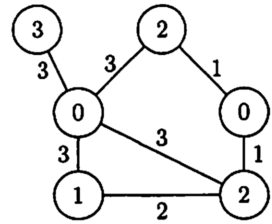

Figure 1: Example of a graph G with $\text{gap}_{\text{ad}}(G) = 3$.

If $l(v) \neq l(w)$ for vertices $v, w \in V$, we call f gap vertex distinguishing or a gap-k-colouring, a notion defined by M. A. Tahraoui, E. Duchêne, and H. Kheddouci in [5]. The minimum number k, such that G has a gap-k-colouring, is called the gap chromatic number gap(G) of G. The variant of this colouring problem, which we address here, is to find the minimum number k, such that the induced function l is a proper vertex colouring of G, i.e., neighbour distinguishing. The corresponding graph parameter is denoted by $\text{gap}_{\text{ad}}(G)$ (cf. Figure 1). This neighbour distinguishing version of the gap colouring problem was also introduced in [5]. Note, that graphs with isolated edges do not have distinguishing induced colourings because both endpoints of these edges will always get the same induced colour. Moreover, we restrict our investigations mainly to connected graphs because the gap-adjacent-chromatic number of a graph is just the maximum of its components' values. Another basic observation is that $\chi(G) - 1 \leq \text{gap}_{\text{ad}}(G) \leq \text{gap}(G)$ where $\chi(G)$ is the chromatic number of G.

A closely related notion in the literature is the general neighbour distinguishing index, investigated in [3], [2], and [1]. This index is the minimum number k for which an edge labelling $f : E \to \{1, \cdots, k\}$ exists such that adjacent vertices get different sets of incident edge labels. Since adjacent label sets are different when their gap is different, it is easy to see, that for graphs with minimum degree two, the gap-adjacent-chromatic number is as least as large as the general neighbour distinguishing index. The same problem, with the additional constraint that the edge labelling should be a proper edge colouring, was at first investigated in [6]. Other neighbour distinguishing induced colouring functions have been considered in the literature for example the sum of incident edge labels in the preprint [4].

In Section 2, we show an easy approach to obtain a gap-neighbour-distinguishing colouring of trees and state a greedy procedure to colour bipartite graphs. The third section deals with the gap colourings of 3-chromatic graphs and in the last section, we give an estimation of $\text{gap}_{\text{ad}}(G)$ for arbi-

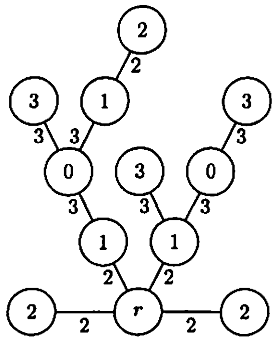

Figure 2: Example of the colouring of a tree.

trary graphs in terms of the chromatic number.

2 Bipartite Graphs

In this section we investigate bipartite graphs and their gap-neighbour-distinguishing colourings. We start with a very easy colouring procedure for trees.

Colouring approach for trees
Input: A tree $T = (V, E)$.

- Choose an arbitrary root r.
- Label the edges of distance $0, 3 \bmod 4$ to the root with two.
- Label the remaining edges with three.

As a result of this procedure (cf. Figure 2), vertices, except leaves, of odd distance to the root are coloured with one and vertices of even distance to the root with two. Leaves get two or three as induced vertex colour. Therefore, we obtain a gap-neighbour-distinguishing labelling and, in particular, no conflicts between inner vertices and leaves can occur.

Figure 3 shows that for bipartite graphs, even for trees, the gap-adjacent-chromatic number is not bounded by two. The tree T in Figure 3 is symmetric to its edge $e^* = uv$. If it was possible to get a neighbour distinguishing labelling with highest used label two, only the values zero and one would be potential induced colours for the inner vertices of T. Let us without

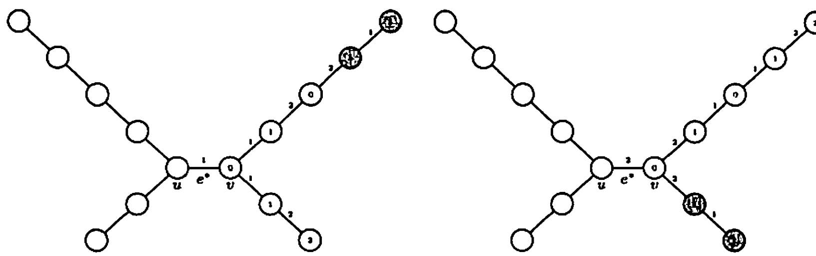

Figure 3: A tree T with $\text{gap}_{\text{ad}}(T) = 3$.

loss of generality assume that vertex v is coloured with zero. There are two possibilities to obtain this colour, both shown in Figure 3. In both situations, the labelling cannot be extended to a gap-neighbour-distinguishing colouring of the complete tree, because a conflict between the colour of a leaf and an inner vertex will occur.
As the next step, we prove an estimation of the gap-neighbour-distinguishing colouring number for bipartite graphs.

Theorem 1. *Let a bipartite graph $G = (V, E)$ without isolated edges be given. Then*

$$\text{gap}_{\text{ad}}(G) \leq 3.$$

Proof. At first, we state a greedy approach for labelling the edges of bipartite graphs.

Greedy colouring approach for bipartite graphs
Input: A connected bipartite graph $G = (C_1 \cup C_2, E)$ with colour classes C_1 and C_2 and $|E| > 1$.

- Choose greedily $v_1, v_2, \cdots \in C_1$ and colour (at least) two of their incident edges:
  - such that v_i gets one and three as incident edge labels.
  - such that vertices of C_2 get either no incident edge labels or only ones or only threes as incident edge labels.
- Stop, if there are no more vertices whose incident edges can be labelled in the way mentioned above.
- Label the remaining edges with two, except edges incident to vertices of degree one in C_2. Label them with three (or one).

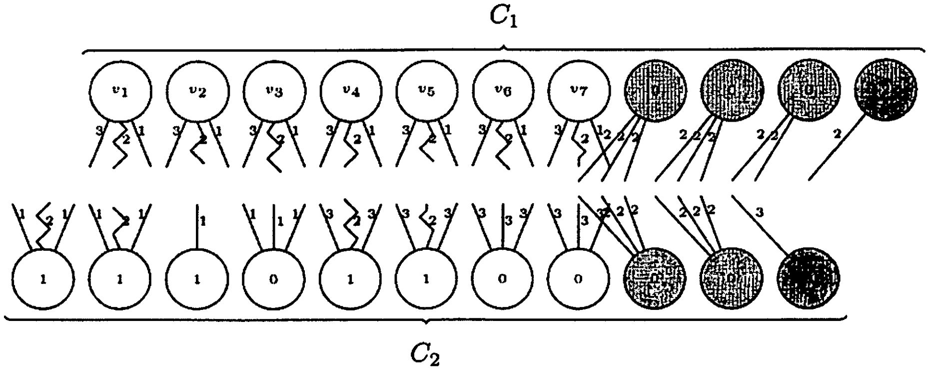

Figure 4: Structure of the greedy approach.

The resulting structure of the bipartite graph is shown in Figure 4. All white vertices have incident edges labelled by the greedy step and might be incident to edges labelled after the greedy procedure (zigzag edges). The remaining vertices (grey and dark grey) are incident to edges coloured after the greedy step.
For $i = 1, \cdots, k$ we obtain $l(v_i) = 2$, because all greedily chosen vertices have one and three as incident edge label. Furthermore, no vertex of C_2 gets the induced colour two. Hence, we do not obtain any conflicts here. The remaining (grey) vertices of C_1 with degree at least two are solely connected to vertices of C_2 which have, from the greedy step, either ones or threes as incident edge labels. Otherwise, more vertices would have been chosen by the greedy procedure. These (grey) vertices of C_1 get zero and their neighbours (white) get one as induced colour, except the vertices of degree one (dark grey). They are coloured with two in C_1 and can be adjacent to white or grey vertices of C_2, which get colour one or zero. In summary, we get a gap-neighbour-distinguishing colouring and the Theorem is proved. □

The decision problem, whether an arbitrary graph has gap-adjacent-chromatic number two or three, is NP-complete. This is also true for the related parameter, the general neighbour distinguishing index (cf. [2]). Here, we can use the same argument, namely that the problem can be reduced to the 2-colourability of hypergraphs. Let a bipartite graph $G = (C_1 \cup C_2, E)$ with minimum degree at least two be given. Identify C_1 with the vertices and C_2 with the edges of a hypergraph $H = (C_1, C_2)$. A vertex $v \in C_1$ is incident to an edge $e \in C_2$ iff v and e are neighbours in G. Then $\text{gap}_{\text{ad}}(G) \leq 2$ iff H resp. its dual hypergraph H^* (exchange the roles of

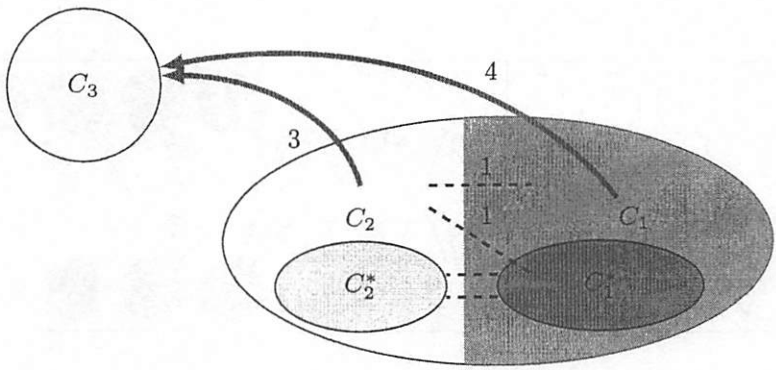

Figure 5: Structure of the colouring for graphs with chromatic number three.

edges and vertices) has a vertex colouring with two colours such that no edge is monochromatic. In order to explain this equivalence suppose at first that H is colourable with two colours, red and blue, such that no edge is monochromatic. Let $R, B \subseteq C_1$ be the colour classes. We label all edges of the original graph G which are incident to vertices of R with one and all other edges with two. As a result, we obtain that vertices of R resp. B get zero and all other vertices get one as induced colour. Hence, $\text{gap}_{\text{ad}}(G) \leq 2$. For the other direction, suppose that $\text{gap}_{\text{ad}}(G) \leq 2$. Choose an optimum gap-neighbour-distinguishing edge labelling. Since G has minimum degree at least two, one of its colour classes, say C_1, has induced colour one and the other class zero. Colour the vertices of C_1, which are incident to edge label one, red and the other vertices of C_1 blue. This is a proper 2-colouring of the hypergraph H.

3 3-chromatic graphs

In this section, we show the following result:

Theorem 2. *Let a 3-chromatic graph $G = (V, E)$ without isolated edges be given. Then*

$$\text{gap}_{\text{ad}}(G) \leq 4.$$

Proof. Assume that G is a 3-chromatic graph without isolated edges. For a subset U of the vertices, we denote by $N(U)$ the set of vertices in $V \setminus U$ adjacent to some $u \in U$. Choose a proper 3-colouring $V = C_1 \cup C_2 \cup C_3$ of G such that the sequence $|C_1|, |C_2|, |C_3|$ is lexicographically maximal and denote by $E(C_i, C_j)$ the set of edges between C_i and C_j. This special kind of decomposition has already been used by Horňák and Soták in [3] to estimate the general neighbour distinguishing index. It has the property that each vertex in C_3 is adjacent to C_2 and C_1 and each vertex in C_2 is adjacent to C_1.

We start by labelling the edges in $E(C_1, C_2), E(C_2, C_3), E(C_1, C_3)$ by 1,3 and 4, respectively. In what follows, we are going to relabel some of the edges in order to obtain an induced colouring which is a proper vertex colouring of G, i.e., neighbour distinguishing.

Denote by C_2^* the set of vertices in C_2 which are not adjacent to a vertex in C_3 and let $C_1^* := N(C_2^*)$. Only labels of edges in $E(C_2^*, C_1^*)$ may be changed below (see Figure 5).

Note that, before we start to relabel, the induced vertex colouring has the following properties: Vertices in C_3 are coloured by one, those in $C_2 \setminus C_2^*$ by two. Vertices in C_2^* are coloured one if they have degree 1 and zero else. Vertices in C_1 may have colour zero, one, three or four, where colour one can only occur if the vertex has degree 1 in G. From this it follows that the only conflicts with the desired colouring property arise from edges in $E(C_2^*, C_1^*)$ where both endpoints received either colour 1 or 0. We now apply the following procedure:

Relabelling procedure for $E(C_2^*, C_1^*)$

- Select a vertex $c \in C_2^*$ with colour zero which has a neighbour coloured with three and a neighbour coloured with zero. Choose a neighbour d of colour three and change the label of cd to two. As a result, the colour of c is changed to one. If cd has been the only edge incident to d with label 1, the colour of d will change to two. Otherwise the colour of d remains unchanged, in particular if d is adjacent to vertices of $C_2 \setminus C_2^*$. All other colours remain unchanged, too. If there is a neighbour a of c with colour one, then a has degree one in G and we change the label of ca to two as well. This step is iterated as often as possible. Then we go to the next step.

- If possible, choose a vertex $c \in C_2^*$ having a neighbour coloured with zero and no neighbour with colour three. Change all incident edge labels from one to four. As a result, the colour of c remains unchanged and its neighbours get one of the colours three or four. In particular, colour zero cannot arise for a neighbour d since otherwise all the neighbours of d except c must lie in C_3 which implies that d had colour three before relabelling. Since new vertices of colour three may be generated, we return to the previous step.

- If none of the steps can be applied to some $c \in C_2^*$, the algorithm stops.

Suppose that after relabelling there is an edge cd such that c and d have the same colour. We may assume $c \in C_2^*$ and $d \in C_1^*$. Then c was never selected in our relabelling procedure and we have two possible cases: If c has colour one, it has degree one in G and d cannot have degree one since G has no isolated edges. If c has colour zero, then it has no neighbours of colour zero, contradiction. Thus, the Theorem is proved.

□

4 A general bound

In this section, we extend our approach for 3-colourable graphs to k-chromatic graphs. We are going to prove:

Theorem 3. *Let $G = (V, E)$ be a graph without isolated edges, then*

$$\chi(G) - 1 \leq \text{gap}_{\text{ad}}(G) \leq \chi(G) + 5.$$

Proof. The case $\chi(G) \leq 3$ was treated above, so let a k-chromatic graph $G = (V, E)$ with $k \geq 4$ and without isolated edges be given. Again, we use a lexicographically maximal decomposition of its vertex set V into k colour classes $C_1, C_2, \cdots, C_k$. In this section we define

$$C_2^* = \{c \in C_2 \mid N(\{c\}) \cap (C_3 \cup \cdots \cup C_k) = \emptyset\}$$

and $N(C_2^*) =: C_1^*$. The proof is similar to the previous one. Again we first describe an edge labelling to start with:
Edges between C_1 and C_2 are labelled by one and all edges connecting C_1 to the colour classes $C_3, \cdots, C_k$ by four. Furthermore, edges between the colour classes $C_3, \cdots, C_k$ are also labelled with four. Edges between C_2 and C_3 are labelled with three. Edges between C_2 and C_i are labelled with $i + 5$ for $i = 4, \cdots, k$. This labelling is shown in Figure 6 and the resulting colours are listed in Table 1 and Table 2.
It is now easy to check that the Relabelling procedure for $E(C_2^*, C_1^*)$ from the proof of our result for 3-chromatic graphs also resolves all possible conflicts in the present situation. Therefore, the Theorem follows. □

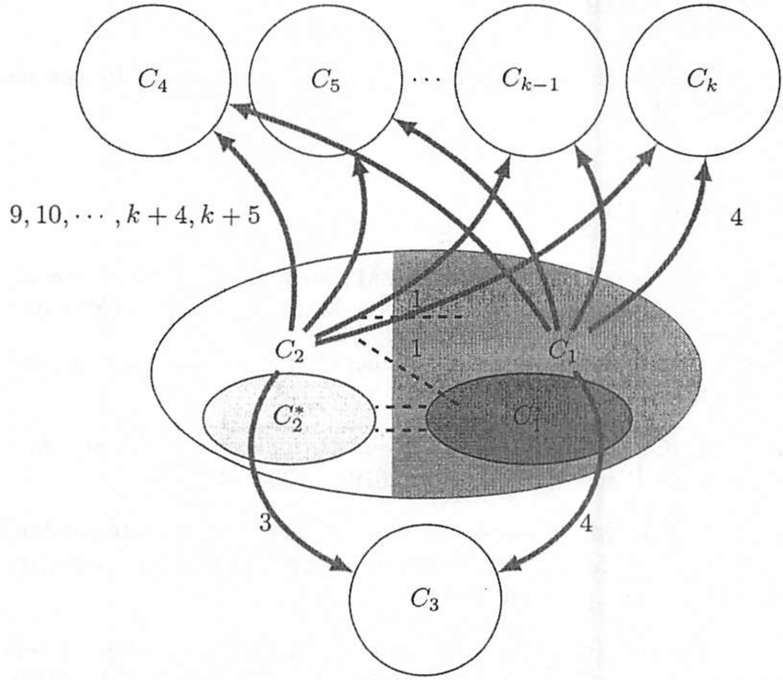

Figure 6: Structure of the colouring for graphs with chromatic number $k \geq 4$.

$l(v)$	C_1^*	$C_1 \setminus C_1^*$	C_2^*	$C_2 \setminus C_2^*$
$N(\{v\}) \cap (C_3 \cup C_4 \cup \cdots) = \emptyset$	$0, 1, 2, 3, 4$	$0, 1$	$0, 1$	X
$N(\{v\}) \cap (C_3 \cup C_4 \cup \cdots) \neq \emptyset$	$0, 2, 3$	$0, 3, 4$	X	$2, 8, \cdots = \max -1$

Table 1: Induced colours of the colour classes C_1 and C_2.

$l(v)$	C_3	C_4	C_5	$\cdots$	C_{k-1}	C_k
	1	5	6	$\cdots$	k	$k+1$

Table 2: Induced colours of the colour classes $C_3, C_4, \cdots, C_k$.

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