On edge-3-equitability of $\overline{K}_n$–union of helms.

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Abstract

A k-edge labeling of a graph G is a function f from the edge set $E(G)$ to the set of integers $\{0, \ldots, k-1\}$. Such a labeling induces a labeling f on the vertex set $V(G)$ by defining $f(v) = \sum f(e)$, where the summation is taken over all the edges incident on the vertex v and the value is reduced modulo k. Cahit calls this labeling edge-k-equitable if f assigns the labels $\{0, \ldots, k-1\}$ equitably to the vertices as well as edges.

If $G_1, \ldots, G_T$ is a family of graphs each having a graph H as an induced subgraph, then by H-union G of this family we mean the graph obtained by identifying all the corresponding vertices as well as edges of the copies of H in $G_1, \ldots, G_T$.

In this paper, which is a sequel to the paper entitled 'On edge-3-equitability of $\overline{K}_n$–union of gears', we prove that $\overline{K}_n$-union of copies of helm H_n is edge-3-equitable for all $n \geq 6$.

INTRODUCTION

All the graphs we consider are simple and without loops. For a graph G, by $V(G)$ and $E(G)$, we mean the vertex set and the edge set of the graph G respectively. Let $G_1, G_2, \ldots, G_T$ be a family of graphs. Let $H_i, 1 \leq i \leq T$, be an induced subgraph of $G_i, 1 \leq i \leq T$, such that each H_i is isomorphic to a fixed graph H. If $v \in V(H)$, the vertex corresponding to it in H_i is denoted by v^i. Similarly if $e \in E(H)$, the edge corresponding to it in H_i is denoted by e^i.

Definition 1: By the **H-union** G of the family $G_1, G_2, \ldots, G_T$, we mean the graph obtained by identifying $v^1, v^2, \ldots v^T$ for every $v \in V(H)$ and identifying $e^1, e^2, \ldots, e^T$ for every $e \in E(H)$.

If $H = K_1$, the H-union is called the one point union. Similarly, if $H = K_2$, the H-union is called one edge union.

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By a k-edge labeling of a graph G we mean a map $f : E(G) \to \{0, 1, \ldots, k-1\}$. A k-edge labeling f induces a labeling, also denoted by f, on the vertex set $V(G)$ of G by defining $f(x) := \sum f(e)$, where the summation is taken over all the edges incident on the vertex x and the value is reduced modulo k. For a k-edge labeling f, by $v_f(j)$ (respectively $e_f(j)$), we mean the number of vertices (respectively edges) which are assigned the label j by the labeling f. These are called the vertex numbers, (respectively the edge numbers) of f. By $v_f(0, 1, \ldots k-1)$ and $e_f(0, 1, \ldots, k-1)$ we mean the k-tuples $(v_f(0), v_f(1), \ldots, v_f(k-1))$ and $(e_f(0), e_f(1), \ldots, e_f(k-1))$.

Definition 2: A k-edge labeling f is said to be **edge-k-equitable** if $|v_f(i) - v_f(j)| \leq 1, |e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$.

In this paper we prove that $\overline{K}_n$-union of copies of helm H_n is edge-3-equitable for all $n \geq 6$.

HELMS

Definition 3: The **helm** graph $\mathbf{H_n}$ is defined as follow:

$V(H_n) = \{v_0, v_1, \ldots, v_n, w_1, \ldots, w_n\}$ and
$E(H_n) = \{v_0 v_i \mid 1 \leq i \leq n\} \bigcup \{v_i v_{i+1} \mid 1 \leq i \leq n\} \bigcup \{v_i w_i \mid 1 \leq i \leq n\}$
where $(i+1)$ is taken modulo n.

The vertices $\{w_1, \ldots, w_n\}$ are the **pendant** vertices. The vertices $\{v_0, v_1, \ldots, v_n\}$ are the **non-pendant** vertices. The edge $v_0 v_i$ is denoted by $e_i, 1 \leq i \leq n$ and is called a **spoke**. The edge $v_i v_{i+1}$ is denoted by $c_i, 1 \leq i \leq n$, where by v_{n+1} we mean v_1. These edges are referred to as the **cyclic edges.** The edges $\{p_i = v_i w_i ; 1 \leq i \leq n\}$ are called **pendant edges**. The vertex v_0 is called the **hub**. An helm H_n is said to be of **type** $\mathbf{i, 1 \leq i \leq 3}$, if $n = 3y + i$ for some $y \in \mathbb{N}$.

For an edge-3-labeling f, by $e_f(0,1,2)$ we mean $(e_f(0), e_f(1), e_f(2))$, where $e_f(r)$ is the number of edges with the label $r \in \{0, 1, 2\}$. By $v_f^1(0,1,2)$ we mean the triple $(v_f^1(0), v_f^1(1), v_f^1(2))$ where $v_f^1(r)$ is the number of non-pendant vertices with the label r. Similarly, By $v_f^2(0,1,2)$ we mean the triple $(v_f^2(0), v_f^2(1), v_f^2(2))$ where $v_f^2(r)$ is the number of pendant vertices with the label r.

Definition 4: Let $G_1, G_2, \ldots, G_k$ be k copies of H_n. The vertices of

G_j are denoted by $\{v_{j,0}, v_{j,1}, \ldots, v_{j,n}, w_{j,1}, \ldots, w_{j,n}\}$. A $\overline{K}_n$ -union $H_{n,k}$ of $G_1, G_2, \ldots, G_k$ is k is obtained by identifying the pendant vertices of $G_i, 1 \leq i \leq k$ in a cyclic order. After indentification these vertices are called $w_1, \ldots, w_n$.

If G is $\overline{K}_n$ -union of k copies of H_n, then $|V(G)| = k(n+1) + n$ and $|E(G)| = 3kn$.

LABELINGS OF HELMS

We first give some labelings of $H_n, n \geq 6$. We believe that the results proved subsequently are true for $n = 3, 4, 5,$. However we have to create separate labelings for these values. Since the techniques are same we avoid constructing those labelings here. If f is a labeling of a helm H_n by f^r we mean a labeling obtained by keeping the labels of some w_i and v_i same and reversing the order of all the other vertices while assigning the labels used by f. The vertex w_i is called the **pivotal** vertex.

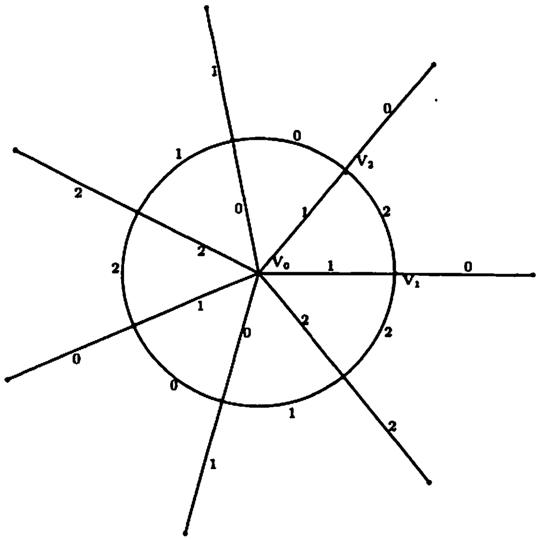
Helms of type 1

Let $n = 3x + 1, n \geq 7$. We first define various labelings.

Labeling $\mathbf{f_1}$: One defines the labeling f_1 as follows:

i	$f_1(e_i)$	$f_1(c_i)$	$f_1(p_i)$
1	1	2	0
$2 \leq i \leq n, i \equiv 1 \mod 3$	2	2	2
$2 \leq i \leq n, i \equiv 2 \mod 3$	1	0	0
$2 \leq i \leq n, i \equiv 0 \mod 3$	0	1	1

It can be checked that $e_{f_1}(0,1,2) = (3x+1, 3x+1, 3x+1), v_{f_1}^1(0,1,2) = (x, x+1, x+1), v_{f_1}^2(0,1,2) = (x+1, x, x)$. This is an edge-3-equitable labeling. The pendant vertices $w_1, w_2, \ldots, w_n$ get the labels $0, 0, 1, 2, 0, 1, 2 \ldots 0, 1, 2$. The non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $2, 0, 2, 1, 0, 2, 1, \ldots, 0, 2, 1$ and $f_1(v_0) = 1$.



The labeling f_1 for H_7

Labeling $\mathbf{g_1}$: One defines the labeling g_1 as follows:

i	$g_1(e_i)$	$g_1(c_i)$	$g_1(p_i)$	$5 \leq i \leq n$	$g_1(e_i)$	$g_1(c_i)$	$g_1(p_i)$
1	1	0	0	$i \equiv 1 \mod 3$	1	1	2
2	2	2	2	$i \equiv 2 \mod 3$	2	2	0
3	0	1	1	$i \equiv 0 \mod 3$	0	0	1
4	2	1	0	–	–	–	–

It can be checked that $e_{g_1}(0,1,2) = (3x+1, 3x+1, 3x+1), v_{g_1}^1(0,1,2) = (x, x+1, x+1), v_{g_1}^2(0,1,2) = (x+1, x, x)$. This labeling is an edge-3-equitable labeling. $g_1(v_0) = 2$ and the pendant vertices $w_1, w_2, \ldots, w_n$ get the labels

$$0, 2, 1, 0, 0, 1, 2, 0, 1, 2, 0, 1, 2 \ldots, 0, 1, 2.$$

The non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels

$$2, 0, 1, 1, 2, 0, 1, 2, 0, 1, \ldots, 2, 0, 1.$$

Labeling $\mathbf{h_1}$: One defines the labeling h_1 as follows:

i	$h_1(e_i)$	$h_1(c_i)$	$h_1(p_i)$	$5 \leq i \leq n$	$h_1(e_i)$	$h_1(c_i)$	$h_1(p_i)$
1	0	2	0	$i \equiv 1 \bmod 3$	1	1	1
2	0	1	0	$i \equiv 2 \bmod 3$	2	0	0
3	1	2	1	$i \equiv 0 \bmod 3$	0	2	2
4	2	1	2	-	-	-	-

The non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels

$$2, 2, 1, 0, 2, 1, 0, 2, 1, 0 \ldots 2, 1, 0.$$

The following table gives the information about these labelings for $H_{n,k}$.

Labeling	Labels of pendant vertices	$v^1(0,1,2)$	Equitable or Not
f_1	$0,0,1,2,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	Equitable
g_1	$0,2,1,0,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	Equitable
h_1	$0,0,1,2,0,2,1,\ldots,0,2,1$	$(x+2, x, x)$	Non-equitable
k_1	$0,0,2,1,0,2,1,\ldots,0,2,1$	$(x, x+1, x+1)$	Equitable

The following proposition gives a clue about how to construct the labelings.

Proposition1: If G is $\overline{K}_n$ union of k helms of type $1, 1 \leq k \leq 6$ then G is edge-3-equitable.
Proof: If G is $\overline{K}_n$ union of k helms of type 1, by $f_{k,1}$ we mean the edge-3-equitable labeling we construct for G. Let $n = 3x + 1$. If $k = 1$, we have an equitable labeling g_1

The following table gives the labelings $f_{k,1}, 2 \leq k \leq 6$ and the vertex numbers. The labels of identified vertices for $f_{k,1}$ are obtained by just adding those of previous rows:

$f_{k,1}$:	Labels of identified vertices	$v^1_{f_{k,1}}(0,1,2)$	$v^2_{f_{k,1}}(0,1,2)$
f_1	$0,0,1,2,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	$(x+1, x, x)$
f_1	$0,0,1,2,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	$(x+1, x, x)$
$f_{2,1}$:	$0,0,2,1,0,2,1\ldots,0,2,1$	$(2x, 2x+2, 2x+2)$	$(x+1, x, x)$
f_1	$0,0,1,2,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	$(x+1, x, x)$
g_1	$0,2,1,0,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	$(x+1, x, x)$
h_1	$0,0,1,2,0,2,1,\ldots,0,2,1$	$(x+2, x, x)$	$(x+1, x, x)$
$f_{3,1}$:	$0,2,0,1,0,1,2\ldots,0,1,2$	$(3x+2, 3x+2, 3x+2)$	$(x+1, x, x)$
$f_{3,1}$:	$0,2,0,1,0,1,2\ldots,0,1,2$	$(3x+2, 3x+2, 3x+2)$	$(x+1, x, x)$
f_1	$0,0,1,2,0,1,2,\ldots,0,1,2$	$(x, x+1, x+1)$	$(x+1, x, x)$
$f_{4,1}$:	$0,2,1,0,0,2,1\ldots,0,2,1$	$(4x+2, 4x+3, 4x+3)$	$(x+1, x, x)$
$f_{4,1}$:	$0,2,1,0,0,2,1\ldots,0,2,1$	$(4x+2, 4x+3, 4x+3)$	$(x+1, x, x)$
k_1	$0,0,2,1,0,2,1,\ldots,0,2,1$	$(x, x+1, x+1)$	$(x+1, x, x)$
$f_{5,1}$:	$0,2,0,1,0,1,2\ldots,0,1,2$	$(5x+2, 5x+4, 5x+4)$	$(x+1, x, x)$

$f_{k,1}$:	Labels of identified vertices	$v^1_{f_{k,1}}(0,1,2)$	$v^2_{f_{k,1}}(0,1,2)$
$f_{3,1}$	$0,2,0,1,0,1,2\ldots,0,1,2$	$(3x+2,3x+2,3x+2)$	$(x+1,x,x)$
$f_{3,1}$	$0,2,0,1,0,1,2\ldots,0,1,2$	$(3x+2,3x+2,3x+2)$	$(x+1,x,x)$
$f_{6,1}$:	$0,1,0,2,0,2,1\ldots,0,2,1$	$(6x+4,6x+4,6x+4)$	$(x+1,x,x)$

The edges are labeled equally by all the labelings. The last two columns, after adding, show that $f_{k,1}$ is edge-3-equitable for all $k, 1 \leq k \leq 6$. •

Theorem 2: If G is $\overline{K}_n$ -union of $3t, t \geq 1$ helms of type 1, then G is edge-3-equitable.

Proof: Let $G_1, G_2, \ldots, G_{3t}$ be the copies of a helm H_n, where $n = 3x+1$. Let G be the $\overline{K}_n$ -union of them. Clearly, $|V(G)| = 3t(3x+2)+3x+1 = 9xt+6t+3x+1, |E(G)| = 3t(9x+3) = 27xt+9t$. Form triple helms $K_1, K_2, \ldots K_t$ and assign the labeling $f_{3,1}$ to K_1 and also to K_s when s is even. Let $f^r_{3,1}$ be the labeling obtained by reversing $f_{3,1}$ with w_1 as the pivotal vertex. Assign this labeling $f^r_{3,1}$ to $K_s, s > 1$ when s is odd. The labels assigned by $f^r_{3,1}$ to the identified pendent vertices are $0,2,1,0,2,1,0,\ldots 2,1,0,1,0,2$ with
$v^1_{f^r_{3,1}}(0,1,2) = (3x+2,3x+2,3x+2), v^2_{f^r_{3,1}}(0,1,2) = (x+1,x,x)$.

The following table gives the sequence of labels of the identified vertices for G when $1 \leq t \leq 7$. The values of edge numbers as well as $v^1_{f_{3t,1}}(0,1,2)$ are not mentioned in this table since they are all equitably labeled.

Labeling	Sequence of labels of the identified vertices
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f_{6,1}$:	$0,1,0,2,0,2,1,\ldots,0,2,1,0,2,1$
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f^r_{3,1}$:	$0,2,1,0,2,1,0,\ldots,2,1,0,1,0,2$
$f_{9,1}$:	$0,0,1,2,2,0,1,\ldots,2,0,1,1,2,0$
$f_{9,1}$:	$0,0,1,2,2,0,1,\ldots,2,0,1,1,2,0$
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f_{12,1}$:	$0,2,1,0,2,1,0,\ldots,2,1,0,1,0,2$
$f_{12,1}$:	$0,2,1,0,2,1,0,\ldots,2,1,0,1,0,2$
$f^r_{3,1}$:	$0,2,1,0,2,1,0,\ldots,2,1,0,1,0,2$
$f_{15,1}$:	$0,1,2,0,1,2,0,\ldots,1,2,0,2,0,1$

Labeling	Sequence of labels of the identified vertices
$f_{15,1}$:	$0,1,2,0,1,2,0,\ldots,1,2,0,2,0,1$
$f_{3,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
$f_{18,1}$:	$0,0,2,1,1,0,2,\ldots,1,0,2,2,1,0$
$f_{18,1}$:	$0,0,2,1,1,0,2,\ldots,1,0,2,2,1,0$
$f_{3,1}^r$:	$0,2,1,0,2,1,0,\ldots,2,1,0,1,0,2$
$f_{21,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$

One can see that after this, all the sequences are going to repeat. Thus, we will get the sequences as given in the following table.

r where $t \equiv r \bmod 6$	Sequence of labels of identified vertices
1	$0,2,0,1,0,1,2,\ldots,0,1,2$
2	$0,1,0,2,0,2,1,\ldots,0,2,1$
3	$0,0,1,2,2,0,1,\ldots 2,0,1,1,2,0$
4	$0,2,1,0,2,1,0,\ldots 2,1,0,1,0,2$
5	$0,1,2,0,1,2,0,\ldots 1,2,0,2,0,1$
6	$0,0,2,1,1,0,2,\ldots 1,0,2,2,1,0$

This shows that $v^1_{f_{3t,1}}(0,1,2) = (3xt+2t, 3xt+2t, 3xt+2t)$ and $v^2_{f_{3t,1}}(0,1,2) = (x+1, x, x)$, that is, $f_{3t,1}$ is an edge-3-equitable labeling for all $t \in \mathbb{N}$. •

Theorem 3: If G is $\overline{K}_n$ -union of $3t+1$ or $3t+2$ helms of type 1, then G is edge-3-equitable.

Proof: Let $k = 3t+1$ or $3t+2$. Let G be $\overline{K}_n$ -union of k helms $G_1, G_2, \ldots, G_k$ which are copies of the helm H_{3x+1} on $6x+3$ vertices. First form $\overline{K}_n$ -union of $G_1, \ldots, G_{3t}$. Assign the labeling $f_{3t,1}$ to this union.

Case 1: $k = 3t+1$. Clearly, $|V(G)| = (3t+1)(3x+2)+3x+1 = 9xt+6t+6x+3$, $|E(G)| = (3t+1)(9x+3)$. For G_{3t+1} assign the labeling f_1 if t is odd and k_1 if t is even. The following table gives the vertex numbers of these labelings. The edge numbers, being all equal, are not mentioned.

r, where $t \equiv r \bmod 6$	Labeling	Sequence
1	$f_{3t,1}$:	$0,2,0,1,0,1,2,\ldots,0,1,2,0,1,2$
	f_1 :	$0,0,1,2,0,1,2,\ldots,0,1,2,0,1,2$
	$f_{3t+1,1}$:	$0,2,1,0,0,2,1,\ldots,0,2,1,0,2,1$
2	$f_{3t,1}$:	$0,1,0,2,0,2,1,\ldots,0,2,1,0,2,1$
	k_1 :	$0,0,2,1,0,2,1,\ldots,0,2,1,0,2,1$
	$f_{3t+1,1}$:	$0,1,2,0,0,1,2,\ldots,0,1,2,0,1,2$

r, where $t \equiv r \bmod 6$	Labeling	Sequence
3	$f_{3t,1}$:	$0,0,1,2,2,0,1,\ldots,2,0,1,1,2,0$
	f_1 :	$0,0,1,2,0,1,2,\ldots,0,1,2,0,1,2$
	$f_{3t+1,1}$:	$0,0,2,1,2,1,0,\ldots,2,1,0,1,0,2$
4	$f_{3t,1}$:	$0,2,1,0,2,1,0,\ldots,2,1,0,1,0,2$
	k_1 :	$0,0,2,1,0,2,1,\ldots,0,2,1,0,2,1$
	$f_{3t+1,1}$:	$0,2,0,1,2,0,1,\ldots,2,0,1,1,2,0$
5	$f_{3t,1}$:	$0,1,2,0,1,2,0,\ldots,1,2,0,2,0,1$
	f_1 :	$0,0,1,2,0,1,2,\ldots,0,1,2,0,1,2$
	$f_{3t+1,1}$:	$0,1,0,2,1,0,2,\ldots,1,0,2,2,1,0$
6	$f_{3t,1}$:	$0,0,2,1,1,0,2,\ldots,1,0,2,2,1,0$
	k_1 :	$0,0,2,1,0,2,1,\ldots,0,2,1,0,2,1$
	$f_{3t+1,1}$:	$0,0,1,2,1,2,0,\ldots,1,2,0,2,0,1$

This table clearly shows that
$v^1_{f_{3t+1,1}}(0,1,2) = (3xt+2t+x, 3xt+2t+x+1, 3xt+2t+x+1)$,
$v^2_{f_{3t+1,1}}(0,1,2) = (x+1,x,x)$, that is, $f_{3t+1,1}$ is an edge-3-equitable labeling.

Case 2: $k = 3t+2$. Clearly, $|V(G)| = (3t+2)(3x+2)+3x+1 = 9xt+6t+9x+5, |E(G)| = (3t+2)(9x+3)$. First we note that for $\overline{K}_n$ -union of two helms, if we assign the labeling f_1 to one and the labeling k_1 to the other, then the resulting labeling θ has the following properties:

Labeling	Sequence of labels of identified vertices	$v^1(0,1,2)$
f_1 :	$0,0,1,2,0,1,2,\ldots,0,1,2$	$(x,x+1,x+1)$
k_1 :	$0,0,2,1,0,2,1,\ldots,0,2,1$	$(x,x+1,x+1)$
θ :	$0,0,0,0,0,0,0,\ldots,0,0,0$	$(2x,2x+2,2x+2)$

Let G be $\overline{K}_n$ -union of $G_1, G_2, \ldots, G_{3t+2}$ where each of them is a copy of the helm H_{3x+1}. First form $\overline{K}_n$ -union of $G_1, G_2, \ldots, G_{3t}$ and assign the labeling $f_{3t,1}$ to it. For the remaining two helms assign the labeling θ. Call the resulting labeling $f_{3t+2,1}$. This simply means that the sequences of the identified vertices in $f_{3t,1}$ and $f_{3t+2,1}$ are same. One can check that
$v^1_{f_{3t+2,1}}(0,1,2) = (3x+2t+2x, 3x+2t+2x+2, 3x+2t+2x+2)$ and
$v^2_{f_{3t+2,1}}(0,1,2) = (x+1,x,x)$, that is $f_{3t+2,1}$ is an edge-3-equitable labeling.
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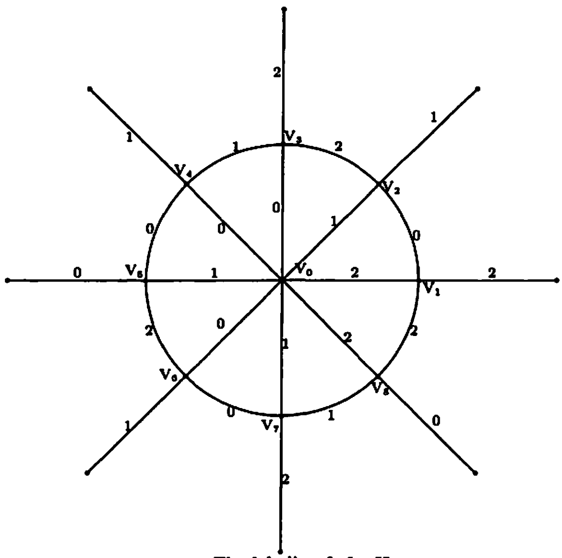
Helms of Type 2:

Let $n = 3x + 2, n \geq 8$.

Labeling $\mathbf{f_2}$: One defines the labeling f_2 as follows:



The labeling f_2 for H_8

i	$f_2(e_i)$	$f_2(c_i)$	$f_2(p_i)$	$6 \leq i \leq n$	$f_2(e_i)$	$f_2(c_i$	$f_2(p_i)$
1	2	0	2	$i \equiv 1 \bmod 3$	1	1	2
2	1	2	1	$i \equiv 2 \bmod 3$	2	2	0
3	0	1	2	$i \equiv 0 \bmod 3$	0	0	1
4	0	0	1				
5	1	2	0				

It can be checked that $e_{f_2}(0,1,2) = (3x+2, 3x+2, 3x+2), v_{f_2}^1(0,1,2) = (x+1, x+1, x+1), v_{f_2}^2(0,1,2) = (x, x+1, x+1)$. This is edge-3-equitable labeling. $f_2(v_0) = 1$. The pendant vertices $w_1, w_2, \ldots, w_n$ get the labels $2,1,2,1,0,1,2,0 \ldots 1,2,0$. The non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $0,1,2,2,0,0,1,2,\ldots,0,1,2$.

Labeling $\mathbf{g_2}$: One defines the labeling g_2 as follows:

i	$g_2(e_i)$	$g_2(c_i)$	$g_2(p_i)$	$2 \leq i \leq n-4$	$g_2(e_i)$	$g_2(c_i)$	$g_2(p_i)$
1	2	2	2	$i \equiv 1 \bmod 3$	0	2	1
$n-3$	1	0	0	$i \equiv 2 \bmod 3$	2	1	0
$n-2$	0	1	1	$i \equiv 0 \bmod 3$	1	0	2
$n-1$	0	2	2				
n	1	0	1				

In fact, g_2 is obtained from f_2 by reversing the order with w_1 as the pivotal. Hence, $e_{g_2}(0,1,2) = (3x+2, 3x+2, 3x+2)$ and $v^1_{g_2}(0,1,2) = (x+1, x+1, x+1), v^2_{g_2}(0,1,2) = (x, x+1, x+1)$. This is an edge-3-equitable labeling. $g_2(v_0) = 1$. The pendant tip vertices $w_1, w_2, \ldots, w_n$ get the labels $2,0,2,1,0,2,1,0\ldots,1,2,1$. The non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $0,2,1,0,2,1\ldots0,2,1,0,0,2,2,1$.

Labeling $\mathbf{h_2}$: One defines the labeling h_2 as follows:

i	$h_2(e_i)$	$h_2(c_i)$	$h_2(p_i)$	$6 \leq i < n-1$	$h_2(e_i)$	$h_2(c_i)$	$h_2(p_i)$
1	0	1	1	$i \equiv$ 1mod 3	0	2	1
2	0	2	2	$i \equiv$ 2mod 3	2	1	0
3	1	0	1	$i \equiv$ 0mod 3	1	0	2
4	2	2	2				
5	2	1	0				
n	1	0	0				

It can be checked that $e_{h_2}(0,1,2) = (3x+2, 3x+2, 3x+2)$ and $v^1_{h_2}(0,1,2) = (x+1, x+1, x+1), v^2_{h_2}(0,1,2) = (x, x+1, x+1)$. This is an 3-equitable labeling. $h_2(v_0) = 1$ and the pendant vertices $w_1, w_2, \ldots, w_n$ get the labels $1,2,1,2,0,2,1,0,2,1,0\ldots,2,1,0$ respectively. The non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $2,2,1,0,2,1,0,2,\ldots1,0,2,1,0,0$ respectively. The labeling h_2 is obtained by triple left shift of g_2.

The following table shows the details of these labelings which are all edge-3-equitable.

Labeling	Labels of pendant vertices	$v^1(0,1,2)$	$v^2(0,1,2)$
f_2 :	$2,1,2,1,0,1,2,0,\ldots,1,2,0.$	$(x+1,x+1,x+1)$	$(x,x+1,x+1)$
g_2 :	$2,0,2,1,0,2,1,0,\ldots,1,2,1.$	$(x+1,x+1,x+1)$	$(x,x+1,x+1)$
h_2 :	$1,2,1,2,0,2,1,0,\ldots,2,1,0.$	$(x+1,x+1,x+1)$	$(x,x+1,x+1)$

Theorem 4: If G is $\overline{K_n}$ - Union of t helms of type 2, then G is edge-3-equitable.
Proof: Let $G_1,\ldots G_t$ be t copies of helm H_n where $n = 3x+2$. Let G be the combined helm of $G_1,\ldots G_t$. Clearly $|E(G)| = 9xt+6t$ and
$|V(G)| = 3xt+3x+3t+2$.

Assign the labeling f_2 to G_1 and also to G_s when s is even. Assign the labeling h_2 to $G_s, s > 1$ when s is odd. Call the resulting labeling $f_{t,2}$. One can check that $e_{f_{t,2}}(0,1,2) = (3xt+2t, 3xt+2t, 3xt+2t)$. Moreover, $v^1_{f_{t,2}}(0,1,2) = (xt+t, xt+t, xt+t), v^2_{f_{t,2}}(0,1,2) = (x, x+1, x+1)$.

The sequences of labels of the pendant tips in $G_1, G_2, G_3, \ldots, G_t$ are as follows:

Case 1: t is even.

For $G_1 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0$
For $G_2 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0$
For $G_3 : h_2 : 1,2,1,2,0,2,1,0\ldots2,1,0.$
For $G_4 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0.$
For $G_5 : h_2 : 1,2,1,2,0,2,1,0\ldots2,1,0$
For $G_6 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0.$
⋮
For $G_t : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0.$

It can be checked that after adding the respective labels, the identified pendant tips get the labels $1,2,1,2,0,2,1,0\ldots2,1,0$.
Thus, $v^2_{f_{t,2}}(0,1,2) = (x, x+1, x+1)$.

Case 2: t is odd.

For $G_1 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0.$
For $G_2 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0.$
For $G_3 : h_2 : 1,2,1,2,0,2,1,0\ldots2,1,0.$
For $G_4 : f_2 : 2,1,2,1,0,1,2,0\ldots1,2,0.$

For $G_5 : h_2 : 1, 2, 1, 2, 0, 2, 1, 0 \ldots 2, 1, 0.$

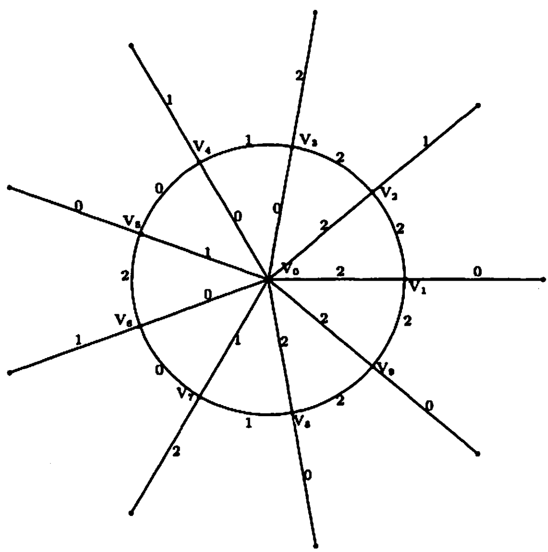
$\vdots$

For $G_t : h_2 : 1, 2, 1, 2, 0, 2, 1, 0 \ldots 2, 1, 0.$

It can be checked that after adding the respective labels, the identified pendant tips get the labels $2, 1, 2, 1, 0, 1, 2, 0 \ldots 1, 2, 0$. Thus, $v^2_{f_{t,2}}(0, 1, 2) = (x, x+1, x+1)$. Hence, in both the cases the labeling $f_{t,2}$ is edge-3-equitable. •

Helms of Type 3

Let $n = 3x, n \geq 6$.



The labeling f_3 for H_9

Labeling f₃ : One defines the labeling f_3 as follows:

i	$f_3(e_i)$	$f_3(c_i)$	$f_3(p_i)$
$i \equiv 1 \bmod 3$	2	2	0
$i \equiv 2 \bmod 3$	0	0	1
$i \equiv 0 \bmod 3$	1	1	2

It can be checked that $e_{f_3}(0, 1, 2) = (3x, 3x, 3x), v^1_{f_3}(0, 1, 2) = (x + 1, x, x)$, $v^2_{f_3}(0, 1, 2) = (x, x, x)$. This is edge-3-equitable labeling. The value $f_3(v_0) = 0$ and the pendant vertices $w_1, w_2, \ldots, w_n$ get the labels $0, 1, 2, 0, 1, 2 \ldots$, respectively and the non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $2, 0, 1, 2, 0, 1 \ldots,$ respectively.

Labeling $\mathbf{g_3}$: One defines the labeling g_3 as follows:

i	$g_3(e_i)$	$g_3(c_i)$	$g_3(p_i)$	$1 \leq i \leq n-3$	$g_3(e_i)$	$g_3(c_i)$	$g_3(p_i)$
$n-2$	0	1	0	$i \equiv 1 \bmod 3$	0	2	0
$n-1$	2	0	1	$i \equiv 2 \bmod 3$	1	0	1
n	2	1	2	$i \equiv 0 \bmod 3$	2	1	2

It can be checked that $e_{g_3}(0,1,2) = (3x, 3x, 3x), v^1_{g_3}(0,1,2) = (x-1, x+1, x+1), v^2_{g_3}(0,1,2) = (x, x, x)$. This is not edge-3-equitable labeling though it labels the vertices equitably. $g_3(v_0) = 1$ and the pendant vertices $w_1, w_2, ..,$ w_n get the labels $0, 1, 2, 0, 1, 2 \ldots$ respectively and the non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $0, 1, 2, 0, 1, 2, \ldots, 2, 1, 2$ respectively.

Labeling $\mathbf{h_3}$ One defines the labeling h_3 by reversinf f_3 with w_1 as the pivotal vertex:

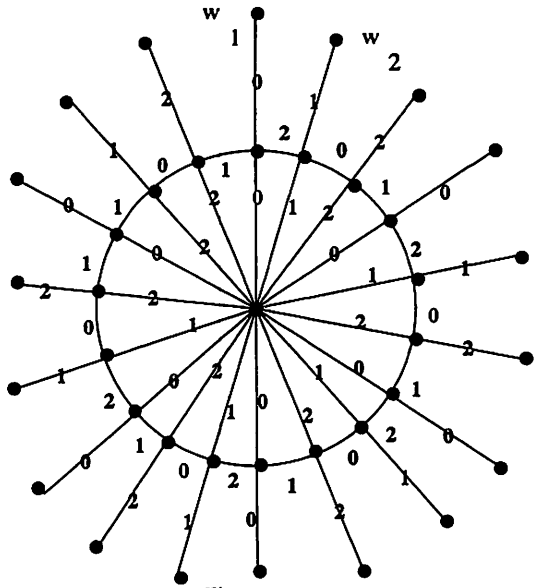
$1 \leq i \leq n$	$h_3(e_i)$	$h_3(c_i)$	$h_3(p_i)$
$i \equiv 1 \bmod 3$	2	1	0
$i \equiv 2 \bmod 3$	1	0	2
$i \equiv 0 \bmod 3$	0	2	1

Clearly, $e_{h_3}(0,1,2) = (3x, 3x, 3x), v^1_{h_3}(0,1,2) = (x+1, x, x), v^2_{h_3}(0,1,2) = (x, x, x)$. This is edge-3-equitable labeling. $h_3(v_0) = 0$ and the pendant vertices $w_1, w_2, \ldots, w_n$ get the labels $0, 2, 1, 0, 2, 1, \ldots$ respectively and the non-pendant vertices $v_1, v_2, \ldots, v_n$ get the labels $2, 1, 0, 2, 1, 0 \ldots$ respectively.

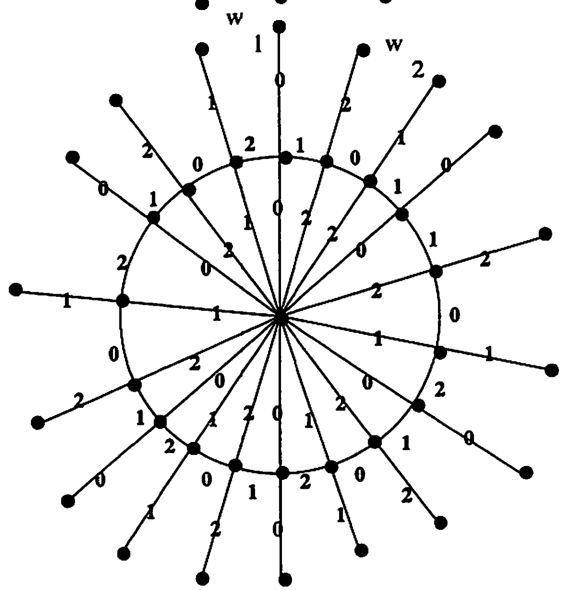
Labeling $\mathbf{k_3}$: One defines the labeling k_3 by reversing g_3. The sequence of labels of pendent vertices is $0, 2, 1, \cdots, o, 2, 1$.

i	$k_3(e_i)$	$k_3(c_i)$	$k_3(p_i)$	$4 \leq i \leq n$	$k_3(e_i)$	$k_3(c_i)$	$k_3(p_i)$
1	0	1	0	$i \equiv 1 \bmod 3$	0	1	0
2	2	0	2	$i \equiv 1 \bmod 3$	2	0	2
3	2	1	1	$i \equiv 1 \bmod 3$	1	2	1

g_3



k_3



The following table gives complete data of these labelings:

	Labels of pendent vertices	$v^1(0,1,2)$	Equitable or not
f_3 :	$0,1,2,\ldots,0,1,2$	$(x+1,x,x)$	Equitable
g_3 :	$0,1,2,\ldots,0,1,2$	$(x-1,x+1,x+1)$	Not equitable
h_3 :	$0,2,1,\ldots,0,2,1$	$(x+1,x,x)$	Equitable
k_3 :	$0,2,1,\ldots,0,2,1$	$(x-1,x+1,x+1)$	Not equitable

Theorem 5: If G is $\overline{K}_n$ -union of k copies of the helm H_{3x}, then G is edge-3-equitable for $1 \le k \le 6$.
Proof: Let G be $\overline{K}_n$ -union of k copies $G_1, G_2, \ldots, G_k$ of the helm H_{3x}. If $k = 1$ we have the labeling f_3 which is equitable. If $k \ge 2$, the following table shows the required labeling and vertex numbers. The values of edge numbers as well as $v^2(0,1,2)$ are all equal and hence are not mentioned.

Labeling	Labels of identified vertices	formula	$v^1(0,1,2)$
$f_{2,3}$:	$0,2,1,\ldots,0,2,1$	$f_3 + g_3$	$(2x, 2x+1, 2x+1)$
$f_{3,3}$:	$0,1,2,\ldots,0,1,2$	$f_3 + g_3 + h_3$	$(3x+1, 3x+1, 3x+1)$
$f_{4,3}$:	$0,2,1,\ldots,0,2,1$	$f_{3,3} + f_3$	$(4x+2, 4x+1, 4x+1)$
$f_{5,3}$:	$0,1,2,\ldots,0,1,2$	$f_{4,3} + k_3$	$(5x+1, 5x+2, 5x+2)$
$f_{6,3}$:	$0,2,1,\ldots,0,2,1$	$f_{3,3} + f_{3,3}$	$(6x+2, 6x+2, 6x+2)$

This shows that $f_{k,3}$ is edge-3-equitable for all $1 \le k \le 6$. •

Theorem 6: If G is $\overline{K}_n$ -union of T copies $G_1, G_2, \ldots, G_T$ of the helm H_{3x} of type 3, then G is edge-3-equitable.
Proof: Case 1: $T = 3t$. Let G is $\overline{K}_n$ -union of $3t$ copies $G_1, G_2, \ldots, G_{3t}$ of the helm H_{3x} of type 3. First form triple helms $K_1, K_2, \ldots, K_t$ using the helms $G_1, G_2, \ldots, G_{3t}$. Assign the labeling $f_{3,3}$ to K_1, K_2, K_s whenever s is even and assign the labeling $f_{3,3}^r$ obtained by reversing $f_{3,3}$ with the pivotal point w_1 to K_s whenever $s > 1$ is odd. The following table gives the formula, sequence of labels of identified vertices and $v^1_{f_{3t,3}}(0,1,2)$.

Labeling	Labels of identified Vertices	$v^1(0,1,2)$
$f_{3,3}$:	$0,1,2,\ldots,0,1,2$	$(3x+1,3x+1,3x+1)$
$f_{3,3}$:	$0,1,2,\ldots,0,1,2$	$(3x+1,3x+1,3x+1)$
$f^r_{3,3}$:	$0,2,1,\ldots,0,2,1$	$(3x+1,3x+1,3x+1)$
$f_{9,3}$:	$0,1,2,\ldots,0,1,2$	$(9x+3,9x+3,9x+3)$
$f_{3t,3}$ t even:	$0,2,1,\ldots,0,2,1$	$(3t+t,3t+t,3t+t)$
$f_{3t,3}$ t odd:	$0,1,2,\ldots,0,1,2$	$(3t+t,3t+t,3t+t)$

This shows that $f_{3t,3}$ is edge-3-equitable.

Case 2: $T = 3t+1$. We first form $\overline{K}_n$-union of $3t$ helms and assign it the labeling $f_{3t,3}$. For the remaining helm we assign the labeling h_3 if t is even and the labeling f_3 if t is odd. The following table gives the label numbers as well as the labels of the identified vertices.

Labeling	Labels of identified Vertices	$v^1(0,1,2)$
$f_{3t,3}$ t even:	$0,2,1,\ldots,0,2,1$	$(3t+t,3t+t,3t+t)$
h_3 :	$0,2,1,\ldots,0,2,1$	$(x+1,x,x)$
$f_{3t+1,3}$ t even :	$0,1,2,\ldots,0,1,2$	$(3xt+t,3xt+t,3xt+t)$ $+(x+1,x,x)$
$f_{3t,3}$ t odd:	$0,1,2,\ldots,0,1,2$	$(3t+t,3t+t,3t+t)$
f_3 :	$0,1,2,\ldots,0,1,2$	$(x+1,x,x)$
$f_{3t+1,3}$ t odd :	$0,2,1,\ldots,0,2,1$	$(3xt+t,3xt+t,3xt+t)$ $+(x+1,x,x)$

Case 3: $T = 3t+2$. Again we assign the labeling $f_{3t+1,3}$ to the $\overline{K}_n$-union of the first $3t+1$ helms. The remaining helm is labeled g_3 if t is even and k_3 if t is odd. The following table shows the relevant vertex numbers where edge numbers and $v^2_{f_{3t+2,3}}(0,1,2)$ are not mentioned.

Labeling	Labels of identified vertices	$v^1(0,1,2)$
$f_{3t+1,3}$ t even :	$0,1,2,\ldots,0,1,2$	$(3xt+t, 3xt+t, 3xt+t)$ $+(x+1, x, x))$
g_3 :	$0,1,2,\ldots,0,1,2$	$(x-1, x+1, x+1)$
$f_{3t+2,3}$ t even :	$0,2,1,\ldots,0,2,1$	$(3xt+2x, 3xt+2x, 3xt+2x)$ $+(t, t+1, t+1)$
$f_{3t+1,3}$ t odd :	$0,2,1,\ldots,0,2,1$	$(3xt+t, 3xt+t, 3xt+t)$ $+(x+1, x, x))$
k_3 :	$0,2,1,\ldots,0,2,1$	$(x-1, x+1, x+1)$
$f_{3t+2,3}$ t odd :	$0,1,2,\ldots,0,1,2$	$(3xt+2x, 3xt+x, 3xt+x)$ $+(t, t+1, t+1)$

Hence $f_{T,3}$ is edge-3-equitable for all $T \in \mathbb{N}$. •

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References

[1] Cahit I. and Yilmaz R., E3-cordial graphs, Ars Combinatoria, 54 (2000), 119-127.

[2] Abhaya M. Chitre and Nirmala B. Limaye, On edge-3-equitability of $\overline{K}_n$-union of gears, JCMCC, 83, November (2012), 129-150.

[3] Gallian J. A., A dynamic survey of graph labelings, Electronic Journal of Combinatorics, DS6, (2008).

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