Edge Centered Surface Area for the (n, k)-Star Graph

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Abstract

We suggest the notion of the surface area *centered at an edge* for a network structure, which generalizes the usual notion of surface area of a structure centered at a vertex. As a specific result, we derive explicit expressions of the edge centered surface areas for the edge asymmetric (n, k)-star graph, following a generating function approach, in terms of two different kinds of edges.

1 Introduction

Given a vertex v in a graph G, a question one may ask is *how many vertices are at distance i from v*, $i \in [0, D(G)]$, where $D(G)$ stands for the diameter of G. This quantity has been referred to as the Whitney numbers of the second kind [14] and *the surface area with radius i centered at v* [11], which we adopt in this paper.

The surface area of a (di)graph can find several applications in network performance evaluation, e.g., in computing various bounds for the problem

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of k-neighborhood broadcasting [8] and in spanning tree identification [15]. For solutions to this problem for various networks, including the (n,k)-star graph, readers are referred to [2, 11] and the references cited within.

In this paper, we study a related question: given an edge (v,w) in a graph G, referred to as the reference edge henceforth, how many vertices are at distance i from (v,w), $i \in [0, D(G)]$? We refer to this quantity as the *surface area with radius i centered at* (v,w), denoted as $B^e_G(v,w,i)$. We sometimes refer to $(B^e_G(v,w,0), B^e_G(v,w,1), \ldots, B^e_G(v,w,D(G)))$ as the edge-centered surface area sequence of G. We drop G from this notation, and other relevant ones, when the context is clear.

Let G be a graph, and let $e(v,w)$ be an edge in G. It is clear that $B^e_G(v,w,0) = 2$. Let $i \in [1, D(G)]$, for a vertex u, if $d(u,v) = i$, but $d(u,w) < i$, then the distance from u to (v,w) would be strictly less than i. On the other hand, the very existence of (v,w) implies $d(u,w) \leq i+1$. These facts and a symmetric consideration lead to the following definition.

Definition 1.1 *Let G be a general graph, and let (v,w) be an edge of G. For $i \in [1, D(G)]$,*

$$B^e_G(v,w,i) = E_G(v,w,i) + S_G(v,w,i) + L_G(v,w,i), \qquad (1)$$

where

$$E_G(v,w,i) = |\{u \in G | (d(u,v) = i) \wedge (d(u,w) = d(u,v))\}|, \qquad (2)$$
$$S_G(v,w,i) = |\{u \in G | (d(u,v) = i) \wedge (d(u,w) = d(u,v)+1)\}|, \qquad (3)$$
$$L_G(v,w,i) = |\{u \in G | (d(u,v) = i+1) \wedge (d(u,w) = d(u,v)-1)\}|. \qquad (4)$$

It is clear that, by definition, $S_G(v,w,D(G)) = L_G(v,w,D(G)) = 0$.

This notion of edge-centered surface area of a graph is clearly an immediate generalization of the usual notion of vertex centered surface area, with the most general scenario being the surface area centered at a subgraph of such a graph. Recently, it is suggested that the surface area centered at a path of length 2 plays an important role in exploring techniques of establishing the *conditional diagnosability* of certain networks [16], namely, the number of necessarily detectable faulty vertices in such a network. This notion of edge-centered surface area, besides being an interesting combinatoric quantity, also provides a good starting point along this line of research. Explicit expressions for the edge-centered surface area for the general arrangement graph has been studied in [6], and [5] where a generating function approach is followed to derive the desired result.

Let $(a_0, a_1, \ldots, a_n, \ldots)$ be an infinite sequence of numbers, to seek a "simple representation" of a_n, we construct a formal power series $F(x) = \sum_{i\geq 0} a_i x^i$, often called a generating function of the above sequence. We can then apply algebraic, differential, integral, and other operations to possibly simplify $F(x)$, and finally extract the coefficient of x^n in $F(x)$, denoted as $[x^n]F(x)$, which clearly equals a_n. We can sometimes also derive other properties of a_n, such as its asymptotic approximation. For these reasons, generating function has long been recognized as a powerful tool in carrying out combinatorial studies, particularly in enumeration. For an excellent introduction to generating function, as well as many approachable examples, readers are referred to [10, 17]. On the other hand, [9] provides an encyclopedic treatment of this subject.

In this paper, we study the edge-centered surface areas of the edge asymmetric(n, k)-star graph following a generating function approach, in terms of two kinds of reference edges.

The rest of this paper proceeds as follows: After briefly discussing the (n, k)-star graph and some of its relevant properties in the next section, we characterize the structures of the (n, k)-star graph vertices related to one kind of reference edges in Section 3, and derive an explicit expression of their associated edge-centered surface area in Section 4. The surface area associated with the other kind of reference edges is dealt with in Section 5. We finally present our main results in Section 6 and conclude this paper in Section 7.

2 Edge-centered surface area of the (n, k)-star graph

To address a scalability issue associated with the popular star graph [1], the (n, k)-star graph, denoted as $S_{n,k}$, was put forward in [7]. $S_{n,k}$, with $n!/(n-k)!$ vertices, brings in flexibility when choosing an appropriate interconnection structure for an actual network, while preserving many attractive properties of the star graph, including vertex symmetry. On the other hand, $S_{n,k}$ is not edge symmetric in general.

The vertex set of an $S_{n,k}$ graph, $n \geq 3, k \in [1, n)$, is simply the collection of all the k-permutations on $\langle n \rangle = \{1, 2, \ldots, n\}$, and its edges are defined via symbol replacement regarding the very first position: for any $u, v \in V(S_{n,k})$, $(u, v) \in E(S_{n,k})$ if and only if v can be obtained from u by either 1) applying a transposition $(1, i)$ to u, $i \in [2, k]$ (i-edge); or 2) for some $x \in \langle n \rangle - \{u_i | i \in [1, k]\}$, replacing u_1 with x in u (1-*edge*). It is thus clear that $S_{n,k}$ is $(n-1)$-regular. Notice that $S_{n,n-1}$ is just the usual star graph S_n.

The following diameter result is given in [7].

$$D(S_{n,k}) = \begin{cases} 2k-1, & \text{if } 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ k + \lfloor \frac{n-1}{2} \rfloor & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq k < n; \end{cases}$$

We denote the identity vertex of $S_{n,k}$ as $e_k = 12\cdots k$. We caution that in later sections, we will use nations such as $e_1(e_2)$ to denote external symbols that occur in some specific external cycles, but they occur in a completely different context from where e_k might occur, and thus should not cause any confusion.

Following the terms adopted in [7], we refer to $u \in \langle n \rangle$ as an internal symbol if $u \in [1, k]$; and as an external symbol if $u \in (k, n]$. We also refer to i as an internal position if $i \in [1, k]$; and as an external position if $i \in (k, n]$). Figure 1 shows a $(4, 2)$-star graph, where an external symbol 3 occurs in an internal position 1 in 32. Moreover, $(12, 21)$ is a 2-edge, while both $(12, 32)$ and $(12, 42)$ are 1-edges.

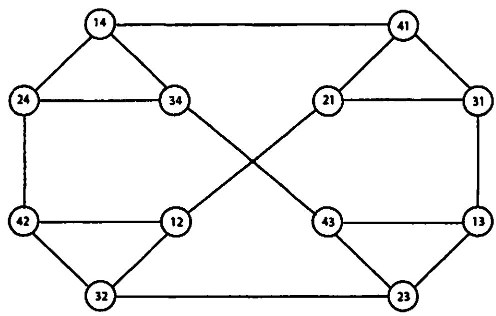

Figure 1: $S_{4,2}$

It is pointed out in [2, 7] that, for a vertex u in $S_{n,k}$, as a partial permutation, an *extended permutation*, u', on $\langle n \rangle$, can be derived, and then be decomposed as a collection of possibly trivial cycles [1], $C(u)$, referred to as the *cycle structure* of u. Such a cycle structure is unique except for the order of these cycles. Furthermore, for $u \in S_{n,k}$, $C(u)$ with $b(u)$ symbols consists of $g_I(u)$ non-trivial internal cycles and $g_E(u)$ non-trivial external cycles. Each non-trivial internal cycle contains at least two internal symbols and no external symbol and each non-trivial external cycle contains exactly one external symbol and at least one internal symbol. We also refer to a non-trivial cycle containing 1 a primary cycle.

For example, given $f = 6351792$, a vertex in $A_{9,7}$, we first derive $f' = 6351792\underline{84}$[2] on $\langle 9 \rangle$ as follows: Since 8 does not occur in f, we let $f'_8 = 8$. On the other hand, since $f_6 = 9, f_1 = 6$, and $f_4 = 1$, but 4 does not occur in f, we set $f'_9 = 4$. Then, it is easy to see that $C(f')$, always denoted by $C(f)$ in this paper, is $(2, 3, 5, 7)(8)(9; 4, 1, 6)$, where $(2, 3, 5, 7)$ is a non-trivial

[1] A cycle is trivial if it contains exactly one symbol, called a fixed point. It is non-trivial otherwise.

[2] We use "$\underline{84}$" to indicate the two symbols that occur in external positions.

internal cycle, (8) is a trivial external cycle, and $(9; 4, 1, 6)$ is a non-trivial primary external cycle, with 9 being its external symbol, as indicated with ';'. As a result, $g_I(f) = g_E(f) = 1$, and $b(f) = 8$. Notice the convention we adopt here, when representing an external cycle, we place its external symbol first. As a result, all the external cycles in a cycle structure are ordered in terms of their respective external symbol, which plays a role in the later enumeration of such cycle structures.

The following distance result between u and e_k in $S_{n,k}$ is given in [2, 7].

Theorem 2.1 *Let $d(u, v)$ stand for the distance between u and v in $S_{n,k}$. Then, for any $u \in S_{n,k}$,*

1. *if u does not contain any non-trivial external cycle, then*

$$d(u, e_k) = \begin{cases} b(u) + g_I(u) - 2, & \text{if } C(u) \text{ contains a primary cycle,} \\ b(u) + g_I(u), & \text{otherwise;} \end{cases} \quad (5)$$

2. *if u does contain at least one non-trivial external cycle, then*

$$d(u, e_k) = \begin{cases} b(u) + g_I(u) - 1 & \text{if } C(u) \text{ contains a primary cycle,} \\ b(u) + g_I(u) + 1, & \text{otherwise.} \end{cases} \quad (6)$$

For example, since $C(f)$ for the above vertex f of $S_{9,7}$ contains a primary external cycle, $d(f, e_7) = 8$, and one of the shortest paths from f to e_7 is given as follows[3]:

$$6351792\underline{84} \overset{(1,2)}{\rightarrow} 3651792\underline{84} \overset{(1,3)}{\rightarrow} 5631792\underline{84} \overset{(1,5)}{\rightarrow} 7631592\underline{84} \overset{(1,7)}{\rightarrow} 2631597\underline{84}$$
$$\overset{(1,2)}{\rightarrow} 62315978\underline{84} \overset{(1,6)}{\rightarrow} 9231567\underline{84} \overset{(1,9)}{\rightarrow} 4231567\underline{89} \overset{(1,4)}{\rightarrow} 1234567\underline{89},$$

where, (p_1, p_2) in "$u \overset{(p_1,p_2)}{\rightarrow} v$" is an edge that connects u and v in $S_{9,7}$. We notice that, in the above path, $(1, 9)$ is a 1-edge, where the symbol 9 that occurs in the first position of 9231567 is switched with 4, which does not occur there; and all the other edges are i-edge, where $i \in [2, 7]$.

To derive the edge-centered surface area for $S_{n,k}$, we have to fix a reference edge (v, w). Since $S_{n,k}$ is vertex symmetric, we choose $v = e_k = 123 \cdots k$. But, since $S_{n,k}$ is not edge symmetric, we cannot choose w arbitrarily. By the structural definition of $S_{n,k}$, it turns out that we have two options: We can obtain w by either switching symbol 1 in v with an external symbol $e \in (k, n]$, reference edge of the first kind henceforth; or switching

[3] For a result on enumerating such shortest paths from a vertex u to e_k in $S_{n,k}$, readers are referred to [3], which is based on a result of [12].

symbol 1 in v with another internal symbol $l \in [2, k]$, thus requiring $k \geq 2$, reference edge of the second kind henceforth.

It is clear that there are $n-k$ choice of e for the first kind of reference edges, and $k-1$ choices of l for the second kind, with a total number of $n-1$ reference edges in all, agreeing with the degree of $S_{n,k}$. For example, as shown in Figure 1, there are two first kind reference edges in $S_{4,2}$: $(12, 32)$ and $(12, 42)$; and one second kind reference edge, i.e., $(12, 21)$.

The above two kinds of reference edges immediately lead to the following respective automorphism: Let $u \in S_{n,k}$, define $\phi(u)$ by swapping symbol 1 and $e(\in (k, n])$ in u, when it is applicable; and define $\varphi(u)$ by swapping symbol 1 and l $(\in (1, k])$ in u, when it is applicable. For example, in $S_{4,2}$, we have two choices for e, either 3 or 4. Letting $e = 3$, we have $\phi(12) = 32$, but, $\phi(42) = 42$. On the other hand, we have exactly one choice for l, i.e., $l = 2$, we have $\varphi(13) = 23$, and $\varphi(42) = 41$.

It is clear that $\phi(\phi(e_k)) = \varphi(\varphi(e_k)) = e_k$. Moreover, since both ϕ and φ are automorphism, for any edge $(u, v) \in S_{n,k}$, $(\phi(u), \phi(v))$ (respectively, $(\varphi(u), \varphi(v))$) is also an edge in $S_{n,k}$. As a result, for any $u \in S_{n,k}$, $d(u, \phi(e_k)) = d(\phi(u), e_k)$, and $d(u, \varphi(e_k)) = d(\varphi(u), e_k)$. Thus, we can use the distance expression result as given in Theorem 2.1 to express the distance between u and $\phi(e_k)$ (respectively, between u and $\varphi(e_k)$).

In the rest of this paper, for notational simplicity, we fix $e = k + 1$, *and* $l = 2$. *It will be clear from our later derivation in the subsequent sections that such choices are immaterial.*

3 Vertex structures related to a first kind reference edge

To enumerate the vertices that are of certain distance to a reference edge in $S_{n,k}$, we need to explore the relationship between the cycle structures of all such vertices in $S_{n,k}$ and their distances to such a reference edge in light of Theorem 2.1. We start the discussion with a reference edge of the first kind, $(e_k, \phi(e_k))$, where with $\phi(e_k)$, we switch 1 and $k + 1$.

For a given vertex u in $S_{n,k}$, $k+1$ and 1 can occur either together in one cycle, or separately in two different cycles in $C(u)$. In the later case, symbol 1 can appear in either an internal cycle, or an external cycle. If we further consider the symbols that may or may not exist after $k + 1$ and 1 in their respective cycle; and, when 1 does not occur in a non-trivial external cycle, whether there exists another non-trivial external cycle in $C(u)$, there are twenty-four cases for us to consider. After a detailed analysis of all these cases, we have reached the following characterization result:

Theorem 3.1 *Let $u \in S_{n,k}, n \geq 3, k \in [1, n)$ such that $v = e_k, w = \phi(v) = (k+1)2\cdots k$, $d(u,(v,w)) = i \in [1, D(S_{n,k})]$. Let A, B and C be possibly empty sequences of internal symbols. Then,*

1. *If $C(u)$ contains $E_1 = (k+1; 1, B)$, where B may be empty, and at least another non-trivial external cycle, besides E_1, then $d(\phi(u), e_k) = d(u, e_k)$.*
2. *If $C(u)$ contains $E_1 = (k+1; 1, B)$, B may be empty, but no other non-trivial external cycles, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*
3. *If $C(u)$ contains $E_2 = (k+1; A, 1, B)$, where A is not empty but B may be empty, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*
4. *If $C(u)$ contains $(k+1)$ and $I_1 = (1, B)$, B may be empty, and at least one non-trivial external cycle, then $d(\phi(u), e_k) = d(u, e_k)$.*
5. *If $C(u)$ contains $(k+1)$ and $I_1 = (1, B)$, B may be empty, but no other non-trivial external cycles, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*
6. *If $C(u)$ contains $E_3 = (k+1; A)$ where A is not empty, and $I_1 = (1, B)$, where B may be empty, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*
7. *If $C(u)$ contains $E_3' = (k+1, A)$, and $E_4' = (e, C, 1, B)$, where e is an external symbol, B may be empty, and either both A and C are empty, or neither is, then $d(\phi(u), e_k) = d(u, e_k)$.*
8. *If $C(u)$ contains $E_3 = (k+1; A)$, where A is not empty, and $E_5 = (e; 1, B)$, where e is an external symbol, B may be empty, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*
9. *If $C(u)$ contains $(k+1)$ and $E_4 = (e; C, 1, B)$, where e is an external symbol, B may be empty, but C is not, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*

Proof: We prove Cases 1 and 2 as follows. Assume $B \neq \epsilon$, i.e., $C(u)$ contains $E_1 = (k+1; 1, b_2, \ldots, b_y)$, which is a non-trivial primary external cycle, thus, by Eq. 6,

$$d(u, e_k) = b(u) + g_I(u) - 1.$$

On the other hand, since $u_1 = b_2, \ldots, u_{b_y} = k+1$, and $u'_{k+1} = 1$, we have that $\phi(u)_1 = b_2, \ldots, \phi(u)_{b_y} = 1$, and $\phi(u)'_{k+1} = k+1$. As a result, $k+1$ becomes a fixed point in $C(\phi(u))$, and the original external cycle E_1 of $C(u)$ turns into a primary internal cycle $(1, b_2, \ldots, b_y)$ in $C(\phi(u))$. Hence, $b(\phi(u)) = b(u) - 1$, and $g_I(\phi(u)) = g_I(u) + 1$.

- If it falls into Case 1, i.e., $C(u)$ contains at least another non-trivial external cycle, besides E_1, then $C(\phi(u))$ contains at least one non-trivial external cycle, and the above primary internal cycle. By Eq. 6,

$$\begin{aligned} i = d(\phi(u), e_k) &= b(\phi(u)) + g_I(\phi(u)) - 1 \\ &= (b(u) - 1) + (g_I(u) + 1) - 1 \\ &= b(u) + g(u) - 1 = d(u, e_k). \end{aligned}$$

- Otherwise, it falls into Case 2, $C(\phi(u))$ contains no non-trivial external cycle, but the above primary internal cycle. By Eq. 5,

$$\begin{aligned} i = d(\phi(u), e_k) &= b(\phi(u)) + g(\phi(u)) - 2 = b(\phi(u)) + g_I(\phi(u)) - 2 \\ &= (b(u) - 1) + (g_I(u) + 1) - 2 = b(u) + g_I(u) - 2 \\ &= d(u, e_k) - 1. \end{aligned}$$

The proof of the $B = \epsilon$ case and those of the other cases are similar. □

We use Table 1 to demonstrate the results as reported in Theorem 3.1 for $S_{4,2}$. We notice that the edge-centered surface area sequence of $S_{4,2}$, centered at $(12, 32)$, is $(2, 3, 5, 2)$.

Table 1: Edge-center surface area of $S_{4,2}$ centered at $(12, 32)$

u	u'	$C(u)$	$d(u, e_2)$	$\phi(u)$	$d(\phi(u), e_2)$	$d(u, (v, w))$	Case
32	32$\underline{14}$	(3; 1)	1	12	0	0	2
12	12$\underline{34}$	(1)(2)(3)(4)	0	32	1	0	5
23	23$\underline{14}$	(3; 1, 2)	2	21	1	1	2
21	21$\underline{34}$	(1, 2)	1	23	2	1	5
42	42$\underline{31}$	(4; 1)	1	42	1	1	7
31	31$\underline{24}$	(3; 2, 1)	2	13	3	2	3
13	13$\underline{24}$	(3; 2)	3	31	2	2	6
24	24$\underline{31}$	(4; 1, 2)	2	24	2	2	7
41	41$\underline{32}$	(4; 2, 1)	2	43	3	2	8
43	43$\underline{21}$	(4; 1)(3; 2)	3	41	2	2	9
34	34$\underline{12}$	(3; 1)(4; 2)	3	14	3	3	1
14	14$\underline{32}$	(4; 2)	3	34	3	3	4

If we refer to the nine mutually exclusive partition as shown in Theorem 3.1 as Cases 1 through 9, Theorem 3.1, Eqs. 2, 3 and 4 tell us the following:

$$
\begin{aligned}
& E_{S_{n,k}}(e_k, \phi(e_k), i) \\
= \;& |\{u \in S_n |\ u \text{ occurs in either Case 1, 4 or 7 and } d(u, e_k) = i\}|\,, \quad (7)\\
& S_{n,k}(e_k, \phi(e_k), i) \\
= \;& |\{u \in S_n |\ u \text{ occurs in either Case 3, 5 or 9 and } d(u, e_k) = i\}|\,, \quad (8)\\
& L_{S_{n,k}}(e_k, \phi(e_k), i) \\
= \;& |\{u \in S_n |\ u \text{ occurs in either Case 2, 6, or 8 and } d(u, e_k) = i+1\}|\,. \quad (9)
\end{aligned}
$$

For all $i \in [1, D(S_{n,k})]$, let $B^{l,1}_{n,k}(e_k, \phi(e_k), i), l \in [1, 9]$, stand for the number of cycle structures falling into Case l of Theorem 3.1, by Eqs. 1 through 4 and Eqs. 7 through 9, the edge-centered surface area of $S_{n,k}$, centered at $(e_k, \phi(e_k))$, is given as follows:

$$
B^{e_1}_{S_{n,k}}(e_k, \phi(e_k), i) = \sum_{l=1}^{9} B^{l,1}_{n,k}(e_k, \phi(e_k), i). \quad (10)
$$

Before evaluating $B^{l,1}_{n,k}(e_k, \phi(e_k), i), l \in [1, 9]$, we will provide a generating function based enumerating model in the next section.

4 Surface area of $S_{n,k}$ centered at a reference edge of first kind

Our task for now is to enumerate cycle structures $C(u)$ containing $b(u)$ symbols, organized in $g_I(u)(\geq 0)$ non-trivial internal cycles and $n-k$ possibly trivial external cycles, that satisfy Cases 1 through 9 and the general setting as given in Theorem 3.1.

For a direct counting approach to enumerate such structures, readers are referred to [6]. We now follow a multivariate generating function approach [2] to enumerate such structures. For a given $u \in S_{n,k}$, since $g_E(u)$ does not play a role in the distance between u and a reference edge, symbol 1 plays a special labeling role in forming such an edge as indicated in Theorem 3.1, and labels are irrelevant in enumerating structures, we use x to mark, exponentially, an internal symbol, excluding 1, that occurs in all the cycles, trivial or not, of $C(u)$; y all the symbols, both internal and external, including 1, in non-trivial cycles; and z a non-trivial internal cycle.

Given $f = 6351792$ in $S_{9,7}$, then $C(f) = (2,3,5,7)(8)(9;4,1,6)$. Since four symbols occur in the non-trivial internal cycle $(2,3,5,7)$, the exponents for the labeling variables x, y and z are 4, 4, and 1, respectively; the trivial external cycle (8) contains exactly one external symbol, thus the exponents for both x and y are 0; and the non-trivial external cycle $(9;4,1,6)$ contains three internal symbols, including 1, and one external symbol, thus the exponents for x and y are 2 and 4, respectively. As a result, the generating function for $\mathcal{C}(f)$ is $x^6y^8z/6!$, as x is an exponential marker. It is clear that this expression tells us that there are eight symbols that occur in non-trivial cycles and one of such cycles is internal.

To form a generating function of a cycle structure, we first consider an internal cycle, C_I, that does not contain 1.

- If C_I is trivial, i.e., C_I contains an internal symbol which is not 1, the associated generating function is $x^1y^0z^0 = x$.
- Otherwise, C_I contains $j \geq 2$ internal symbols, none being 1. The exponent of x, y and z will be j, j and 1, respectively. Moreover, there are $(j-1)!$ different cycles with these j elements. Thus, the generating function for such a non-trivial cycle is the following:

$$\sum_{j\geq 2}\frac{(j-1)!x^jy^jz}{j!} = \sum_{j\geq 2}\frac{x^jy^jz}{j} = z\left[\sum_{j\geq 1}\frac{x^jy^j}{j} - xy\right]$$
$$= z\left[\ln\frac{1}{1-xy} - xy\right].$$

Hence, we have the following generating function for C_I:

$$f_I(x,y,z) = x - xyz + \ln\frac{1}{(1-xy)^z}.$$

We now turn to an external cycle, C_E, that does not contain 1.

- If C_E contains exactly one external symbol, the exponent of x and y will be 0 and 0, respectively, which leads to $x^0y^0 = 1$.
- Otherwise, let C_E contain, besides an external symbol, $j \geq 1$ internal symbols, the exponent of x and y will be j and $j+1$, respectively. Moreover, this many symbols will form exactly $j!$ distinct cycles.

As a result, the generating function for C_E is the following:

$$f_E(x,y) = 1 + \sum_{j\geq 1}\frac{j!x^jy^{j+1}}{j!} = 1 + y\sum_{j\geq 1}x^jy^j = 1 + \frac{xy^2}{1-xy}.$$

We are ready to derive generating functions for cycle structures falling into Cases 1 through 9 as mentioned in Theorem 3.1. We start with those falling into Case 1, where $\mathcal{C}(u)$ contains cycles $E_1 = (k+1; 1, B)$, where B may be empty, and at least another non-trivial external cycle.

We consider E_1 first. Let $|B| = j \geq 0$, then the respective exponent for x and y is j and $j+2$. Since there are exactly $j!$ different cycles for E_1,

$$f_{E_1}(x,y) = \sum_{j\geq 0} \frac{j! x^j y^{j+2}}{j!} = y^2 \sum_{j\geq 0} x^j y^j = \frac{y^2}{(1-xy)}.$$

To construct the generating function for the requirement that there exist at least one non-trivial external cycle, we notice that the generating function for such a cycle is simply $xy^2/(1-xy)$, as discussed above. Let $j \in [1, n-k-1]$, there are $\binom{n-k-1}{j}$ ways to choose these $j \geq 1$ non-trivial external cycles out of a total of $n-k-1$ remaining external cycles, other than E_1, while the remaining $n-k-1-j$ external cycles are trivial, each being represented by 1.

Hence, noticing that $\mathcal{C}(u)$, u a vertex in $S_{n,k}$, contains exactly $n-k$ possibly trivial external cycles, ordered by their respective external symbols, and the order of the internal cycles does not matter, the generating function of $\mathcal{C}(u)$, falling into Case 1, is given as follows:

$$\begin{aligned} &f_1(x,y,z) \\ &= \sum_{0\leq l\leq k<n\leq\infty} \frac{[f_I(x,y,z)]^l}{l!} \times f_{E_1}(x,y) \times \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} \left[\frac{xy^2}{(1-xy)}\right]^j \\ &= \frac{y^2 e^{x-xyz}}{(1-xy)^z} \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} \frac{x^j y^{2j}}{(1-xy)^{j+1}}. \end{aligned}$$

Since the external cycle E_1 is both non-trivial and primary, by Case 1 of Theorem 3.1, Eq. 7, and Theorem 2.1, $i = d(u, e_k) = b(u) + g_I(u) - 1$, i.e., $b(u) + g_I(u) = i + 1$. Thus, to enumerate the cycle structures falling into the above Case 1, we set z to y in $f_1(x,y,z)$ so that the exponent of y becomes the number of symbols, both internal and external, in all non-trivial cycles plus the total number of non-trivial internal cycles, i.e., $b(u) + g_I(u)$. Finally, since x is an exponential marker of all the internal symbols, other than 1, it is immediate that

$$\begin{aligned} &B^{1,1}_{n,k}(e_k, \phi(e_k), i) = \left[\frac{x^{k-1}y^{i+1}}{(k-1)!}\right] f_1(x,y,y) \\ &= (k-1)! \left[x^{k-1}y^{i+1}\right] \frac{e^{x-xy^2}}{(1-xy)^y} \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} \frac{x^j y^{2j+2}}{(1-xy)^{j+1}} \\ &= (k-1)! \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} \left[x^{k-(j+1)} y^{i-(2j+1)}\right] \frac{e^{x-xy^2}}{(1-xy)^y (1-xy)^{j+1}} \end{aligned}$$

To simplify the representation of such results, for $a, b, c, d \geq 0$, we introduce the following notation:

$$\begin{aligned} & V_{a,b,c,d}(n,k,i) \\ = \quad & (k-1)!\,[x^{k-a}y^{i-b}]\,\frac{e^{x-xy^2}}{(1-xy)^y(1-xy)^c}\left[1+\frac{xy^2}{1-xy}\right]^{n-k-d}. \quad (11) \end{aligned}$$

Then, for $i \in [3, D(S_{n,k})]$[4],

$$B^{1,1}_{n,k}(e_k, \phi(e_k), i) \quad = \quad \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} V_{j+1,2j+1,j+1,n-k}(n,k,i). \quad (12)$$

We notice that the above process can be generalized as follows: *to derive $B^{l,1}_{n,k}(e_k, \phi(e_k), i)$, we simply derive the generating functions for those basic cycles as contained in Case l in Theorem 3.1, collected in Table 2, assemble these pieces together with $f_I(x,y,z)$ and $f_E(x,y)$, with exponent appropriate for the case, to form a generating function $f_l(x,y,z)$, and finally extract the coefficient of the term appropriate for this case in $f_l(x,y,y)$.*

Table 2: Generating functions for the basic cycles (I)

Index	Cycle	Generating function
I	$(A), A \neq \emptyset, 1 \notin A$	$x - xyz + \ln \frac{1}{(1-xy)^z}$
E	$(e; A)$, e fixed, $\emptyset \subseteq A, 1 \notin A$	$1 + \frac{xy^2u}{1-xy}$
I_1	$(1, B), B \neq \emptyset$	$\frac{xy^2z}{1-xy}$
E_1	$(k+1; 1, B), \emptyset \subseteq B$	$\frac{y^2}{(1-xy)}$
E_2	$(k+1; A, 1, B), A \neq \emptyset, \emptyset \subseteq B$	$\frac{xy^3}{(1-xy)^2}$
E_3	$(k+1; A), A \neq \emptyset$	$\frac{xy^2}{(1-xy)}$
E_4	$(e; C, 1, B), e \in (k+1, n]; \emptyset \subseteq B, C \neq \emptyset$	$(n-k-1)\frac{xy^3}{(1-xy)^2}$
E_5	$(e; 1, B), e \in (k+1, n], \emptyset \subseteq B$	$(n-k-1)\frac{y^2}{(1-xy)}$

We have gone through such a general process and obtained the following results:

$$B^{2,1}_{n,k}(e_k, \phi(e_k), i) \quad = \quad V_{1,0,1,n-k}(n,k,i). \quad (13)$$

$$B^{3,1}_{n,k}(e_k, \phi(e_k), i) \quad = \quad V_{2,2,2,1}(n,k,i). \quad (14)$$

[4]We notice that, for this case, there exist at least four symbols in $C(u)$, thus, by Theorem 2.1, $d(u, (e_k, \phi(e_k)) = d(u, e_k) \geq 4 - 1 = 3$.

$$B^{4,1}_{n,k}(e_k, \phi(e_k), i) = \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} V_{j+1,2j+1,j,n-k}(n,k,i) + \sum_{j=1}^{n-k-1} \binom{n-k-1}{j} V_{j+2,2j+2,j+1,n-k}(n,k,i) \quad (15)$$

$$B^{5,1}_{n,k}(e_k, \phi(e_k), i) = V_{1,0,0,n-k}(n,k,i) + V_{2,1,1,n-k}(n,k,i). \quad (16)$$

$$B^{6,1}_{n,k}(e_k, \phi(e_k), i) = V_{2,2,1,1}(n,k,i) + V_{3,3,2,1}(n,k,i). \quad (17)$$

$$B^{7,1}_{n,k}(e_k, \phi(e_k), i) = (n-k-1)(V_{1,1,1,2}(n,k,i) + V_{3,4,3,2}(n,k,i)). \quad (18)$$

$$B^{8,1}_{n,k}(e_k, \phi(e_k), i) = B^{9,1}_{n,k}(e_k, \phi(e_k), i) = (n-k-1)V_{2,2,2,2}(n,k,i). \quad (19)$$

We postpone the derivation of $V_{a,b,c,d}(n,k,i)$, $B^{l,1}_{n,k}(e_k, \phi(e_k), i)$, and that of $B^{e_1}_{S_{n,k}}(e_k, \phi(e_k), i)$ until Section 6.

5 Surface area of $S_{n,k}$ centered at a second kind reference edge

We now similarly explore the relationship between the cycle structure of a vertex in $S_{n,k}$ and its distance to a reference edge of second kind, where we switch the symbol 1 in e_k with 2.

Since 1 and 2 can be located in one cycle or two cycles, which can be either internal or external, and additional symbols can be present or absent after both 1 and 2, and the applicable external symbols, it turns out that we have to consider fifty-six cases in all. After a detailed analysis by cases, we have also obtained the following result. Its proof is the same as that for Theorem 3.1.

Theorem 5.1 *Let $u \in S_{n,k}, n \geq 3, k \in [1,n)$ such that $v = e_k, w = \varphi(v) = 21\cdots k$, $d(u,(v,w)) = i \in [1, D(S_{n,k})]$. Let A, B, C and D be possible empty sequences of internal symbols. Then,*

1. *If $C(u)$ contains $I_2 = (1, A, 2)$, where A may be empty, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*

2. *If $C(u)$ contains $I_3 = (1, A, 2, B)$, where A may be empty, but $B \neq \emptyset$, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*

3. *If $C(u)$ contains $E_6 = (e; C, 1, A, 2, B)$, where A, B and C may be empty, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*

4. *If $C(u)$ contains $E_7 = (e; C, 2, 1, A)$, where both A and C may be empty, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*

5. *If $C(u)$ contains $E_8 = (e; C, 2, B, 1, A)$, where both A and C may be empty, but $B \neq \emptyset$, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*

6. *If $C(u)$ contains $I_4 = (1, A)$, and (2), A may be empty, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*

7. *If $C(u)$ contains $I_4 = (1, A)$, and $(2, B)$, A may be empty, but $B \neq \emptyset$, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*

8. *If $C(u)$ contains $I_4 = (1, A)$ and $E_9 = (e; D, 2, B)$, where A, B, D may be empty, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*

9. *If $C(u)$ contains $E_9 = (e; C, 1, A)$ and (2), where both A and C may be empty, then $d(\phi(u), e_k) = d(u, e_k) + 1$.*

10. *If $C(u)$ contains $E_9 = (e; C, 1, A)$ and $I_4 = (2, B)$, here both A and C may be empty, but $B \neq \emptyset$, then $d(\phi(u), e_k) = d(u, e_k) - 1$.*

11. *If $C(u)$ contains $E_9 = (e_1; C, 1, A)$ and $E_9 = (e_2, D, 2, B)$, where A, B, C and D may be empty, then $d(\phi(u), e_k) = d(u, e_k)$.*

In general, referring to the eleven mutually exclusive partitions of cycle structures as shown in Theorem 5.1 as Cases 1 through 11, Theorem 5.1, Eqs. 2, 3 and 4 tell us the following:

$$\begin{aligned} & E_{S_{n,k}}(e_k, \varphi(e_k), i) \\ = \; & |\{u \in S_n |\; u \text{ occurs in Case 11}, d(u, e_k) = i\}|, \quad (20) \\ & S_{S_{n,k}}(e_k, \varphi(e_k), i) \\ = \; & |\{u \in S_n |\; u \text{ occurs in Case 2, 3, 5, 6, 9}, d(u, e_k) = i\}|, \quad (21) \\ & L_{S_{n,k}}(e_k, \varphi(e_k), i) \\ = \; & |\{u \in S_n |\; u \text{ occurs in Case 1, 4, 7, 8, 10}, d(u, e_k) = i + 1\}|. \quad (22) \end{aligned}$$

Let $B^{l,2}_{n,k}(e_k, \varphi(e_k), i), l \in [1, 11]$, stand for the number of cycle structures falling into Case l of Theorem 5.1, by Eqs. 1 through 4 and Eqs. 20 through 22, we express the edge-centered surface area of $S_{n,k}$, centered at a reference edge of second kind, $(e_k, \varphi(e_k))$, as follows:

$$B^{e_2}_{S_{n,k}}(e_k, \varphi(e_k), i) \;=\; \sum_{l=1}^{11} B^{l,2}_{n,k}(e_k, \varphi(e_k), i)). \quad (23)$$

When deriving generating functions for this case, we follow essentially the same procedure as we did for the previous case, except that, considering the special role that both symbols 1 and 2 play for this case as characterized in Theorem 5.1, we will now use x to mark, exponentially, an internal symbol, excluding both 1 and 2, in all the cycles; y still marks all the symbols, both internal and external, including both 1 and 2, in non-trivial cycles; and z still marks a non-trivial internal cycle.

We notice that both x and y used in this current case subsume their respective role in the previous case: when x marks an internal symbol, excluding both 1 and 2, it certainly does not mark 1; and when y marks all the symbols that occur in non-trivial cycles, including both 1 and 2, it certainly includes 1. Thus, the two general results for f_I and f_E, as shown in Table 2, also hold for the current case, if we take I (respectively, E) to be an internal (respectively, external) cycle, containing neither 1 nor 2.

Table 3: Generating functions for the basic cycles (II)

Index	Cycle	Generating function
I_2	$(1, A, 2), \emptyset \subseteq A$	$\frac{y^2 z}{1-xy}$
I_3	$(1, A, 2, B), \emptyset \subseteq A, B \neq \emptyset$	$\frac{xy^3 z}{(1-xy)^2}$
I_4	$(1, A), A \neq \emptyset$	$\frac{xy^2 z}{1-xy}$
E_6	$(e; C, 1, A, 2, B), e \in (k, n], \emptyset \subseteq A, B, C$	$\frac{(n-k)y^3}{(1-xy)^2} + \frac{(n-k)xy^4}{(1-xy)^3}$
E_7	$(e; C, 2, 1, A), A, e \in (k, n], \emptyset \subseteq A, C$	$\frac{(n-k)y^3}{(1-xy)^2}$
E_8	$(e; C, 2, B, 1A), e, \in (k, n], B \neq \emptyset, \emptyset \subseteq A, C$	$\frac{(n-k)xy^4}{(1-xy)^3}$
E_9	$(e; C, 1, A), e \in (k, n], \emptyset \subseteq A, C$	$\frac{(n-k)y^2}{(1-xy)^2}$

Once the generating functions of the basic cycle structures as shown in Theorem 5.1 are derived, listed in Table 3, it is straightforward to follow the above general process to obtain results for this case as follows: for $n \geq 3, k \in (1, n)$, and $i \in [1, D(S_{n,k})]$,

$$B^{1,2}_{n,k}(e_k, \varphi(e_k), i) = \frac{1}{k-1} \sum_{j=1}^{n-k} \binom{n-k}{j} V_{j+2,2j+1,j+1,n-k}(n, k, i) + \frac{1}{k-1} V_{2,0,1,n-k}(n, k, i). \tag{24}$$

$$B^{2,2}_{n,k}(e_k, \varphi(e_k), i) = \frac{1}{k-1} \sum_{j=1}^{n-k} \binom{n-k}{j} V_{j+3,2j+3,j+2,n-k}(n, k, i) + \frac{1}{k-1} V_{3,2,2,n-k}(n, k, i) \tag{25}$$

$$B^{3,2}_{n,k}(e_k, \varphi(e_k), i) = \frac{n-k}{k-1} (V_{2,2,2,1}(n, k, i) + V_{3,3,3,1}(n, k, i)) \tag{26}$$

$$B^{4,2}_{n,k}(e_k,\varphi(e_k),i) = \frac{n-k}{k-1}V_{2,1,2,1}(n,k,i). \tag{27}$$

$$B^{5,2}_{n,k}(e_k,\varphi(e_k),i) = \frac{n-k}{k-1}V_{3,3,3,1}(n,k,i). \tag{28}$$

$$\begin{aligned} B^{6,2}_{n,k}(e_k,\varphi(e_k),i) &= \frac{1}{k-1}\sum_{j=1}^{n-k}\binom{n-k}{j}V_{j+2,2j+1,j,n-k}(n,k,i) \\ &+\frac{1}{k-1}\sum_{j=1}^{n-k}\binom{n-k}{j}V_{j+3,2j+2,j+1,n-k}(n,k,i) \\ &+\frac{1}{k-1}\left(V_{2,0,0,n-k}(n,k,i)+V_{3,1,1,n-k}(n,k,i)\right) \end{aligned} \tag{29}$$

$$\begin{aligned} B^{7,2}_{n,k}(e_k,\varphi(e_k),i) &= \frac{1}{k-1}\sum_{j=1}^{n-k}\binom{n-k}{j}V_{j+3,2j+3,j+1,n-k}(n,k,i) \\ &+\frac{1}{k-1}\sum_{j=1}^{n-k}\binom{n-k}{j}V_{j+4,2j+4,j+2,n-k}(n,k,i) \\ &+\frac{1}{k-1}\left(V_{3,2,1,n-k}(n,k,i)+V_{4,3,2,n-k}(n,k,i)\right) \end{aligned} \tag{30}$$

$$B^{8,2}_{n,k}(e_k,\varphi(e_k),i)) = \frac{n-k}{k-1}\left(V_{2,2,2,1}(n,k,i)+V_{3,3,3,1}(n,k,i)\right). \tag{31}$$

$$B^{9,2}_{n,k}(e_k,\varphi(e_k),i) = \frac{n-k}{k-1}V_{2,1,2,1}(n,k,i). \tag{32}$$

$$B^{10,2}_{n,k}(e_k,\varphi(e_k),i) = \frac{n-k}{k-1}V_{3,3,3,1}(n,k,i). \tag{33}$$

$$B^{11,2}_{n,k}(e_k,\varphi(e_k),i) = \frac{(n-k)(n-k-1)}{k-1}V_{2,3,4,2}(n,k,i). \tag{34}$$

The only thing that remains is to evaluate $V_{a,b,c,d}(n,k,i)$, as defined in Eq. 11, which we will do in the next section.

6 Edge-centered surface areas for $S_{n,k}$

We are now ready to derive $V_{a,b,c,d}(n,k,i)$ and present our main results for this paper, i.e., the edge-centered surface areas for the (n,k)-star graphs in terms of both kinds of reference edges.

6.1 Derivation of $V_{a,b,c,d}(n,k,i)$

The following results are well known [10]:

$$\begin{aligned}\frac{1}{(1-z)^c} &= \sum_{q\geq 0}\binom{c+q-1}{c-1}z^q, \text{ where } c\geq 1; \text{ and}\\ \frac{1}{(1-z)^w} &= \sum_{p\geq r\geq 0}\left[\begin{matrix}p\\ r\end{matrix}\right]w^r\frac{z^p}{p!},\end{aligned}$$

where, $\left[\begin{matrix}p\\ r\end{matrix}\right]$ stands for the Stirling numbers of the first kind [10, §6.1], i.e., the number of ways of arranging p objects in r cycles.

If we substitute z with xy, and w with y in the last two equations, we have

$$\begin{aligned}\frac{1}{(1-xy)^c} &= \sum_{q\geq 0}\binom{c+q-1}{c-1}x^qy^q;\ c\geq 1, \text{ and},\\ \frac{1}{(1-xy)^y} &= \sum_{p\geq r\geq 0}\left[\begin{matrix}p\\ r\end{matrix}\right]\frac{x^py^{p+r}}{p!}.\end{aligned}$$

It is clear, by definition, that

$$\begin{aligned}e^{x-xy^2} &= \sum_{s\geq 0}\frac{x(1-y)^2)^s}{s!} = \sum_{s\geq 0}\frac{x^s}{s!}\sum_{t\geq 0}(-1)^t\binom{s}{t}y^{2t}\\ &= \sum_{s\geq 0}\sum_{t\geq 0}\frac{(-1)^t}{s!}\binom{s}{t}x^sy^{2t}.\end{aligned}$$

It is also not difficult to obtain the following result based on the well-known binomial theorem: for $d\in[0,n-k]$,

$$\left[1+\frac{xy^2}{1-xy}\right]^{n-k-d} = 1+\sum_{\alpha=1}^{n-k-d}\sum_{\beta\geq 0}\binom{n-k-d}{\alpha}\binom{\alpha+\beta-1}{\alpha-1}x^{\alpha+\beta}y^{2\alpha+\beta}.$$

We notice that, when $d=n-k$, the above equals 1.

Therefore, for $n\geq 3, k\in[1,n), i\geq 1, a\in[1,k), b\in[0,i], c\geq 1$, and $d\in[0,n-k)$,

$$[x^{k-a}y^{i-b}]\,\frac{e^{x-xy^2}}{(1-xy)^y(1-xy)^c}\left[1+\frac{xy^2}{1-xy}\right]^{n-k-d}$$

$$
\begin{aligned}
&= [x^{k-a}y^{i-b}] \sum_{p\geq r\geq 0} \left[\begin{matrix} p \\ r \end{matrix} \right] \frac{x^p y^{p+r}}{p!} \frac{e^{x-xy^2}}{(1-xy)^c} \left[1 + \frac{xy^2}{1-xy} \right]^{n-k-d} \\
&= \sum_{p\geq r\geq 0} \frac{1}{p!} \left[\begin{matrix} p \\ r \end{matrix} \right] [x^{k-a-p}y^{i-b-p-r}] \sum_{q\geq 0} \binom{c+q-1}{c-1} x^q y^q \\
&\qquad \left[1 + \frac{xy^2}{1-xy} \right]^{n-k-d} e^{x-xy^2} \\
&= \sum_{p\geq r\geq 0} \sum_{q\geq 0} \frac{1}{p!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{c+q-1}{c-1} [x^{k-a-p-q}y^{i-b-p-r-q}] \\
&\qquad \left[1 + \sum_{\alpha=1}^{n-k-d} \sum_{\beta\geq 0} \binom{n-k-d}{\alpha} \binom{\alpha+\beta-1}{\alpha-1} x^{\alpha+\beta} y^{2\alpha+\beta} \right] e^{x-xy^2} \\
&= \sum_{p\geq r\geq 0} \sum_{q\geq 0} \frac{1}{p!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{c+q-1}{c-1} [x^{k-a-p-q}y^{i-b-p-r-q}] \\
&\qquad \sum_{s\geq 0} \sum_{t\geq 0} \frac{(-1)^t}{s!} \binom{s}{t} x^s y^{2t} \\
&\quad + \sum_{p\geq r\geq 0} \sum_{q\geq 0} \frac{1}{p!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{c+q-1}{c-1} [x^{k-a-p-q}y^{i-b-p-r-q}] \\
&\qquad \left[\sum_{\alpha=1}^{n-k-d} \sum_{\beta\geq 0} \binom{n-k-d}{\alpha} \binom{\alpha+\beta-1}{\alpha-1} x^{\alpha+\beta} y^{2\alpha+\beta} \right] e^{x-xy^2} \\
&= V_1 + V_2.
\end{aligned}
$$

The first term is pretty easy to resolve.

$$
\begin{aligned}
V_1 &= \sum_{p\geq r\geq 0} \sum_{q\geq 0} \frac{1}{p!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{c+q-1}{c-1} [x^{k-a-p-q}y^{i-b-p-r-q}] \\
&\qquad \sum_{s\geq 0} \sum_{t\geq 0} \frac{(-1)^t}{s!} \binom{s}{t} x^s y^{2t} \\
&= \sum_{p\geq r\geq 0} \sum_{q\geq 0} \frac{(-1)^{\frac{i-b-p-r-q}{2}}}{p!(k-a-p-q)!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{c+q-1}{c-1} \binom{k-a-p-q}{\frac{i-b-p-r-q}{2}}. \qquad (35)
\end{aligned}
$$

From $k-a-p-q \geq 0$ and $q \geq 0$, we have $p \in [0, k-a]$. We already know that $r \in [0, p]$. To get the upper bound of q, we notice $k-a-p-q \geq (i-b-p-r-q)/2$, leading to $q \in [0, 2k-i-2a+b-p+r]$. Finally, we notice that $(i-b-p-r-q)$ has to be positive and even for the associated binomial coefficient not equal to zero.

By going through the same process, we derive the second term as follows:

$$V_2 = \sum_{p\geq r\geq 0} \sum_{\alpha=1}^{n-k-d} \sum_{\beta\geq 0} \sum_{q\geq 0} \frac{(-1)^{i-b-p-r-q-2\alpha-\beta}}{p!(k-a-p-q-\alpha-\beta)!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{c+q-1}{c-1} \binom{n-k-d}{\alpha} \binom{\alpha+\beta-1}{\alpha-1} \binom{k-a-p-q-\alpha-\beta}{\frac{i-b-p-r-q-2\alpha-\beta}{2}}. \quad (36)$$

Since $k-a-p-q-\alpha-\beta \geq 0$, $\alpha \geq 1, p \geq 0$, and $q \geq 0$, we have $p \in [0, k-a-1]$. Bounds for both r and α are known. From $i-b-p-r-q-2\alpha-\beta \geq 0$ and $q \geq 0$, we obtain $\beta \in [0, i-b-p-r-2\alpha]$. Finally, from $k-a-p-q-\alpha-\beta \geq (i-b-p-r-q-2\alpha-\beta)/2$, we obtain the bound for q as follows: $q \in [2k-i-2a+b-p+r-\beta]$. Again, since we only need to sum up non-zero terms, $(i-b-p-r-q-2\alpha-\beta)$ has to be positive and even.

We notice that, when $n-k=d$, $\left(1+\frac{xy^2}{1-xy}\right)^{n-k-d} = 1$. As a result, for $n \geq 3, k \in [1,n), i \geq 1, a \in [1,k), b \in [0,i], c \geq 1$, by Eqs. 35 and 36,

$$V_{a,b,c,d}(n,k,i) = (k-1)!\,(V_1 + [n-k \neq d]V_2), \quad (37)$$

where [**P**] refers to the Iverson's convention [10, §2.1] where [**P**] = 1, if the predicate **P** is true; 0 otherwise.

Regarding the computational complexity, we notice that $r \leq p \leq k-a \leq k$, and $q \leq 2k+b+r \leq 3k+D(A_{n,k})$, since $b \leq i \leq D(A_{nk}) = O(n)$. We notice that the Stirling numbers can be represented as an explicit formula itself [11, Eqs. 5 and 6] in terms of factorials. As a result, V_1 can be calculated in $O(k^2 n)$, if we take factorial as a standard operation as often done in practice [10]. Similarly, V_2 can be calculated in $O(k^2 n^3)$. Therefore, it is computationally feasible to calculate $V_{a,b,c,d}(n,k,i)$.

Finally, we notice that, for $n \geq 3, k \in [1,n), i \geq 1, a \in [1,k), b \in [0,i]$, and $d \in [0, n-k]$,

$$V_{a,b,0,d}(n,k,i) = (k-1)! \sum_{p\geq r\geq 0} \frac{(-1)^{i-b-p-r}}{p!(k-a-p)!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{k-a-p}{\frac{i-b-p-r}{2}}$$
$$+ [n-k \neq d]\,(k-1)! \sum_{p\geq r\geq 0} \sum_{\alpha=1}^{n-k-d} \sum_{\beta\geq 0} \frac{(-1)^{i-b-p-r-2\alpha-\beta}}{p!(k-a-p-\alpha-\beta)!} \left[\begin{matrix} p \\ r \end{matrix} \right] \binom{n-k-d}{\alpha} \binom{\alpha+\beta-1}{\alpha-1} \binom{k-a-p-\alpha-\beta}{\frac{i-b-p-r-2\alpha-\beta}{2}}, \quad (38)$$

where the bounds of p, r, and β, are the same as before.

6.2 Edge-centered surface area results

We can readily derive the edge-centered surface area of $S_{n,k}, n \geq 3, k \in [1, n-1]$, in terms of the first kind of reference edges via Eq. 10, Eqs. 12 through 19, as well as the just derived Eqs. 37 and 38. Table 4 shows $B^{e_1}_{S_{8,k}}(12\cdots k, (k+1)2\cdots k, i), k \in [1,7], i \in [0, D(S_{8,k})]$.

Table 4: Sample data for $B^{e_1}_{S_{8,k}}(12\cdots k, (k+1)2\cdots k, i), k \in [1,7]$

	i										
k	0	1	2	3	4	5	6	7	8	9	10
1	2	6	0	0	0	0	0	0	0	0	0
2	2	7	17	30	0	0	0	0	0	0	0
3	2	8	32	90	156	48	0	0	0	0	0
4	2	9	45	168	451	687	306	12	0	0	0
5	2	10	56	252	844	1,964	2,578	990	24	0	0
6	2	11	65	330	1,270	3,610	6,808	6,699	1,335	30	0
7	2	12	72	390	1,640	5,220	11,538	14,628	6,188	630	0

We note that in Table 4, the column corresponding to $i = 1$, i.e., the number of vertices of distance 1 from a reference edge of first kind, contains a sequence of consecutive numbers. In general, it is clear that the number of vertices are of distance 1 from an edge $e(v, w)$ is given as follows:

$$B^e_G(v, w, 1) = d(v) + d(w) - 2 - T_G(v, w), \tag{39}$$

where $d_G(v)$ and $T_G(v, w)$ refer to the degree of v, and the number of vertices adjacent to both v and w, in G.

Let $v = e_k = 12\cdots k, w = \phi(v) = (k+1)2\cdots k$. Let $u = u_1u_2\cdots u_k$ be a vertex in $S_{n,k}$ and (u, v) is an edge in $S_{n,k}$. Assume (u, v) is an i-edge, then, $u = i2\cdots 1\cdots k, i \in [2, k]$. Obviously, u is not adjacent to w. Thus, (u, v) has to be a 1-edge. Indeed, let $e \in (k, n]$, then $u = e2\cdots k$ is adjacent to both v and w. In other words, when (v, w) is a reference edge of the first kind, $T_{S_{n,k}}(v, w) = n - k - 1$. Since $S_{n,k}$ is an $(n-1)$-regular graph, by Eq. 39,

$$B^{e_1}_{S_{n,k}}(e_k, \phi(e_k), 1) = 2(n-1) - 2 - (n-k-1) = n + k - 3.$$

In particular, for all $k \in [1, 7]$, $B^{e_1}_{S_{8,k}}(e_8, \phi(e_8), 1) = k + 5$, consistent with the above observation.

We can also derive the edge-centered surface area of $S_{n,k}, k \in [1, n-1]$, in terms of the second kind of reference edges via Eq. 23, Eqs. 31 through 34, as well as the just derived Eqs. 37 and 38. Table 5 shows $B^{e2}_{S_{8,k}}(12\cdots k, 21\cdots k, i), k \in [2,7], i \in [0, D(S_{8,k})]$:

Table 5: Sample data for $B^{e^2_2}_{S_{8,k}}(12\cdots k, 21\cdots k, i), k \in [2,7]$

	i										
k	0	1	2	3	4	5	6	7	8	9	0
2	2	12	12	30	0	0	0	0	0	0	0
3	2	12	32	110	120	60	0	0	0	0	0
4	2	12	48	198	472	624	300	24	0	0	0
5	2	12	60	282	928	2,034	2,460	924	18	0	0
6	2	12	68	350	1,360	3,852	7,114	6,532	864	6	0
7	2	12	72	390	1,640	5,220	11,538	1,4628	6,188	630	0

We also notice that the column corresponding to $i = 1$ in Table 5 contains a constant of 12. Indeed, let $v = 12\cdots k, w = \varphi(e_k) = l2\cdots 1\cdots k, l \in [2,k]$. Let $u = u_1u_2\cdots u_k$ be a vertex in $S_{n,k}$. Assume that (u,v) is an i-edge, then, $u = i2\cdots 1\cdots k, i \in [2,k]$. If $l = i$, $u = w$; otherwise, u is not adjacent to w. Thus, (u,v) has to be a 1-edge, i.e., $u = e2\cdots k, e \in (k,n]$, which is not adjacent to w, either. Thus, $T_{S_{n,k}}(v,w) = 0$. In other words,

$$B^{e_2}_{S_{n,k}}(e_k, \varphi(e_k), 1) = 2(n-1) - 2 = 2(n-2).$$

In particular, for all $k \in [2,7]$, $B^{e_2}_{S_{8,k}}(, e_8, \varphi(e_8), 1) = 12$.

We notice that the only case when $B^{e_1}_{S_{n,k}}(e_k, \phi(e_k), 1)$ and $B^{e_2}_{S_{n,k}}(e_k, \varphi(e_k), 1)$ agree is when $k = n-1$, i.e., when the (n,k)-star graph degrades into an n-star graph.

7 Conclusion

We proposed a general notion of edge-centered surface area for the general graphs, and, following a generating function approach, we derived explicit expressions of the edge-centered surface area of the general asymmetric (n,k)-star graphs in terms of two kinds of reference edges.

We showed that, when solving a subgraph centered surface area problem for a network structure based on symmetric groups where a distance formula is known, a general process can be summarized as follows: 1) characterize the cycle structures related to nodes in such a network in terms of their distances to the involved subgraph; 2) identify generating functions for general cycles, as well as those specific cycles as appearing in the above characterization result; 3) for each case of the above characterization result, assemble the applicable generating functions corresponding to the relevant cycle structures; and 4) extract the coefficient of an appropriate term of the related generating function.

We believe that, when solving the subgraph centered surface area problem and some of the other problems, such an inspiring general process can also be applied to other Cayley graphs, particularly those defined on symmetric groups, when a distance formula is available.

Readers may wonder whether it is possible to derive such explicit formulas (with polynomially bounded number of terms) via elementary techniques like those for the usual surface area problems developed in [6, 11]. The answer is yes but it involves additional enumerating arguments. (Here, these arguments are taken care of automatically via generating functions.) Indeed, we have done so although the expressions look somewhat different. Moreover, all such results were verified using shortest path algorithms on a set of graphs.

We end this paper with some rather concrete observations: Since some of the special cases of the (n, k)-star graph are isomorphic to some of the other graphs, the edge-centered surface area results obtained in this paper for the general (n, k)-star graph can be immediately applied to these graphs. For example, the star graph with n dimensions, S_n, is isomorphic to $S_{n,n-1}$ [7, Lemma 4], thus, the general results as we obtained in this paper, when plugging $n-1$ into placeholder of k, immediately leads to an explicit expression for the edge-centered surface area for the star graphs. In particular, the last row in both Table 4 and 5 give the edge-centered surface area for S_8.

It is recently shown [4] that the alternating group network of n dimensions [13], denoted as AN_n, is isomorphic to $S_{n,n-2}$. Hence, the general results as we obtained in this paper, when plugging $n-2$ into the placeholder of k, also lead to explicit expressions of the edge-centered surface area for the alternating group networks in terms of two kinds of reference edges. In particular, the second last row in Table 4 and 5 give the edge-centered surface area for AN_8 in terms of the two kinds of reference edges. Indeed, as shown in Figure 1, there are three vertices, 21, 23, 42 of distance 1 from (12, 32), but four, 31, 32, 41, 42, of distance 1 from (12, 21), consistent with our findings as made in §6.2.

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