A NOTE CONCERNING KERNELS OF STAIRCASE STARSHAPED SETS IN $\mathbb{R}^d$

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ABSTRACT. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$, and let $S = C_1 \cup \ldots \cup C_n$. Assume that S is staircase starshaped. If the intersection graph of $\mathcal{C}$ is a tree, then the staircase kernel of S, Ker S, will be staircase convex. However, an example in $\mathbb{R}^3$ reveals that, without this requirement on the intersection graph of $\mathcal{C}$, components of Ker S need not be staircase convex. Thus the structure of the kernel in higher dimensional staircase starshaped sets provides a striking contrast to its structure in planar sets.

1. INTRODUCTION

We begin with some definitions from [2]. A set B in $\mathbb{R}^d$ is called a *box* if and only if B is a convex polytope (possible degenerate) whose edges are parallel to the coordinate axes. A nonempty set S in $\mathbb{R}^d$ is an *orthogonal polytope* if and only if S is a connected union of finitely many boxes. An orthogonal polytope in $\mathbb{R}^2$ is an *orthogonal polygon.* Let λ be a simple polygonal path in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes. For x, y in S, the path λ is called an $x - y$ *path* in S if and only if λ lies in S and has endpoints x and y. The $x - y$ path λ is *a staircase path* if and only if, as we travel along λ from x to y, no two edges of λ have opposite directions. That is, for each standard basis vector e_i, $1 \leq i \leq d$, either each edge of λ parallel

Mathematics Subject Classification:Primary 52.A30,52.A35.
Keywords: Orthogonal polytopes, staircase paths, staircase starshaped sets, staircase kernels.

to e_i is a positive multiple of e_i or each edge of λ parallel to e_i is a negative multiple of e_i.

For points x and y in a set S, we say x *sees* y (x is *visible* from y) *via staircase paths* if and only if there is a staircase path in S that contains both x and y. A set S is *staircase convex (orthogonally convex)* if and only if, for every pair x, y in S, x sees y via staircase paths. Similarly, a set S is *staircase starshaped (orthogonally starshaped)* if and only if, for some point p in S, p sees each point of S via staircase paths. The set of all such points p is the *staircase kernel* of S, Ker S.

We also will use a few standard terms from graph theory. For $F = \{C_1, \ldots, C_n\}$ a finite collection of distinct sets, the *intersection graph* G of F has vertex set $\{c_1, \ldots, c_n\}$. Moreover, for $1 \leq i < j \leq n$, the points c_i, c_j determine an edge in G if and only if the corresponding sets C_i, C_j in F have nonempty intersection. A graph G is a *tree* if and only if G is connected and acyclic.

Many results in convexity that involve the usual concept of visibility via straight line segments have interesting analogues that instead use the idea of visibility via staircase paths. For example, the familiar Krasnosel'skii theorem [8] states that, for a nonempty compact set S in the plane, S is starshaped via segments if and only if every three points of S see via segments in S a common point. In the staircase analogue [1], for a nonempty simply connected orthogonal polygon S in $\mathbb{R}^2$, S is staircase starshaped if and only if every two points of S see via staircase paths in S a common point. Moreover, in an interesting study involving rectilinear spaces, Chepoi [4] has generalized the planar result to any finite union S of boxes in $\mathbb{R}^d$ whose corresponding intersection graph is a tree. As he observes, every simply connected orthogonal polygon may be expressed as such a union.

Some other analogues of segment visibility results involve the kernel of a staircase starshaped set. When set S is starshaped via segments, it is easy to show that its kernel is a convex set. In a staircase analogue ([3, Theorem 2]), when the orthogonal polygon S is starshaped via staircase paths, then every component of Ker S is orthogonally convex. Moreover, as [3, Theorem 1] reveals, when S is a simply connected orthogonal polygon and S is starshaped via staircase paths, then Ker S itself will be orthogonally convex.

We might expect similar results to hold in higher dimensions, when S is an orthogonal polytope in $\mathbb{R}^d$. Indeed, Theorem 1 will show that, when S is a finite union of boxes whose intersection graph is a tree and S is staircase starshaped, then Ker S is staircase convex, and we have a direct analogue of the planar result. However, Example 1 will reveal that, without this requirement on the intersection graph, components of Ker S need not be staircase convex, providing a striking contrast to the planar situation.

Readers may refer to Valentine [10], to Lay [9], to Danzer, Grűnbaum, Klee [5], and to Eckhoff [6] for discussions concerning visibility via straight line segments and starshaped sets. Readers may consult Harary [7] for information on intersection graphs, trees, and other graph theoretic concepts.

2. THE RESULTS.

Theorem 1. *Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^d$ whose intersection graph is a tree, and let $S = C_1 \cup \ldots \cup C_n$. If S is staircase starshaped, then Ker S is staircase convex.*

Proof. Let x, y belong to Ker S and let λ be any $x - y$ staircase in S. We will show that $\lambda \subseteq$ Ker S. Select any point z in S, and let $T = \{x, y, z\}$. Following an approach from [2, Theorem 1], let $U(a, b)$ denote the union of all staircase $a - b$ paths in S, and let $U = \cup\{U(a, b) : a, b \text{ in } T\}$. For each $C_i, 1 \leq i \leq n$, select a smallest box B_i (possibly empty) such that $B_i \subseteq C_i$ and $B_i \cap U = C_i \cap U$. That is, B_i is the smallest sub-box of C_i containing $C_i \cap U$. Remove any empty B_i sets and assume that $B_1, \ldots, B_m$ are the remaining B_i sets. By the argument in [2, Theorem 1], $B_1 \cup \ldots \cup B_m$ is a staircase convex union of boxes.

Certainly z and λ lie in $B_1 \cup \ldots \cup B_m$, so z sees each point of λ via staircase paths in S. Since this is true for every z in S, it follows that $\lambda \subseteq \text{Ker } S$. Hence Ker S is staircase convex, finishing the proof. □

As in the planar case, without the requirement that the intersection graph of the boxes be a tree, Ker S need not be staircase convex. (See [3, Example 1].) However, for $d = 3$, not even the components of Ker S need be convex, providing a striking departure from the planar results.

Consider the following example.

Example 1. Let S be a box, a union of five planar surfaces, without a lid. (See Figure 1.) The set S is staircase starshaped, and its kernel consists of the bottom of the box, together with four additional edges. The kernel resembles a four legged table. (See Figure 2.) While Ker S is connected, it is not staircase convex. Points on distinct legs (and not on the table top) cannot see each other via staircase paths in Ker S. The box may be adapted easily to a box with full 3-dimensional sides.

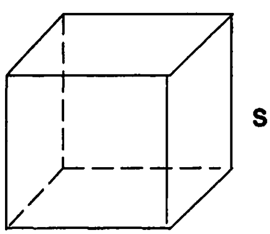

Figure 1

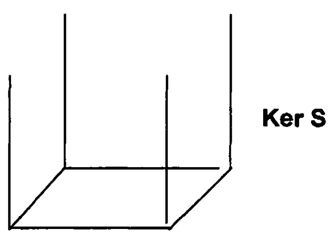

Figure 2

REFERENCES

[1] Marilyn Breen, *An improved Krasnosel'skii-type theorem for orthogonal polygons which are starshaped via staircase paths*, Journal of Geometry **51** (1994), 31-35.

[2] ———, *Characterizing certain staircase convex sets in* $\mathbb{R}^d$, Beiträge zur Algebra und Geometrie **51** (2010), 251-261.

[3] ______, *Staircase kernels in orthogonal polygons*, Archiv der Mathematik **59** (1992), 588-594.
[4] Victor Chepoi, *On staircase starshapedness in rectilinear spaces*, Geometriae Dedicata **63** (1996), 321-329.
[5] Ludwig Danzer, Branko Grűnbaum, and Victor Klee, *Helly's theorem and its relatives*, Convexity, Proc. Sympos. Pure Math. **7** (1962), Amer. Math. Soc., Providence, RI, 101-180.
[6] Jűrgen Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of Convex Geometry vol. A, ed. P.M. Gruber and J.M. Wills, North Holland, New York (1993), 389-448.
[7] Frank Harary, *Graph Theory*, Addison-Wesley, Reading (1969).
[8] M.A. Krasnosel'skii, *Sur un critère pour qu'un domaine soit étoilé*, Math. Sb. (61) **19** (1946), 309-310.
[9] Steven R. Lay, *Convex Sets and Their Applications*, John Wiley, New York, (1982).
[10] F.A. Valentine, *Convex Sets*, McGraw-Hill, New York (1964).

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