Graphs whose complement and cube are isomorphic

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Abstract

We study cube-complementary graphs, that is, graphs whose complement and cube are isomorphic. We prove several necessary conditions for a graph to be cube-complementary, describe ways of building new cube-complementary graphs from existing ones, and construct infinite families of cube-complementary graphs.

Keywords: Graph cube, cubeco graph, Graph complement, Graph isomorphism, Circulant graph, Radius and Diameter of a graph.

AMS Subject Classification: 05C12, 05C25, 05C40.

1 Introduction

Given a graph G and a positive integer d, a new graph G^d, called the d^{th} power of G, is defined as vertex set $V(G^d) = V(G)$ and two distinct vertices

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x and y are adjacent in G^d if the distance between x and y, $d(x,y)$ is at most d. Recall that a graph G is called square complementary if the graph G^2, called the squeco of G, is isomorphic to the complement of G, $\overline{G}$. That is, $G^2 \cong \overline{G}$.

Square-complementary graphs were extensively studied, see [1], [2], [3], [4], and [5]

Motivated by the study of square-complementary graphs, we define and study the cube-complementary graphs,(cubeco for short). These graphs are defined as graphs G for which G^3 is isomorphic to the complement of G, i.e. $G^3 \cong \overline{G}$. Of course, also we will have $G \cong \overline{G^3}$.

After introducing the necessary basic terms and definitions, we provide in Section 2 basic examples of cubeco graphs. In Section 3, we give an upper-bound on n and show that there exist no cubeco circulant graphs of certain jumps for n larger than this upper-bound, this upper-bound improves computations significantly. In Section 4, we describe a method of constructing new cubeco graphs from existing ones, allowing us to construct infinite families of cubeco graphs. Basic properties in terms of connectivity, radius, and diameter are studied In Section 5. We finally end up with some possible open problems.

Unless stated otherwise, all graphs considered in the paper will be finite, simple and undirected. Let G be a graph. A $k-vertex$ of G is a vertex of degree k in G. An $n-vertex$ graph is a graph of order n, that is, a graph on exactly n vertices. We denote by $n(G)$ the number of vertices of G and by $m(G)$ the number of its edges. Given a vertex v in a graph G, we denote by $\deg(v,G)$ its degree, that is, the size of its neighborhood $N_G(v) := \{u \in V(G) : uv \in E(G)\}$. The closed neighborhood of v is the set $N_G(v) := N_G(v) \cup \{v\}$. By $\Delta(G)$ and $\delta(G)$ we denote the maximum and the minimum degree of a vertex in G respectively. For two vertices u, v in a graph G, we denote by $d_G(u,v)$ the distance between u and v, that is, the number of edges on a shortest path connecting u and v; if there is no path connecting the two vertices, then the distance is defined to be infinite.

The eccentricity $ecc_G(u)$ of a vertex u in a graph G is maximum of the numbers $d_G(u,v)$ where $v \in V(G)$. The radius of a graph G, denoted $radius(G)$, is the minimum of the eccentricities of the vertices of G. The diameter of a graph G, denoted $diam(G)$, is the maximum of the eccentricities of the vertices of G, or, equivalently, the maximum distance between any two vertices in G. The girth of a graph G, denoted $girth(G)$, is the length of a shortest cycle in G (or infinity, if G has no cycles).

Given two graphs G and H, an isomorphism between G and H is a bijective mapping $\phi : V(G) \to V(H)$ such that for every two vertices $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. If there exists an isomorphism between graphs G and H, we say that G and H are isomorphic, and denote this relation by $G \cong H$. An automorphism of a graph G is an isomorphism between G and itself. The complement of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$, in which two distinct vertices are adjacent if and only if they are not adjacent in G.

2 Examples

In this section, we provide some examples of cubeco graphs:

Lemma 1 *If the cycle, C_n, is cubeco graph, then $n = 9$.*

Proof:

$$\begin{aligned} K_n &= C_n \cup \overline{C_n} \\ m(K_n) &= m(C_n) + m(\overline{C_n}) \\ &= m(C_n) + m(C_n^3) \\ &= n + 3n \\ \text{So, } n(n-1) &= 8n, \\ \text{Hence, } n &= 9 \end{aligned}$$

□

Theorem 1 *C_9 is cubeco graph.*

Proof:

Since C_9^3 is a graph of degree 6, $\overline{C_9^3}$ is a regular graph of degree $9-6-1 = 2$. So, $\overline{C_9^3}$ is a regular graph of degree 2, i.e. graph $\overline{C_9^3} \cong C_9$.

□

Recall that the graph G is called a circulant graph if it is a Cayley graph over the cyclic graph of order n denoted by $C_n(D)$, where $D \subseteq [\lfloor \frac{n}{2} \rfloor] := \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In fact, the circulant $C_n(D)$ is the graph with vertex set $\{0, 1, \ldots, n-1\}$ and two distinct vertices $i, j \in [0, 1, \ldots, n-1]$ are adjacent

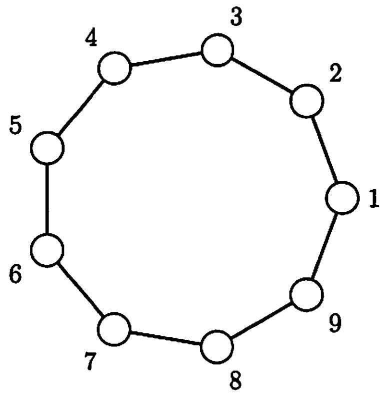

Figure 1: C_9

if $|i - j| \in D$. The cycle C_n is the circulant graph $C_n\{1\}$. By Theorem 1, C_9 is a cubeco graph. Other circulant graphs are also cubeco graphs. In fact, it is known that the two circulant graphs $C_n(D)$ and $C_n(D')$ are isomorphic if there is a unit u in the ring Z_n with $uD = D'$.

The following are examples of circulant cubeco graphs:

1. $C_{18}\{1, 8\}$ is a cubeco graph.
The cube of the graph $C_{18}\{1, 8\}$ is the graph $C_{18}\{1, 2, 3, 6, 7, 8\}$, and so its complement is $C_{18}\{4, 5\}$, since 5 is a unit in Z_{18} and $5.\{1, 8\} = \{5, 4\}$, we have $C_{18}\{1, 8\}$ and $C_{18}\{4, 5\}$ are isomorphic.

2. $C_{27}\{1, 8, 10\}$ is a cubeco graph.
The cube of the graph $C_{27}\{1, 8, 10\}$ is the graph $C_{27}\{1, 2, 3, 6, 7, 8, 9, 10, 11, 12\}$, and so its complement is $C_{27}\{4, 5, 13\}$, since 4 is a unit in Z_{27} and $4.\{1, 8, 10\} = \{4, 5, 13\}$, we have $C_{27}\{1, 8, 10\}$ and $C_{27}\{4, 5, 13\}$ are isomorphic.

3. $C_{29}\{1, 12\}$ is a cubeco graph.
The cube of the graph $C_{29}\{1, 12\}$ is the graph $C_{29}\{1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14\}$, and so its complement is $C_{29}\{8, 9\}$, since 8 is a unit in Z_{29} and $8.\{1, 12\} = \{8, 9\}$, we have $C_{29}\{1, 12\}$ and $C_{29}\{8, 9\}$ are isomorphic.

The same technique can be used to show that the graphs $C_{27}\{1, 5\}$, $C_{36}\{1, 8, 10, 17\}$, $C_{43}\{1, 6, 7\}$, $C_{45}\{4, 5, 13, 14, 22\}$, $C_{61}\{1, 5, 24\}$, and $C_{63}\{1, 5, 25\}$

are non-isomorphic cubeco graphs.

3 An upper-bound

It is obvious from the previous examples that if the cube of the circulant graph $C_n(D)$ is $C_n(D')$, then the elements of D' are generated from additions and subtractions of some elements in D, this gives an upper-bound on n.

Theorem 2 *If $C_n(D)$ is cubeco then $n \leq 2d(d^2 + d + 2)$ where $d = |D|$.*

Proof:

Suppose that $D = \{a_1, a_2, \ldots, a_d\}$, and $C_n^3(D) = C_n(D')$, then elements of D' are in one of the following forms: x, $x + y$, or $x + y + z$ where $x \in D$ and $y, z \in \{\pm a_i | a_i \in D'\}$. In the first form, there are d possibilities. When $x = a_1$ in the second form, there are d possible values of y, and when $x = a_2$, there are d possible values of y. Note that the possible redundant values $a_1 + a_2$ were removed and the values $a_1 - a_2$ were added. Continuing with this computation we get d^2 possible values of this form. The third form is the same as the second form, it contains d^3 elements. Since many elements may be repeated in the forms, one can conclude that $|D'| \leq d^3 + d^2 + d$. Therefore $n \leq 2(d^3 + d^2 + 2d)$.

□

It should be mentioned that the computation of searching for cubeco circulant graphs for a given n is now improved significantly. Additionally, the upper-bound of n for cubeco circulant graph in the form $C_n\{1, k\}$ is 32, the largest cubeco circulant graph we found of this form is $C_{29}\{1, 12\}$. The upper-bound of n for cubeco circulant graph of the form $C_n\{1, k_1, k_2\}$ is 84, the largest one we found of this form is $C_{63}\{1, 5, 25\}$. The upper-bound of n for $C_n\{1, k_1, k_2, k_3\}$ is 176 and the largest one we found of this form is $C_{117}\{1, 13, 17, 55\}$.

4 Construction

In this section we discuss ways of building new cubeco graphs from known ones. In particular, this will imply the existence of arbitrarily large cubeco graphs. Here we should indicate that this method was originally defined by Milanič in [2] to construct square complementary graphs.

Given an n-vertex graph G with vertices labeled $v_1, \dots, v_n$ and positive integers $k_1, \dots, k_n$, we denote by $G[k_1, k_2, \dots, k_n]$ the graph obtained from G by replacing each vertex v_i of G with a set U_i of nonadjacent k_i (new) vertices and joining vertices $u_i \in U_i$ and $u_j \in U_j$ with an edge if and only if v_i and v_j are adjacent in G. If $k_1 = \dots = k_n = k$, then we write $G[k]$ instead of $G[k_1, \dots, k_n]$, see [2].

Theorem 3 *The graph $C_9[k_1, k_2, k_2, k_3, k_2, k_2, k_3, k_2, k_2]$ is cubeco graph for any positive integers k_1, k_2, and k_3.*

Proof:

Let $G := C_9[k_1, k_2, k_2, k_3, k_2, k_2, k_3, k_2, k_2]$ and let $U_1, \dots, U_9$ be the corresponding nine sets of vertices partitioning $V(G)$. Define the isomorphism $\psi : V(G) \to V(G)$ between G and $\overline{G^3}$ by $\psi(v_i^l) = v_{\phi(i)}^l$, such that, ϕ is the isomorphism between C_9 and $\overline{C_9^3}$ where:

$$\begin{aligned}
\phi(2) &= 6\\
\phi(6) &= 8\\
\phi(8) &= 9\\
\phi(9) &= 5\\
\phi(5) &= 3\\
\phi(3) &= 2\\
\phi(4) &= 7\\
\phi(7) &= 4\\
\phi(1) &= 1
\end{aligned}$$

$\psi(v_i^l)$ is an isomorphism by construction between G and $\overline{G^3}$, therefore G is cubeco.

□

Theorem 4 *For any $n \geq 9$, there exists a cubeco graph with n vertices.*

Proof:

Using Theorem 3, take $k_1 = n - 8,\ k_2 = k_3 = 1$

□

Theorem 5 *If G is nontrivial cubeco graph, then G contains no isolated points.*

Proof: Assume that G contains an isolated point v. Then v is adjacent to all vertices in $\overline{G^3}$, thus, $\overline{G^3}$ has no isolated points. This contradicts the fact that $G \cong \overline{G^3}$.

□

Theorem 6 *If $G \cong \overline{G^3}$, $G[k]$ defined as above, $u_i^p \in U_i$ and $u_j^q \in U_j$, then for any $i, j \in V(G)$, $d_G(i,j) \leq 3$ if and only if $d_{\overline{G[k]^3}}(u_i^p, u_j^q) \leq 3$.*

Proof:

($\Longrightarrow$) Suppose that $d(i,j) \leq 3$ then we consider the following two cases:

Case1: if $i = j$, then since G contains no isolated vertex, by Theorem 5 there is a vertex $t \in V(G)$ such that i and t are adjacent. Thus, u_i^p and u_j^q are adjacent to each element in $U_t = \{u_t^l\}_{l=1}^k$, and therefore, $d_{\overline{G[k]^3}}(u_i^p, u_j^q) \leq 3$.

Case2: If $i \neq j$, then the shortest path between i and j can be mapped to an equal long path in $G[k]$.

($\Longleftarrow$) Suppose that $d_{\overline{G[k]^3}}(u_i^p, u_j^q) \leq 3$. We want to show that $d_G(i,j) \leq 3$.

if $i = j$ then nothing to do.

if $i \neq j$ then the shortest path from u_i^p to u_j^q along u_t^1 can be mapped to shortest path between i and j.

□

Theorem 7 *For every nontrivial cubeco graph G and positive integer k, the graph $G[k]$ is a cubeco graph.*

Proof: Let G denote a nontrivial cubeco n-vertex graph.

Let $V(G) = \{i\}_{i=1}^n$ and denote $\{U_i\}_{i=1}^n$ to be the sets of k-elements that will replace the corresponding vertex i in G. Assume $U_i = \{u_i^l\}_{l=1}^k$ and let $\phi : V(G) \to V(G)$ be the isomorphism from G to $\overline{G^3}$. We extend ϕ to a function $\psi : V(G[k]) \to V(G[k])$ by $\psi(u_i^l) = u_{\phi(i)}^l$. Since ϕ is a bijection, then ψ is also a bijection.

To show that ψ is an isomorphism from $G[k]$ to $\overline{G[k]^3}$; let u_i^p and u_j^q be two adjacent vertices in $G[k]$. We want to show that $\psi(u_i^p)$ and $\psi(u_j^q)$ are adjacent in $\overline{G[k]^3}$.

i and j are adjacent in G

$\iff \phi(i)$ and $\phi(j)$ are adjacent in $\overline{G^3}$

$\iff d_G(\phi(i), \phi(j)) > 3$, by definition of G^3

$\iff d_{G[k]}\left(u_{\phi(i)}^p, u_{\phi(j)}^q\right) > 3$, by Theorem 6

$\iff u_{\phi(i)}^p$ and $u_{\phi(j)}^q$ are adjacent in $\overline{G[k]^3}$

$\iff \psi(u_i^p)$ and $\psi(u_j^q)$ are adjacent in $\overline{G[k]^3}$

□

It should be mentioned that the $k's$ in $G[k_1, k_2, \ldots, k_n]$ need not be equal. For example $C_9^+ = C_9[2, 1, 1, 1, 1, 1, 1, 1, 1]$ is cubeco

5 Properties of cubeco graphs

In this section we reveal several necessary conditions that every cube-complementary graph must satisfy. We start with some connectivity and distance-related conditions.

Theorem 8 *Every cubeco graph G is connected and its complement $\overline{G}$ is connected.*

Proof:

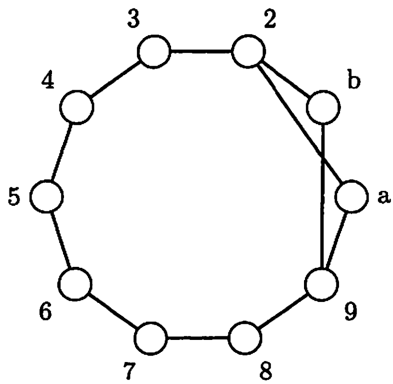

Figure 2: C_9^+

Let G denote a cubeco graph. Suppose that G is disconnected, and let C denote a connected component of G. Then every vertex of $V(C)$ is adjacent to every vertex of $V(G)\backslash V(C)$ in $\overline{G^3}$, and so, in particular, $\overline{G^3}$ is connected, which contradicts the fact that G and $\overline{G^3}$ are isomorphic. On the other hand, if $\overline{G}$ is disconnected, then G has at least two vertices, G^3 is a complete graph and hence G^3 is the edgeless graph, a contradiction with the fact that G is connected.

□

Theorem 9 *Let G be a cubeco graph. For every nonempty proper subset of S of V(G) there exists a* $u \in S$ *and* $v \in V(G)\backslash S$ *such that* $d_G(u, v) \geq 4$.

Proof:

Suppose that for every $u \in S$ and $v \in V(G)\backslash S$ such that $d_G(u, v) \leq 3$ in G, then, every vertex of S is adjacent to every vertex of $V(G)\backslash S$ in G^3, this means that there are no edges between $V(S)$ and $V(G)\backslash S$ in $\overline{G^3}$, hence, $\overline{G^3}$ is disconnected which contradicts Theorem 8.

□

Theorem 10 *If G is a nontrivial cubeco graph, then* $4 \leq radius(G) \leq diam(G) \leq 6$

Proof:

Let G be a nontrivial cubeco graph. For every $v \in V(G)$, applying Theorem 9 to the set $S = \{v\}$ we see that $ecc_G(v) \geq 4$. Hence, G is of radius at least 4. For every graph G it holds that $radius(G) \leq diam(G)$. Hence, it only remains to show that $diam(G) \leq 5$.

Suppose for a contradiction that G is a cubeco graph with diameter at least 6. Let u and v be two vertices such that $d_G(u,v) = 7$. We will verify that the eccentricity of u in $\overline{G^3}$ is at most 2. Let x be any vertex in $V(G)$:
Case1: If $d_G(u,x) \geq 4$ then $d_{\overline{G^3}}(u,x) = 1$.
Case2: If $d_G(u,x) \leq 3$ then $d_G(u,v) = 7 \leq d_G(u,x)+d_G(v,x)$, or $d_G(v,x) \geq 4$ or equivalently, v and x are adjacent in $\overline{G^3}$. Consequently, $d_{\overline{G^3}}(x,u) = 2$. This contradicts the fact that radius of G is at least 4.

□

6 Summary

In this paper we introduced the notion of cube-complementary graphs, we were able to prove several necessary conditions for a graph to be cube complementary, described ways of building new cube-complementary graphs from existing ones, constructed infinite families of cube-complementary graphs and showed some examples.

Results obtained in this paper motivate a further study of cubeco graphs. Since a complete characterization of cubeco graphs seems perhaps too challenging, we pose the following:

Open problem:

- Is there a cubeco graph with diameter equals to 5 or 6?

Conjectures:

- A cubeco graph could have a cut vertex.

- Given the set D, there exists an $n \leq 2d(d^2+d+2)$ such that $C_n(D)$ is cubeco where $|D| = d$.

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