Generalized 3-connectivity and 3-edge-connectivity for the Cartesian products of some graph classes*

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Abstract

The generalized k-connectivity $\kappa_k(G)$ of a graph G was introduced by Chartrand et al. in 1984. As a natural counterpart of this concept, Li et al. in 2011 introduced the concept of generalized k-edge-connectivity. In this paper, we completely determine the precise values of the generalized 3-connectivity and generalized 3-edge-connectivity for the Cartesian products of some graph classes.

Keywords: Generalized 3-connectivity, generalized 3-edge-connectivity, Cartesian product

AMS Subject Classification 2000: 05C40, 05C76

1 Introduction

All graphs in this paper are undirected, finite and simple. We follow the notations and terminology of [1] for those not defined in this paper. Connectivity is one of the most important concepts in graph theory and its applications, both in a combinatorial sense and an algorithmic sense. In theoretical computer science, connectivity is a basic measure of reliability of networks. By the well-known Menger's theorem, the (vertex) *connectivity* of a graph $G = (V(G), E(G))$, denoted $\kappa(G)$, can be defined as the minimum $\kappa(\{u, v\})$ over all 2-subsets $\{u, v\}$ of $V(G)$, where $\kappa(\{u, v\})$ denotes the

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maximum number of internally disjoint u-v paths in G. In [2], Chartrand et al. introduced the following generalized connectivity. Let G be a graph of order $n \geq 2$ and let k be an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for $1 \leq i < j \leq \ell$. (Note that these trees are vertex-disjoint in $G \setminus S$.) A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called a *set of internally disjoint trees connecting S*. The *generalized k-connectivity* of G is then defined as

$$\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G) \text{ and } |S| = k\}.$$

Thus $\kappa_2(G) = \kappa(G)$ and $\kappa_k(G) = 0$ when G is disconnected. As a natural counterpart of the generalized connectivity, recently Li et al. [11] introduced the following concept of generalized edge-connectivity. Let $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $S \subseteq V(T_i)$ for $1 \leq i \leq \ell$. The *generalized k-edge-connectivity* of G is defined as

$$\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}.$$

Thus $\lambda_2(G) = \lambda(G)$ is the usual edge-connectivity, and $\lambda_k(G) = 0$ when G is disconnected. Clearly, we have $\kappa_k(G) \leq \lambda_k(G)$.

The generalized connectivity and edge-connectivity are also called the *tree connectivities*. In addition to being a natural combinatorial measure, the tree connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one wants to connect a set S of nodes of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of nodes is usually called a *Steiner tree*, and popularly used in the physical design of VLSI [15]. Usually, one wants to consider how reliable (or tough) a network can be for the connection of a set of vertices. Then the number of totally independent ways to connect them is a measure for this purpose. The tree connectivities can serve for measuring the capability of a network G to connect any k vertices in G. The reader is referred to a recent survey [10] on the state-of-the-art of research on tree connectivities.

Products of graphs occur naturally in discrete mathematics as tools in combinatorial constructions, they give rise to important classes of graphs and deep structural problems. Cartesian product is one of the most important graph products and plays a key role in design and analysis of networks. Many researchers have investigated the (edge) connectivity of the Cartesian product graphs in the past several decades [4, 6, 7, 8, 14, 16, 20]. Specially, the exact formula for $\kappa(G \Box H)$ was obtained.

Theorem 1.1 *[13, 16] Let G and H be graphs on at least two vertices. Then* $\kappa(G\Box H) = \min\{\kappa(G)|H|, \kappa(H)|G|, \delta(G)+\delta(H)\}$.

This theorem was first stated by Liouville [13]. However, the proof never appeared. In the meantime, several partial results were obtained until Špacapan [16] provided the proof. Theorem 1.1 in particular implies the following result of Sabidussi [14]:

Theorem 1.2 *[14] Let G and H be connected graphs. Then* $\kappa(G\Box H) \geq \kappa(G)+\kappa(H)$.

Li, Li and Sun [9] investigated the generalized 3-connectivity of the Cartesian product graphs and get the following result which could be seen as an extension of Theorem 1.2.

Theorem 1.3 *[9] Let G and H be connected graphs such that* $\kappa_3(G) \geq \kappa_3(H)$. *We have*
(i) If $\kappa_3(G) < \kappa(G)$, *then* $\kappa_3(G\Box H) \geq \kappa_3(G)+\kappa_3(H)$. *Moreover, the bound is sharp;*
(ii) If $\kappa_3(G) = \kappa(G)$, *then* $\kappa_3(G\Box H) \geq \kappa_3(G)+\kappa_3(H)-1$. *Moreover, the bound is sharp.*

In [17], we continue the research on the tree connectivity of the Cartesian product graphs and get the following result for the generalized 3-edge-connectivity of Cartesian product.

Theorem 1.4 *[17] Let G and H be two connected graphs, we have* $\lambda_3(G\Box H) \geq \lambda_3(G)+\lambda_3(H)$. *Moreover, the bound is sharp.*

With a similar but more complicated argument, we get the following result for the generalized 3-edge-connectivity of the strong product graphs.

Theorem 1.5 *[18] For any two connected graphs G and H, we have* $\lambda_3(G\boxtimes H) \geq \min\{2\lambda_3(G)+\lambda_3(H), \lambda_3(G)+2\lambda_3(H)\}$, *where* $G\boxtimes H$ *is the strong product of G and H. Moreover, the bound is sharp.*

Note that in the sequel we use K_m, C_m, T_m to denote a complete graph, a cycle and a tree of orders m, respectively. In this paper, we investigate the generalized 3-connectivity and generalized 3-edge-connectivity for the Cartesian products of some special graph classes and get the following result.

Theorem 1.6 *Let $m, n \geq 3$, then the following holds:*
(i) $\kappa_3(K_m \square K_n) = \lambda_3(K_m \square K_n) = m + n - 3$;
(ii) $\kappa_3(K_m \square C_n) = \lambda_3(K_m \square C_n) = m$;
(iii) $\kappa_3(K_m \square T_n) = \lambda_3(K_m \square T_n) = m - 1$;
(iv) $\kappa_3(C_m \square C_n) = \lambda_3(C_m \square C_n) = 3$;
(v) $\kappa_3(C_m \square T_n) = \lambda_3(C_m \square T_n) = 2$;
(vi) $\lambda_3(T_m \square T_n) = 2$; *if both T_m and T_2 are paths,* $\kappa_3(T_m \square T_n) = 1$ *and otherwise,* $\kappa_3(T_m \square T_n) = 2$.

The proof of Theorem 1.6 consists of Lemmas 3.1, 3.2, 3.5, 3.6 and Propositions 3.4, 3.8. In this theorem, we completely determine the precise values of the generalized 3-(edge)-connectivities for the Cartesian products of any two graphs which belong to the following three graph classes: complete graphs, trees and cycles.

2 Preliminaries

The *Cartesian product* of two graphs G and H, denoted by $G \square H$, is defined to have the vertex set $V(G) \times V(H)$ such that (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. Note that this product is commutative, that is, $G \square H = H \square G$.

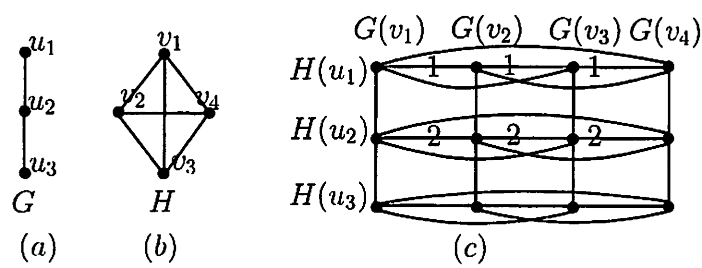

Figure 2.1 Graphs G, H and their Cartesian product.

We use the following useful notion of *projection* which was used in [6, 7]. The mappings $p_G : (u, v) \mapsto u$ and $p_H : (u, v) \mapsto v$ from $V(G \square H)$ into $V(G)$, resp. $V(H)$, are weak homomorphisms from $G \square H$ onto the factors G, resp. H. They are called *projections* in the literature.

Let G and H be two graphs with $V(G) = \{u_i | 1 \leq i \leq n\}$ and $V(H) =$

$\{v_j|1 \leq j \leq m\}$. We use $G(v_j)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v_j)|1 \leq i \leq n\}$ where $1 \leq j \leq m$, and use $H(u_i)$ to denote the subgraph of $G\Box H$ induced by the vertex set $\{(u_i, v_j)|1 \leq j \leq m\}$ where $1 \leq i \leq n$. Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. For example, as shown in Figure 2.1, $G(v_j) \cong G$ for $1 \leq j \leq 4$ and $H(u_i) \cong H$ for $1 \leq i \leq 3$. For $1 \leq j_1 \neq j_2 \leq m$, the vertices (u_i, v_{j_1}) and (u_i, v_{j_2}) belong to the same graph $H(u_i)$ where $u_i \in V(G)$, we call (u_i, v_{j_2}) the *vertex corresponding to* (u_i, v_{j_1}) in $G(v_{j_2})$; for $1 \leq i_1 \neq i_2 \leq n$, we call (u_{i_2}, v_j) the vertex corresponding to (u_{i_1}, v_j) in $H(u_{i_2})$ [9]. Similarly, we can define the path and the tree corresponding to some path and tree, respectively. For example, in the graph (c) of Figure 2.1, let P_1, resp. P_2 be the paths whose edges are labelled 1, resp. 2 in $H(u_1)$, resp. $H(u_2)$. Then P_2 is called the path corresponding to P_1 in $H(u_2)$. Clearly, P_1 and P_2 correspond to the path v_1, v_2, v_3, v_4 in H.

Chartrand et al. [3] got the precise value of the generalized k-connectivity for the complete graph K_n.

Theorem 2.1 *[3] For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$.*

Li, Mao and Sun obtained the explicit value for $\lambda_k(K_n)$ and a sharp lower bound of $\lambda_3(G)$ for a general graph G as follows.

Theorem 2.2 *[11] For every two integers n and k with $2 \leq k \leq n$, $\lambda_k(K_n) = n - \lceil \frac{k}{2} \rceil$.*

Theorem 2.3 *[11] Let G be a connected graph with order n and edge-connectivity $\lambda(G) = 4s + r$, where s and r are integers with $s \geq 0$ and $0 \leq r \leq 3$. Then $\lambda_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$ and the bound is sharp. In particular, $\lambda_3(G) \geq \frac{3\lambda(G)-2}{4}$.*

We still need the following two results.

Lemma 2.4 *[12] Let G be a connected graph of order n. If there exist two adjacent vertices of degree $\delta(G)$, then $\lambda_k(G) \leq \delta(G) - 1$ for every integer k with $3 \leq k \leq n$, and moreover the bound is sharp.*

Theorem 2.5 *[9] Let G be a connected graph and T be a tree. The following assertions holds:*
(i) If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G\Box T) \geq \kappa_3(G)$. Moreover, the bound is

sharp;
(ii) If $1 \le \kappa_3(G) < \kappa(G)$*, then* $\kappa_3(G\Box H) \ge \kappa_3(G)+1$*. Moreover, the bound is sharp.*

3 Proof of Theorem 1.6

In the following argument, we consider graphs with orders at least three. Firstly, we investigate the Cartesian product of two complete graphs and get the following result.

Lemma 3.1 $\kappa_3(K_m\Box K_n) = \lambda_3(K_m\Box K_n) = m+n-3$.

Proof. Since $K_m\Box K_n$ is a $(m+n-2)$-regular graph, we have $\kappa_3(K_m\Box K_n) \le \lambda_3(K_m\Box K_n) \le m+n-3$ by Lemma 2.4. Thus, we need to find at least $m+n-3$ internally disjoint trees connecting S for any set $S=\{x,y,z\}$ in the following argument. Let $G=K_m, H=K_n$ and $x \in V(G(v_\alpha)), y \in V(G(v_\beta)), z \in V(G(v_\gamma))$, where $1 \le \alpha,\beta,\gamma \le n$. In order to prove this lemma, we need the following three claims.

Claim 1. For the case that α,β,γ *are distinct, we can construct at least* $m+n-3$ *internally disjoint trees connecting* S*.*

Proof of Claim 1. Without loss of generality, we assume that $x \in V(G(v_1)) \cap V(H(u_1)), y \in V(G(v_2)), z \in V(G(v_3))$. Furthermore, let y', z' be the vertices corresponding to y, z in $G(v_1)$, x', z'' be the vertices corresponding to x, z in $G(v_2)$ and x'', y'' be the vertices corresponding to x, y in $G(v_3)$. Our proof consists of the following three cases.

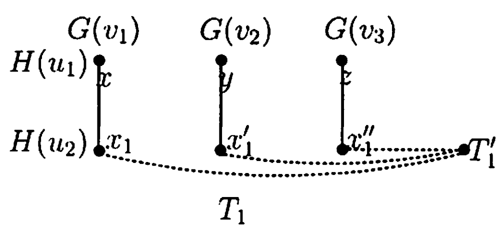

Figure 3.1 The graph of Case 1.

Case 1. $p_G(x) = p_G(y) = p_G(z)$.
Now we have that x, y', z' are the same vertex in $G(v_1)$. Let x_1 be a

neighbor of x in $G(v_1)$. Without loss of generality, we assume that $x_1 \in H(u_2)$. Let x'_1 and x''_1 be the corresponding vertices of x_1 in $G(v_2)$ and $G(v_3)$, respectively. Clearly, $yx'_1 \in E(G(v_2)), zx''_1 \in E(G(v_3))$. Let T_1 be the tree obtained from T'_1 and edges xx_1, yx'_1, zx''_1, where T'_1 is a tree connecting $\{x_1, x'_1, x''_1\}$ in $H(u_2)$ (see Figure 3.1). Since x has at least $m-1$ neighbors in $G(v_1)$, we can find $m-1$ such trees. Thus, we get at least $(m-1)+(n-2) = m+n-3$ trees connecting S totally since there are $n-2$ internally disjoint trees connecting S in $H(u_1)$. It is easy to show that any two of these trees are internally disjoint.

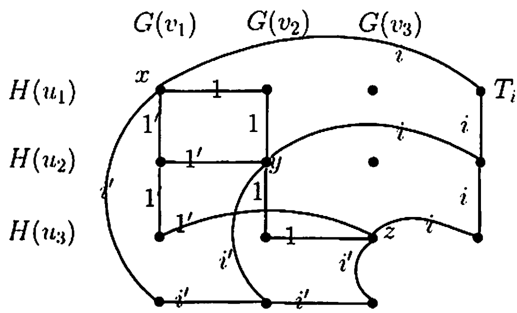

Figure 3.2 The graph of Case 2.

Case 2. $p_G(x), p_G(y), p_G(z)$ are three distinct vertices.
Without loss of generality, we assume that $x = V(G(v_1)) \cap V(H(u_1))$, $y = V(G(v_2)) \cap V(H(u_2))$ and $z = V(G(v_3)) \cap V(H(u_3))$. As shown in Figure 3.2, let the edges labelled by i belong to the tree $T_i (1 \le i \le m-1)$, then we have $m-1$ such trees. Similarly, let the edges labelled by i' belong to the tree $T'_i (1 \le i' \le n-2)$, we have $n-2$ such trees. It is easy to show that any two of these $m+n-3$ trees are internally disjoint.

Case 3. Two of $p_G(x), p_G(y), p_G(z)$ are the same vertices.
Without loss of generality, we assume that $x = V(G(v_1)) \cap V(H(u_1))$, $y = V(G(v_2)) \cap V(H(u_2))$ and $z = V(G(v_3)) \cap V(H(u_2))$. With a similar argument to that of Case 2, we get $m+n-3$ internally disjoint trees as shown in Figure 3.3.

The argument of the following claim is similar to that of Case 2 of Claim 1, we can get $m+n-3$ internally disjoint trees as shown in Figure 3.4.

Claim 2. For the case that exactly two of α, β, γ are distinct, we can construct at least $m+n-3$ internally disjoint trees connecting S.

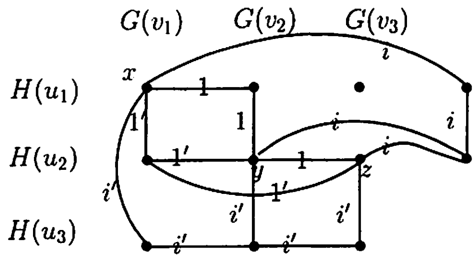

Figure 3.3 The graph of Case 3.

The final case that α, β, γ are the same is similar to Case 1 of Claim 1, thus the following result holds.

Claim 3. For the case that α, β, γ are the same, we can construct at least $m+n-3$ internally disjoint trees connecting S. ∎

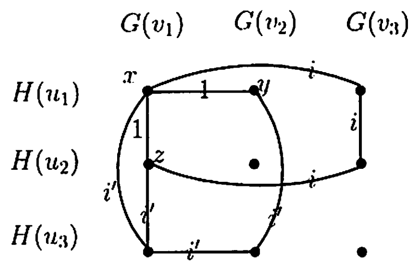

Figure 3.4 The graph of Claim 2.

With a similar argument to that of Lemma 3.1, we get the following result about the Cartesian product of a complete graph and a cycle.

Lemma 3.2 $\kappa_3(K_m \Box C_n) = \lambda_3(K_m \Box C_n) = m.$

For $m \geq 3$, the *wheel graph* W_m is defined as a graph constructed by joining a new vertex to every vertex of a cycle C_m. The following result concerns the Cartesian products of connected graphs with minimum degrees 1 and some special graph classes.

Lemma 3.3 *[17] Let G be a connected graph with $\delta(G) = 1$ and order $n \geq 3$.*

(i) If H is a connected graph with $\delta(H) = 1$ and order $m \geq 3$, then $\lambda_3(G\Box H) = 2$;
(ii) If H is a cycle, then $\lambda_3(G\Box H) = 2$;
(iii) If H is a wheel graph, then $\lambda_3(G\Box H) = 3$;
(iv) If H is a complete graph with order $m \geq 3$, then $\lambda_3(G\Box H) = m - 1$.

By Lemma 3.3, the following proposition holds:

Proposition 3.4 *(i) $\lambda_3(T_m\Box T_n) = 2$;*
(ii) $\lambda_3(K_m\Box T_n) = m - 1$;
(iii) $\lambda_3(C_m\Box T_n) = 2$.

Lemma 3.5 *(i) $\kappa_3(C_m\Box T_n) = 2$;*
(ii) $\kappa_3(W_m\Box T_n) = 3$;
(iii) $\kappa_3(K_m\Box T_n) = m - 1$.

Proof. Since $\kappa_3(C_m) = 1 < 2 = \kappa(C_m)$, $\kappa_3(W_m) = 2 < 3 = \kappa(W_m)$ and $\kappa_3(K_m) = m - 2 < n - 1 = \kappa(K_m)$, we have $\kappa_3(C_m\Box T_n) \geq 2$, $\kappa_3(W_m\Box T_n) \geq 3$ and $\kappa_3(K_m\Box T_n) \geq m - 1$ by Theorem 2.5. Thus, by Lemma 3.3 and the fact that $\kappa_k(G) \leq \lambda_k(G)$, (i)-(iii) hold. ∎

Lemma 3.6

$$\kappa_3(T_m\Box T_n) = \begin{cases} 1, & if\ both\ T_m\ and\ T_n\ are\ paths; \\ 2, & otherwise. \end{cases}$$

Proof. If both T_m and T_n are paths, we know that $\delta(T_m\Box T_n) = 2$, and let S be a set of three vertices of degree two since there are four such vertices. It is not hard to show that the number of internally disjoint trees connecting S is 1 and we have $\kappa_3(T_m\Box T_n) = 1$.

Otherwise, with a similar argument to that of Lemma 3.1, we can get that there are at least two internally disjoint trees connecting S for any set S of three vertices. Thus, $\kappa_3(T_m\Box T_n) = 2$. ∎

Let $C_n : a_1, a_2, \cdots, a_n, a_1$ and $C_m : b_1, b_2, \cdots, b_m, b_1$ be two cycles, and let r be an integer with $0 \leq r \leq m-1$. The *r-pseudo-cartesian product* [5] of C_n and C_m, denoted by $C_n\Box_r C_m$, is the graph obtained from $C_n\Box C_m$ by replacing the edge set $\{(a_1, b_i)(a_n, b_i) : 1 \leq i \leq m\}$ by $\{(a_1, b_{i+r})(a_n, b_i) : 1 \leq i \leq m\}$ with subscripts of b's modulo m. By definition, we have $C_n\Box_r C_m = C_n\Box C_m$ if $r = 0$ or m. In [19], we investigated the tree connectivities of r-pseudo-cartesian product of two cycles. Especially, the following result was obtained.

Lemma 3.7 *[19]* $\kappa_3(C_n\Box_r C_m) = \lambda_3(C_m\Box_r C_n) = 3$.

By Lemma 3.7, the following proposition clearly holds. However, since [19] has not yet been published, we need to give a proof of this result.

Proposition 3.8 $\kappa_3(C_n\Box C_m) = \lambda_3(C_n\Box C_m) = 3$.

Proof. It is not hard to show that $\lambda(C_n\Box C_m) = 4$, then $\lambda_3(C_n\Box C_m)\geq \frac{3\lambda(C_n\Box C_m)-2}{4} = \frac{5}{2}$ by Theorem 2.3. We also have $\lambda_3(C_n\Box C_m) \leq \delta(C_n\Box C_m) - 1 = 3$ by Lemma 2.4 . Thus $\lambda_3(C_n\Box C_m) = 3$. Similarly, we can prove that $\kappa_3(C_n\Box C_m) = 3$. ∎

Proof of Theorem 1.6. By Lemmas 3.1, 3.2, 3.5, 3.6 and Propositions 3.4, 3.8, the theorem holds. ∎

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