Maximum generalized local connectivities of cubic Cayley graphs on Abelian groups*

Yuefang Sun
Department of Mathematics
Shaoxing University, Zhejiang 312000, P.R. China
E-mail: yfsun2013@gmail.com

Abstract

For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \dots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for $1 \leq i \neq j \leq \ell$ and $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \dots, T_\ell$ in G such that $S \subseteq V(T_i)$ for $1 \leq i \leq \ell$. Similar to the classical maximum local connectivity, H. Li et al. introduced the parameter $\overline{\kappa}_k(G) = \max\{\kappa(S) | S \subseteq V(G), |S| = k\}$, which is called the maximum generalized local connectivity of G. The maximum generalized local edge-connectivity of G which was introduced by X. Li et al. is defined as $\overline{\lambda}_k(G) = \max\{\lambda(S) | S \subseteq V(G), |S| = k\}$. In this paper, we investigate the maximum generalized local connectivity and edge-connectivity of a cubic connected Cayley graph G on an Abelian group. We determine the precise values for $\overline{\kappa}_3(G)$ and $\overline{\lambda}_3(G)$ and also prove some results of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ for general k.

Keywords: Generalized local connectivity, generalized local edge-connectivity, Cayley graph

AMS Subject Classification 2010: 05C40

*Supported by NSFC No. 11401389.

1 Introduction

All graphs in this paper are undirected, finite and simple. Any notation or terminology not defined here, follows that used in [3, 4]. Connectivity is one of the most important concepts in graph theory and its applications, both in a combinatorial sense and an algorithmic sense. In theoretical computer science, connectivity is a basic measure of reliability of networks. By the well-known Menger's theorem, the *connectivity* of a graph $G = (V(G), E(G))$, denoted by $\kappa(G)$, can be defined as the minimum $\kappa(\{u,v\})$ over all 2-subsets $\{u,v\}$ of $V(G)$, where $\kappa(\{u,v\})$ denotes the maximum number of internally disjoint u-v paths in G. In contrast to this parameter, $\overline{\kappa}(G) = \max\{\kappa(\{u,v\})|u,v \in V(G), u \neq v\}$, introduced by Bollobás [5], is called the *maximum local connectivity*. Similarly, $\overline{\lambda}(G) = \max\{\lambda(\{u,v\})|u,v \in V(G), u \neq v\}$ is the *maximum local edge-connectivity*, where $\lambda(\{u,v\})$ denotes the maximum number of edge-disjoint u-v paths in G. The concept of maximum local connectivity and edge-connectivity have obtained wide attention and many results have been worked out (for example, see [5, 6, 10, 20]).

In [7], Hager introduced the following generalized connectivity. Let G be a graph of order $n \geq 2$ and let k be an integer with $2 \leq k \leq n$. For a set S of k vertices of G, let $\kappa(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $V(T_i) \cap V(T_j) = S$ for $1 \leq i \neq j \leq \ell$. Note that these trees are vertex-disjoint in $G \setminus S$. A collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G with this property is called a *set of internally disjoint trees connecting S*. The *generalized k-connectivity* of G is then defined as $\kappa_k(G) = \min\{\kappa(S) : S \subseteq V(G) \text{ and } |S| = k\}$. Thus $\kappa_2(G) = \kappa(G)$ and $\kappa_k(G) = 0$ when G is disconnected. As a natural counterpart of the generalized connectivity, Li et al. in [15] introduced the following concept of generalized edge-connectivity. Let $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees $T_1, T_2, \ldots, T_\ell$ in G such that $S \subseteq V(T_i)$ for $1 \leq i \leq \ell$. The *generalized k-edge-connectivity* of G is defined as $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}$. Thus $\lambda_2(G) = \lambda(G)$ is the usual edge-connectivity, and $\lambda_k(G) = 0$ when G is disconnected. Clearly, we have $\kappa_k(G) \leq \lambda_k(G)$. The generalized connectivity and edge-connectivity are also called the *tree connectivities*. There are more and more researchers investigating this topic, such as [7, 11, 12, 14, 15, 17, 18, 19]. The reader is referred to a survey [13] on the state-of-the-art of research on tree connectivities.

Similar to the classical maximum local connectivity, H. Li et al. [12] introduced the parameter $\overline{\kappa}_k(G) = \max\{\kappa(S)|S \subseteq V(G), |S| = k\}$, which is called the *maximum generalized local connectivity* of G. The *maximum* *generalized local edge-connectivity* of G which was introduced by X. Li et al. in [14] is defined as $\overline{\lambda}_k(G) = \max\{\lambda(S)|S \subseteq V(G), |S| = k\}$.

Cayley graphs have been important objects of study in algebraic graph theory over many years (e.g. [2], [3]). In particular, mathematicians and computer scientists recommend (e.g. [1], [8], [21]) Cayley graphs as models for interconnection networks because they exhibit many properties that ensure high performance. In fact, a number of networks of both theoretical and practical importance, including hypercubes, butterflies, cube-connected cycles, star graphs and their generalizations, are Cayley graphs. The reader is referred to the survey papers [8] and [9] for results pertaining to Cayley graphs as models for interconnection networks. Due to the importance of Cayley graphs in network design and the significance of reliability of networks, it is of interest to understand the maximum generalized local connectivity and edge-connectivity of Cayley graphs. This motivated our study in this paper.

In this paper we investigate the maximum generalized local connectivity and edge-connectivity of a cubic connected Cayley graph G on an Abelian group. We will determine the precise values of $\overline{\kappa}_3(G)$ and $\overline{\lambda}_3(G)$ (Theorem 2.7). Based on this result and the monotonicity of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ (Lemmas 2.8 and 2.9), we will prove a result of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ for general k which shows the changing trend of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ (Theorem 2.12).

Note that in the following we assume that every tree T which connects S is *minimal*, that is, the subgraph which is obtained by deleting any set of vertices or edges of T will not connect S. This assumption will not affect our results.

2 Main results

Lemma 2.1 *For a connected d-regular graph G, $\overline{\kappa}_k(G) = d$ if and only if $K_{k,d}$ is a minor of G.*

Proof. We assume that $\kappa(S) = \overline{\kappa}_k(G)$ where $S \subseteq V(G)$ and $|S| = k$. Let $\mathcal{T} = \{T_j | 1 \leq j \leq d\}$ be a set of d internally disjoint trees connecting S. Since G is d-regular, we know every vertex of S is a leaf in each T_j, then $K_{k,1}$ is a minor of each T_j for $1 \leq j \leq d$. Thus, $K_{k,d}$ is a minor of the subgraph $\bigcup_{j=1}^{d} T_j$ of G.

Let $H = K_{k,d}$ be a minor of G, then H can be formed from G by deleting edges and vertices and by contracting edges such that $V(H) = A \cup B$ where

$A = \{u_i | 1 \leq i \leq k\} \subseteq V(G)$ is one part of the bipartition of $V(H)$ and B is another one. Then, with an inverse procedure, we can obtain the original graph G from H and get d internally disjoint trees connecting A. Thus, $\kappa(A) \geq d$ and we have $\overline{\kappa}_k(G) = \kappa(A) = d$ since $\kappa(A) \leq d$. ∎

In [14], Li and Mao made the following observation which can be deduced by the definitions of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$.

Observation 2.2 *[14] For a connected graph G, we have $\overline{\kappa}_k(G) \leq \overline{\lambda}_k(G)$.*

For a cubic connected graph G, equality holds in the above inequality.

Lemma 2.3 *For a cubic connected graph G, we have $\overline{\kappa}_k(G) = \overline{\lambda}_k(G)$.*

Proof. Since the result clearly holds for the case that $\overline{\lambda}_k(G) = 1$, we assume that $\overline{\lambda}_k(G) \geq 2$ in the following argument. Suppose that $\lambda(S) = \overline{\lambda}_k(G) = \ell$ where $S \subseteq V(G)$ and $|S| = k$. Let $\mathcal{T} = \{T_j | 1 \leq j \leq \ell\}$ be a set of ℓ edge-disjoint trees connecting S. For any two trees T_{j_1} and T_{j_2} where $1 \leq j_1 \neq j_2 \leq \ell$, if they have a common vertex, say v, which does not belong to S, then we must have $deg_G(v) \geq 4$, a contradiction. Thus, any two trees in $\mathcal{T}$ are internally disjoint, so $\overline{\kappa}_k(G) \geq \kappa(S) \geq \ell = \lambda(S) = \overline{\lambda}_k(G)$. Together with Observation 2.2, we complete the proof. ∎

We use $N_G(u)$ to denote the set of neighbors of u in G. For a set $S \subseteq V(G)$, let $N_G(S) = (\bigcup_{u \in S} N_G(u)) \setminus S$.

Lemma 2.4 *Let G be a cubic connected graph with order n. If $k \geq \lceil \frac{3n}{8} \rceil + 1$, then $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) \leq 2$. Moreover, if $k \geq \lceil \frac{3n+2}{4} \rceil + 1$, then $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$.*

Proof. We now prove the first part of the lemma. Since G is cubic, $\overline{\kappa}_k(G) \leq 3$. Suppose that $\overline{\kappa}_k(G) = \kappa_k(S) = 3$ for some subset $S \subseteq V(G)$ with $|S| = k$, then there exists a set of three internally disjoint trees connecting S, say $\mathcal{T} = \{T_i | 1 \leq i \leq 3\}$. Since G is cubic, every vertex of S must be a leaf of each T_i for $1 \leq i \leq 3$ and the degree of each vertex in T_i is at most 3. Thus, $|N_{T_i}(S)| \geq \lceil \frac{k}{3} \rceil$ and $|E(T_i)| = |V(T_i)| - 1 \geq |S| + |N_{T_i}(S)| - 1 \geq k + \lceil \frac{k}{3} \rceil - 1$. Since these three trees are also edge-disjoint, the number of edges in these trees is at least $3(k + \lceil \frac{k}{3} \rceil - 1) \geq 4k - 3 \geq 4(\lceil \frac{3n}{8} \rceil + 1) - 3 > \frac{3n}{2}$, this produces a contradiction as the size of G is $\frac{3n}{2}$. Thus, $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) \leq 2$ by Lemma 2.3.

We next prove the second part of the lemma, that is, $\kappa_k(S) = 1$ if $k \geq \lceil \frac{3n+2}{4} \rceil + 1$, where $S \subseteq V(G)$ and $|S| = k$. Suppose that there are at least two internally disjoint trees connecting S in G, then the number of edges in these trees is at least $2(k-1) > \frac{3n}{2}$, a contradiction. Thus, $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$ by Lemma 2.3. ∎

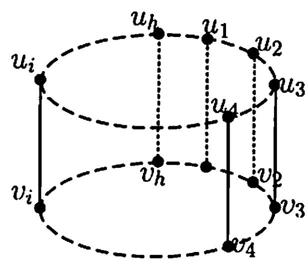

Figure 2.1 A prism with $2h$ vertices.

The *d-dimensional cube* Q_d is the Cartesian product of d copies of the path P_2 of two vertices. The Cartesian product $P(2h) = C_h \Box P_2$ of a cycle C_h of length $h \geq 3$ and P_2 is called a *prism*. As shown in Figure 2.1, $V(P(2h)) = \{u_i, v_j : 1 \leq i, j \leq h\}$, $E(P(2h)) = \{(u_i, v_i) : 1 \leq i \leq h\} \cup \{u_iu_{i+1} : 1 \leq i \leq h\} \cup \{v_jv_{j+1} : 1 \leq j \leq h\}$ with subscripts modulo h. The *Möbius ladder* $M(2h)$ of order $2h$ is the graph obtained from $P(2h)$ by deleting the edges u_1u_h and v_1v_h and adding the edges u_1v_h and u_hv_1.

Let G be a connected d-regular planar graph where $d \geq 3$. Since G is planar, we know G does not contain a $K_{k,d}$ as a minor where $k \geq 3$, so $\overline{\kappa}_k(G) \leq d-1$ by Lemma 2.1. By definition, a prism G is a cubic planar graph, so $\overline{\kappa}_3(G) \leq 2$. Consider the vertex set $S = \{u_1, u_2, u_3\}$ as shown in Figure 2.1, it is not hard to find two internally disjoint trees connecting S, then $\overline{\kappa}_3(G) \geq \kappa(S) = 2$. Thus, $\overline{\kappa}_3(G) = 2$. Moreover, $\overline{\lambda}_3(G)$=2 by Lemma 2.3. Now we have the following result.

Lemma 2.5 *If G is a prism, then $\overline{\kappa}_3(G) = \overline{\lambda}_3(G) = 2$.*

The following result concerns a similar result for a Möbius ladder.

Lemma 2.6 *If G is a Möbius ladder, then $\overline{\kappa}_3(G) = \overline{\lambda}_3(G) = 3$.*

Proof. Let G be a Möbius ladder with vertex set $V(G)) = \{u_i, v_j : 1 \leq i, j \leq h\}$. Since G is a cubic graph, $\overline{\kappa}_3(G) = \overline{\lambda}_3(G) \leq 3$ by Lemma 2.3. We choose $S = \{v_1, u_2, v_h\}$, as shown in Figure 2.2, there are three internally

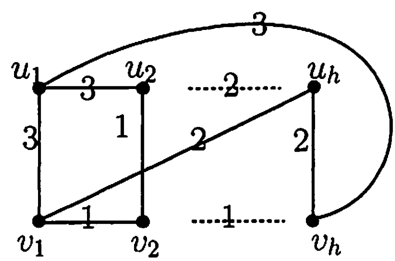

Figure 2.2 Three internally disjoint trees connecting $S = \{v_1, u_2, v_h\}$.

disjoint trees connecting S, where each tree T_i contains edges labeled i for $1 \leq i \leq 3$. Thus, $\overline{\kappa}_3(G) \geq \kappa(S) \geq 3$ and then $\overline{\kappa}_3(G) = \overline{\lambda}_3(G) = 3$. ∎

Let X be a finite group, with operation denoted additively, and A a subset of $X \setminus \{0\}$ such that $a \in A$ implies $-a \in A$, where 0 is the identity element of X. The *Cayley graph* $Cay(X, A)$ is defined to have vertex set X such that there is an edge between x and y if and only if $x - y \in A$.

Now we consider a cubic connected Cayley graph on an Abelian group, let X be an abelian group, $A \subseteq X$ with $1 \notin X, X^{-1} = X$ and $|X| = 3$. By relabeling the elements of X we have the following 4 types[16]: (i) $A = \{a, b, c\}$ and $\langle A\rangle \cong \mathbb{Z}_2^3$; (ii) $A = \{a, ac, c\}$ and $\langle A\rangle = \langle a, c | a^2 = c^2 = 1, ac = ca\rangle \cong \mathbb{Z}_2^2$; (iii) $A = \{a, a^{-1}, c\}$ and $\langle A\rangle = \langle a, c | a^h = c^2 = 1, ac = ca\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_h$; (iv) $A = \{a, a^{-1}, c\}$ and $\langle A\rangle = \langle a, c | a^{2h} = c^2 = 1, a^h = c, ac = ca\rangle \cong \mathbb{Z}_{2h}$.

$Cay(\langle A\rangle, A)$ are isomorphic to Q_3, K_4, $P(2h)$ the prism with $2h$ vertices and $M(2h)$ the Möbius ladder with $2h$ vertices according to type (i), type (ii), type (iii) and type (iv), respectively. Recall that $Cay(X, A)$ is connected if and only if A generates the group X. Thus, any cubic connected Cayley graph on an Abelian group is isomorphic to K_4, Q_3, a prism $P(2h)$ or a Möbius ladder $M(2h)$[16]. Now by Lemmas 2.5 and 2.6, and from the above argument, we can determine the precise values for $\overline{\kappa}_3(G)$ and $\overline{\lambda}_3(G)$.

Theorem 2.7 *If G is a cubic connected Cayley graph on an Abelian group, then*

$$\overline{\kappa}_3(G) = \overline{\lambda}_3(G) = \begin{cases} 3, & \text{if } G \text{ is a Möbius ladder,} \\ 2, & \text{otherwise.} \end{cases}$$

For a cubic connected graph, the following result holds.

Lemma 2.8 *For a cubic connected graph G with order n, we have $\overline{\kappa}_k(G) \geq \overline{\kappa}_{k+1}(G)$, where $3 \leq k \leq n-1$.*

Proof. Assume that $\kappa(S) = \overline{\kappa}_{k+1}(G) = \ell$ where $S \subseteq V(G)$and $|S| = k+1$. Let $\mathcal{T} = \{T_j | 1 \leq j \leq \ell\}$ be a set of ℓ internally disjoint trees connecting S in G. Since G is cubic, we have $1 \leq \ell \leq 3$. The result clearly holds for the case that $\ell = 1$. Thus, we assume that $2 \leq \ell \leq 3$ in the following. Let $v \in S$, we will consider the following two cases.

If v is a leaf of each T_j for $1 \leq j \leq \ell$, then let $S' = S \setminus \{v\}$ and $\mathcal{T}' = \{T_j' | 1 \leq j \leq \ell\}$ where $T_j' = T_j \setminus \{v\}$. Clearly, $\mathcal{T}'$ is a set of ℓ internally disjoint trees connecting S', and $\overline{\kappa}_k(G) \geq \kappa(S') \geq \ell = \overline{\kappa}_{k+1}(G)$.

Otherwise, there exists $1 \leq j_0 \leq \ell$ such that v is not a leaf of T_{j_0}, then $\ell = 2$ since G is cubic. Without loss of generality, we assume that $j_0 = 1$, then v must be a leaf of T_2. Thus, $\mathcal{T}'$ is a set of two internally disjoint trees connecting S', where $S' = S \setminus \{v\}$ and $\mathcal{T}' = \{T_1, T_2 \setminus \{v\}\}$, then $\overline{\kappa}_k(G) \geq \kappa(S') \geq 2 = \overline{\kappa}_{k+1}(G)$. ∎

For a general graph G, the monotonicity of $\overline{\lambda}_k(G)$ holds.

Lemma 2.9 *For a connected graph G with order n, we have $\overline{\lambda}_k(G) \geq \overline{\lambda}_{k+1}(G)$, where $3 \leq k \leq n-1$.*

Proof. Assume that $\lambda(S) = \overline{\lambda}_{k+1}(G) = \ell$ where $S \subseteq V(G)$and $|S| = k+1$. Let $\mathcal{T} = \{T_j | 1 \leq j \leq \ell\}$ be a set of ℓ edge-disjoint trees connecting S in G. We know $\mathcal{T} = \{T_j | 1 \leq j \leq \ell\}$ is also a set of ℓ edge-disjoint trees connecting $S \setminus \{v\}$ in G, where $v \in S$. Then $\overline{\lambda}_k(G) \geq \lambda(S \setminus \{v\}) \geq \lambda(S) = \overline{\lambda}_{k+1}(G)$. ∎

Lemma 2.10 *If G is a prism of order n and $k = \frac{n}{2} + 2$, then $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$.*

Proof. Let $n = 2h, k = \frac{n}{2} + 2$. We use the graph in Figure 2.1 and choose $S = \{u_i, v_1, v_2 | 1 \leq i \leq h\}$. Let T_1 be the path $u_1, v_1, v_2, u_2, \ldots, u_i, \ldots, u_h$ and T_2 be the graph obtained from G by deleting edges of T_1. Clearly, T_1 and T_2 are two internally disjoint trees connecting S, then $\overline{\kappa}_k(G) \geq 2$. By Lemmas 2.5 and 2.8, we have $\overline{\kappa}_k(G) = 2$, and then $\overline{\lambda}_k(G) = 2$ by Lemma 2.3. ∎

For a Möbius ladder G of order $n = 2h$, choose $S = \{u_i, v_1, v_2 | 1 \leq i \leq h\}$, let T_1 be the path $u_1, v_1, v_2, u_2, \ldots, u_i, \ldots, u_h$ and T_2 be the graph obtained from G by deleting edges of T_1. With a similar argument to that of the above lemma, the following result holds.

Lemma 2.11 *If G is a Möbius ladder of order n and $k = \frac{n}{2} + 2$, then* $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$.

Recall that any cubic connected Cayley graph G on an Abelian group is isomorphic to K_4, Q_3, $P(2h)$ or $M(2h)$, we only need to consider the latter two graph classes in our main result.

Theorem 2.12 *Let G be a cubic connected Cayley graph on an Abelian group of order $n \geq 9$.*
(i) If G is a prism, then there exists an integer k_1 such that $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$ for any $3 \leq k \leq k_1$ and $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$ for any $k_1 + 1 \leq k \leq n$; moreover, we have $\frac{n}{2} + 2 \leq k_1 \leq \lceil \frac{3n+2}{4} \rceil$.
(ii) If G is a Möbius ladder, then there exist two integers k_2, k_3 such that $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 3$ for any $3 \leq k \leq k_2$, $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$ for any $k_2 + 1 \leq k \leq k_3$ and $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$ for any $k_3 + 1 \leq k \leq n$; moreover, we have $3 \leq k_2 \leq \lceil \frac{3n}{8} \rceil, \frac{n}{2} + 2 \leq k_3 \leq \lceil \frac{3n+2}{4} \rceil$.

Proof. (*i*) By Theorem 2.7, Lemmas 2.4, 2.8 and 2.9, there exists an integer k_1 such that $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$ for any $3 \leq k \leq k_1$ and $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$ for any $k_1 + 1 \leq k \leq n$. Moreover, $\frac{n}{2} + 2 \leq k_1 \leq \lceil \frac{3n+2}{4} \rceil$ by Lemmas 2.4 and 2.10.

(*ii*) If G is a Möbius ladder, we can get the following result: $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 3$ for $k = 3$ by Theorem 2.7; $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$ for $\lceil \frac{3n}{8} \rceil + 1 \leq k \leq \frac{n}{2} + 2$ by Lemmas 2.4, 2.8, 2.9 and 2.11; $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$ for $k \geq \lceil \frac{3n+2}{4} \rceil + 1$ by Lemma 2.4. Furthermore, by Lemmas 2.8 and 2.9, there exist two integers k_2, k_3 such that $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 3$ for any $3 \leq k \leq k_2$, $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 2$ for any $k_2 + 1 \leq k \leq k_3$ and $\overline{\kappa}_k(G) = \overline{\lambda}_k(G) = 1$ for any $k_3 + 1 \leq k \leq n$. Moreover, $3 \leq k_2 \leq \lceil \frac{3n}{8} \rceil, \frac{n}{2} + 2 \leq k_3 \leq \lceil \frac{3n+2}{4} \rceil$. ∎

3 Remarks

In this paper, we have investigated the maximum generalized local connectivity and edge-connectivity of a cubic connected Cayley graph G on an Abelian group. Based on the structure of Cayley graphs of degree 3 on an Abelian group and the monotonicity of $\overline{\kappa}_k$ and $\overline{\lambda}_k$, we determine the precise values for $\overline{\kappa}_3(G)$ and $\overline{\lambda}_3(G)$ (Theorem 2.7) and get a result of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ for general k which shows the changing trend of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ (Theorem 2.12). It is quite difficult to determine the precise values of $\overline{\kappa}_k(G)$ and $\overline{\lambda}_k(G)$ for each $3 \leq k \leq n$, that is, determine the precise values of k_1, k_2, k_3, and one needs to find other approaches to solve this problem.

Acknowledgements: The author would like to thank the referee for helpful comments and suggestions.

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