A Note on the Omitted Vertex Label of Graceful Pendant Graphs

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Abstract

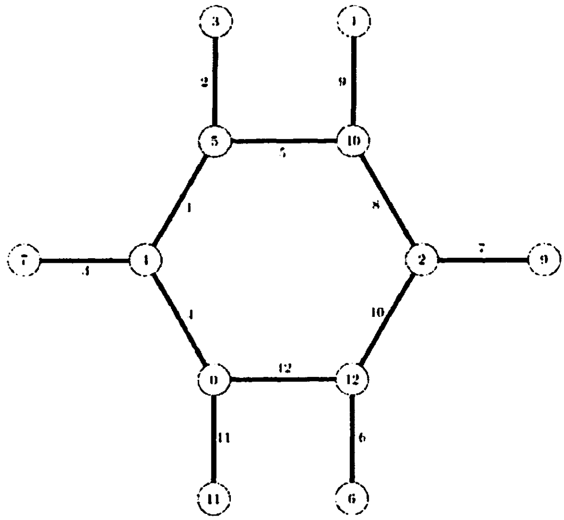
A graceful labeling of a graph G with q edges is an injective assignment from the vertices of G into $\{0, 1, \ldots, q\}$ such that when each edge is assigned the absolute value of the difference of the vertex labels it connects, the resulting edge labels are distinct. In 1978, Frucht conjectured that for gracefully labeled coronas $C_n \odot K_1$ the omitted vertex label is always even. In this paper we will verify Frucht's conjecture.

Let $G = (V(G), E(G))$ be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$ where $e = uv$ if and only if edge e connects vertex u to vertex v. A function f is called a *graceful labeling* of a graph G with q edges if $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.

Frucht [1] investigated a family of graphs, which he described as polygons (= cycles) with pendant points attached, denoted $C_n \odot K_1$, which we will now refer to as pendant graphs. Due to the nature in which the edge labels are induced in a graceful labeling, the largest possible vertex label (q) and the smallest possible vertex label (0) must always be adjacent. Based on this fact, Frucht described a graceful labeling of a pendant graph to be of the:

1. *first kind* if the labels 0 and $q = 2n$ are assigned to adjacent vertices of the n-gon.
2. *second kind* if $q = 2n$ is assigned to a pendant vertex.
3. *third kind* if 0 is assigned to a pendant vertex.

Example 1: Here is a graceful labeling of the first kind for the pendant graph of cycle length 6.



Note that every pendant graph has exactly $2n$ edges and $2n$ vertices. Thus, exactly one vertex label of the $2n+1$ possible vertex labels is omitted in a graceful labeling. Along with proving that all pendant graphs have graceful labelings of the second kind, Frucht [1] noted the vertex label his formulas omitted was even, leading him to conjecture that the omitted label would always be even. In a recent paper by Graf [2], a graceful labeling of the second kind for two congruence classes of pendant graphs also features omitted even vertex labels. That the omitted vertex label should be even is not explicitly stated in the labeling formulas, but can be easily seen. For instance, consider Frucht's result for $n \equiv 0 \pmod 4$ from [1]:

$$f(v_i) = \begin{cases} i-1 & \text{if } \ i = 1,3,5,\ldots,\frac{n}{2}-1 \\ i & \text{if } \ i = \frac{n}{2}+1, \frac{n}{2}+3, \ldots, n-1 \\ 2n+1-i & \text{if } \ i = 2,4,6,\ldots,n \end{cases} \tag{1}$$

$$f(u_i) = \begin{cases} 2n - f(v_i) & \text{if } \ i = 1,2,3,\ldots,\frac{n}{2} \\ 2n+1-f(v_i) & \text{if } \ i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n \end{cases}$$

This labeling produces an equal number of even and odd labels and, as there is an extra even label available, an even vertex label must be therefore omitted. But does this hold, as Frucht conjectured, for all possible graceful labelings of pendant graphs?

To investigate this phenomenon, we wrote a Mathematica program to find all of the graceful labelings for small pendant graphs. On the next page is a table summarizing our observations.

Size of n	Number of Graceful Labelings
3	6
4	46
5	366
6	3404
7	42318

All of these graceful labelings omitted an even vertex label. Convinced that an even label must be omitted, we subsequently found that this is indeed the case.

For any graceful labeling of a pendant graph with $2n$ edges, the sum of the $2n$ edge labels is

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = 2n^2+n. \tag{2}$$

In graceful labelings, the edge sum of a graph can be thought of as a sum of pairs of vertex differences - that is, as a finite sum of integers. From this, our desired conclusion will easily follow.

Theorem 1

The omitted vertex label of any gracefully labeled pendant graph is always even.

Proof. Let $e_i = \{a_i, b_i\}$ for $1 \leq i \leq 2n$ represent the edge connecting a_i and b_i named such that $a_i > b_i$. Thus every vertex v in G of degree x will appear x times using this labeling format. From (2), $\sum_{i=1}^{2n} a_i - b_i = 2n^2 + n$.

Suppose n is odd. Then the right hand side of the above sum is also an odd integer, and thus the left hand side also must contain an odd number of odd terms. As every vertex of G has odd degree, this implies that an odd number of vertices have odd labels. Since the vertex set of G has exactly n odd labels and only one label can be omitted, all n of the odd labels must be used. Therefore the omitted vertex label must be even.

Similarly, suppose n is even. Then the right hand side of the above sum is also an even integer, and thus the left hand side also must contain an even number of odd terms. As every vertex of G has odd degree, this implies that an even number of vertices have odd labels. Since the vertex set of G has exactly n odd labels and only one label can be omitted, all n of the odd labels must be used. Therefore the omitted vertex label must be even. □

It is important to note that graceful labelings of other families of graphs omit different numbers of vertex labels. Paths and caterpillers, for example, require all possible vertex labels to be used since these graphs have more vertices than edges. Generalized Peterson graphs, on the other hand, omit multiple vertex labels, as they possess many more edges than vertices. For these families of graphs, it is not clear how to modify Theorem 1 to determine how many odd or even vertex labels must be omitted.

But for families of gracefully labeled graphs that, like pendant graphs, feature a single omitted vertex label and only odd-degree vertices, the same result holds. One such family is the set of all hairy cycles with an odd number of branches attached to each cycle vertex.

References

[1] R. Frucht, Graceful numbering of wheels and related graphs, *Ann. New York Acad. Sci.,* **319**(1979) 219-229.

[2] A. Graf, A New Graceful Labeling for Pendant Graphs, *Aequationes Math.,* [to appear].