Independence in Function Graphs

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Abstract

Given two graphs G and H and a function $f \subset V(G) \times V(H)$, Hedetniemi [9] defined the *function graph* GfH by $V(GfH) = V(G) \cup V(H)$ and $E(GfH) = E(G) \cup E(H) \cup \{uv | v = f(u)\}$. Whenever $G \cong H$, the function graph was called a *functigraph* by Chen, Ferrero, Gera and Yi [7]. A function graph is a generalization of the α-*permutation graph* introduced by Chartrand and Harary [5]. The independence number of a graph is the size of a largest set of mutually non-adjacent vertices. In this paper, we study independence number in function graphs. In particular, we give a lower bound in terms of the order and the chromatic number, which improves on some elementary results and has a number of interesting corollaries.

Key Words: independence, function graphs, functigraphs, permutation graphs, generalized prisms, generalized Petersen graphs;

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1 Introduction and Definitions

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$, edge set E(G), and order $n = |V(G)|$. Given a subset $S \subset V(G)$, the subgraph *induced* by S is denoted by $G[S]$. A set $I \subseteq V(G)$ is *independent* if no two vertices in I are joined by an edge. Similarly, a set of edges is independent if no two edges share a common vertex. The *independence number* $\alpha(G)$ equals the cardinality of a largest independent set in G. The *chromatic number*, denoted $\chi(G)$, equals the minimum number of independent sets that the vertex set can be partitioned into. The reader is referred to [6] for any additional terminology and notation.

Function graphs, obtained from two graphs with additional edges representing a function from the vertices of one graph to the other, were introduced by Hedetniemi [9]. The main result of this paper is a lower bound for the independence number of a function graph (Theorem 3.3). This result has a number of nice corollaries, one of which improves on the lower bound implied by a result of Chartrand and Frechen, which is described later in the article. Formally, function graphs are defined as follows:

Definition 1.1. *Given two graphs G and H and a function $f \subseteq V(G) \times V(H)$, the* ***function graph*** *GfH is the graph in which*

$$V(GfH) = V(G) \cup V(H) \text{ and } E(GfH) = E(G) \cup E(H) \cup \{uv | v = f(u)\}.$$

Independently Chen, Ferrero, Gera, and Yi introduced *functigraphs* in [7], which differ only in name and notation. Throughout this paper, whenever H is an isomorphic copy of G, we use G and G' to denote the two copies. Thus, GfG' is the function graph of G with respect to the function f.

Function graphs are extensions of α-permutation graphs, introduced by Chartrand and Harary [5]. A *permutation graph* (or α-permutation graph) $P(G)$ is formed from two copies of a graph G by adding nonadjacent edges joining all the vertices of one copy to those of the other according to a permutation. Thus, permutation graphs are simply function graphs when the function is a bijection. Chartrand and Harary studied their planarity in [5]. Note that a prism, typically described as a Cartesian product of the form $G \square K_2$, is a permutation graph. Also, the generalized Petersen graphs are a subclass of the permutation graphs. Recall that *Generalized Petersen Graph* $P(n,k)$ has $V(P(n,k)) = \{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$, and $E(P(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} | 0 \leq i \leq n-1\}$, where the subscripts are expressed as integers modulo n ($n \geq 5$). The independence numbers of generalized Petersen graphs have been investigated in [2, 8].

Finally, we let C_n denote a cycle of length $n \geq 3$, and *id* denote the identity function, $V(G) = \{v_1, v_2, \dots, v_n\}$, and $V(G') = \{v'_1, v'_2, \dots, v'_n\}$. Whenever A is a subset of $V(G)$, we denote its copy in G' by A'. When $G = C_n$, we assume that the vertices of G and G' are labeled cyclically.

2 Bounds on $\alpha(GfG')$

We start with general bounds on the independence numbers of the function graphs GfG' in terms of the independence number of the graph G. A result related to the upper bound of the theorem below, but for the residue instead of the independence number, is given by Amos, Davila and Pepper in [1]. Since any maximum independent set in G (or G') is an independent set in GfG', and since no independent set in GfG' can be larger than the union of maximum independent sets of G and G' we have the following bounds.

Theorem 2.1. *For any graph G and any function $f : V(G) \to V(G')$,*

$$\alpha(G) \leq \alpha(GfG') \leq 2\alpha(G),$$

and these bounds are sharp. Furthermore,

(i) $\alpha(GfG') = \alpha(G)$ *if and only if G is empty and f is a bijection, and*

(ii) $\alpha(GfG') = 2\alpha(G)$ *if and only if G contains (not necessarily disjoint) maximum independent sets A and B such that $f(A) \cap B' = \emptyset$.*

Proof: The proofs of the stated bounds are immediate. To prove sharpness, it suffices to prove (i) and (ii).

First suppose that G is an empty graph and f is a bijection. Since G is empty, $\alpha(G) = |V(G)|$. Since f is a bijection, GfG' consists of $n(G)$ independent edges. Since at most one vertex from each of these edges is in any independent set in GfG', it follows that $\alpha(GfG') = \alpha(G)$.

Conversely, suppose that $\alpha(GfG') = \alpha(G)$. Let I be a maximum independent set in G. If $I \neq V(G)$, then $V(G') - f(I) \neq \emptyset$. In this case, let $v \in V(G') - f(I)$. Then $I \cup \{v\}$ is an independent set in GfG', and $\alpha(GfG') \geq \alpha(G) + 1$, contradicting our assumption. So $I = V(G)$ and G is empty. If f is not a bijection, then $V(G') - f(I) \neq \emptyset$ and we get the same contradiction as before.

Now suppose that $\alpha(GfG') = 2\alpha(G)$. Let I be a maximum independent set in GfG'. Then let $A = I \cap V(G)$ and $B' = I \cap V(G')$. So $\alpha(G) = |A| = |B'| = |B|$, $I = A \cup B'$, and A and B are maximum independent sets in G. Since I is independent, there are no edges between vertices in A and vertices in B', so $f(A) \cap B' = \emptyset$.

For the converse, suppose that G contains (not necessarily disjoint) maximum independent sets A and B where $f(A) \cap B' = \emptyset$. Let $I = A \cup B'$. Then I is an independent set in GfG', and $\alpha(GfG') \geq |I| = |A| + |B'| = |A| + |B| = 2\alpha(G)$. ■

If the graph G is connected, we get a tighter lower bound than the one from the theorem above.

Corollary 2.2. *Let G be a nontrivial connected graph. Then*

$$\alpha(G) + 1 \le \alpha(GfG') \le 2\alpha(G),$$

and the bounds are sharp.

Proof: The upper bound is a corollary of Theorem 2.1 and so is its sharpness.

Let I be a maximum independent set in G. Since G is non-trivial, $V(G) - I \neq \emptyset$. Let $v \in V(G) - I$, and $J = I \cup \{v\}$. Then $J' = I' \cup \{v'\}$. Since $|J'| = |I'| + 1 = |I| + 1 > |f(I)|$, $J' - f(I) \neq \emptyset$. Let $w \in J' - f(I)$. Then $I \cup \{w\}$ is an independent set in GfG' with at least $\alpha(G) + 1$ vertices.

Proposition 2.3 shows the sharpness of the bounds, as well as the realization of the values in between the bounds. ∎

All values between the lower and upper bound above are realizable. It suffices to show realization results for an arbitrary connected graph G as described in Corollary 2.2.

Proposition 2.3. *For every choice of $n > 1$, there exists a connected graph G of order n and a family $\mathcal{F} = \{f_j | 1 \le j \le \alpha(G)\}$ of functions from V to V' such that $\alpha(G\, f_j\, G') = \alpha(G) + j$ for each choice of j.*

Proof: Let $n > 1$ and let $G = K_{1,n-1}$. It suffices to show that for any $j \in \mathbb{Z}$, where $1 \le j \le n - 1$, there exists a function f_j with the property that $\alpha(G\, f_j\, G') = n - 1 + j$. Let G and G' denote the two copies of G. Let the vertices of G be $v_0, v_1, \ldots v_{n-1}$, and label the corresponding vertices of G' as $v'_0, v'_1, \ldots, v'_{n-1}$. Let v_0 and v'_0 be the central vertices of G and G', respectively. For $1 \le j \le n - 1$, construct f_j by mapping v_i to v'_0 for $0 \le i \le j$ and v_i to v'_i for $j + 1 \le i \le n - 1$; note that f_{n-1} maps all of $V(G)$ to v'_0. For each choice of j, one maximum independent set is the set $\{v_0, v_1, \ldots, v_{n-1}, v'_1, \ldots, v'_j\}$. ∎

This shows that every value in the range given by Corollary 2.2 can be achieved. The illustration of Figure 1 shows the case $n = 5$, $j = 2$, with the dashed edges representing f_2.

3 Lower Bounds on $\alpha(GfG')$

In this section we partition the vertex set of G (with any number of sets), and give improved lower bounds on the independence number of the function graph GfG'. First, we present the result of Chartrand and Frechen[1] [4].

[1]We thank the referee for bringing this result to our attention

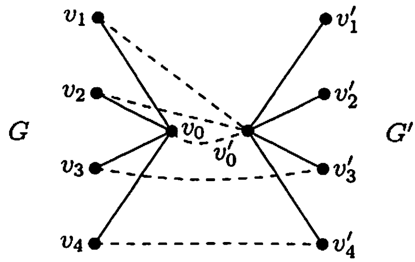

Figure 1: The graph GfG'; the dashed edges represent the function f.

Theorem A: *For any graph G and any permutation graph $P(G)$ of G,*

$$\chi(G) \leq \chi(P(G)) \leq \frac{4\chi(G)}{3}.$$

Since, for any graph G, $\alpha(G) \geq \frac{n(G)}{\chi(G)}$, it follows – for any graph G and any function f – that,

$$\alpha(GfG') \geq \frac{n(GfG')}{\chi(GfG')}. \tag{1}$$

From this we deduce the following corollary to Theorem A.

Corollary 3.1. *For any graph G and any permutation graph $P(G)$ of G,*

$$\alpha(P(G)) \geq \frac{3n(G)}{2\chi(G)}$$

This bound will be improved upon for all non-empty graphs in what follows. Before proceeding, we present a lemma which is needed for the proof of our main theorem.

Lemma 3.2. *Let $A_1, \ldots, A_r$ be pairwise disjoint sets. For any set C,*

$$\sum_{i=1}^{r} |A_i - C| \geq \sum_{i=1}^{r} |A_i| - |C|.$$

Proof: Since the sets $A_1, A_2, \ldots, A_r$ are pairwise disjoint, then the sets $A_1 \cap C, A_2 \cap C, \ldots, A_r \cap C$ are pairwise disjoint subsets of C. It follows that $\sum_{i=1}^{r} |A_i \cap C| \leq |C|$, but then

$$\begin{aligned}\sum_{i=1}^{r} |A_i - C| &= \sum_{i=1}^{r} |A_i - (A_i \cap C)| \\ &= \sum_{i=1}^{r} (|A_i| - |A_i \cap C|) \\ &= \sum_{i=1}^{r} |A_i| - \sum_{i=1}^{r} |A_i \cap C| \\ &\geq \sum_{i=1}^{r} |A_i| - |C|.\end{aligned}$$

Thus the proof is complete. ■

We now present the main theorem of the section, followed by some of its corollaries.

Theorem 3.3. *Let G be a graph, $\{A_1, \ldots, A_r\}$ be a partition of the vertex set $V(G)$, and $f : V(G) \to V(G')$ be any function. Then*

$$\alpha(GfG') \geq \frac{2r-1}{r^2} \sum_{i=1}^{r} \alpha(G[A_i]),$$

and the inequality is sharp.

Proof: Let $\{A_1, \ldots, A_r\}$ be any partition of $V(G)$. Let I_i be a maximum independent set in $G[A_i]$. Note that, for every $i, j \in \{1, \ldots, r\}$, $I_i \cup (I'_j - f(I_i))$ is an independent set in GfG'. So

$$\alpha(GfG') \geq \max_{i,j \in \{1,\ldots,r\}} |I_i \cup (I'_j - f(I_i))|.$$

Therefore,

$$r^2\alpha(GfG') \geq \sum_{i=1}^{r}\sum_{j=1}^{r} |I_i \cup (I_j' - f(I_i))| \tag{2}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} (|I_i| + |I_j' - f(I_i)|) \tag{3}$$

$$= \sum_{i=1}^{r} \left(\sum_{j=1}^{r} |I_i| + \sum_{j=1}^{r} |I_j' - f(I_i)| \right) \tag{4}$$

$$\geq \sum_{i=1}^{r} \left(r|I_i| + \sum_{j=1}^{r} |I_j'| - |f(I_i)| \right) \tag{5}$$

$$= r\sum_{i=1}^{r} |I_i| + r\sum_{j=1}^{r} |I_j'| - \sum_{i=1}^{r} |f(I_i)| \tag{6}$$

$$\geq 2r\sum_{i=1}^{r} |I_i| - \sum_{i=1}^{r} |I_i| \tag{7}$$

$$= (2r-1)\sum_{i=1}^{r} \alpha(G[A_i]). \tag{8}$$

The inequality between lines (3) and (4) follows from Lemma 3.2.

The bound of this theorem is sharp. To see this, consider the function graph depicted in Figure 2, where the dashed lines represent the function. Partitioning G into its four color classes (since $\chi(G) = 4$), we see that the right hand side of the inequality is exactly 7, while $\alpha(GfG') = 7$ as well. Moreover, this can be generalized, for $p \geq 5$, by taking p copies of K_p, instead of 4 copies of K_4 to form the graph G. Thus we have an infinite family of graphs where equality holds. ∎

Notice that when $r = 1$, we obtain the lower bound in Theorem 2.1.

Since every graph G can be partitioned into $\chi(G)$ sets, all of which are independent, the following is an immediate consequence:

Corollary 3.4. *For any graph G with chromatic number $\chi(G)$, and any function $f : V(G) \to V(G')$, we have*

$$\alpha(GfG') \geq \frac{2\chi(G) - 1}{\chi^2(G)}\, n(G),$$

and this bound is sharp.

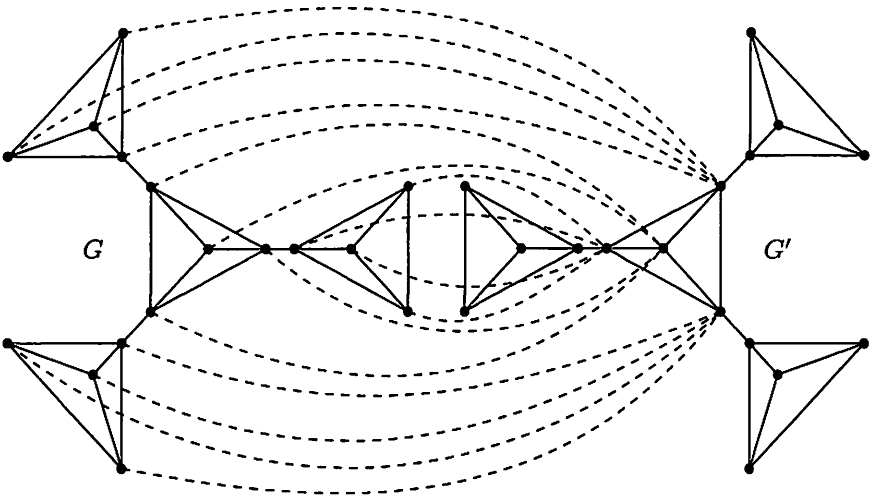

Figure 2: The graph GfG'; the dashed edges represent the function f.

For sharpness, see the class of graphs described at the end of the proof of Theorem 3.3. First, to see that Corollary 3.4 can sometimes be a better lower bound than the trivial bound given in Inequality 1 above, let G and GfG' be the graphs described at the end of the proof of Theorem 3.3. Then,

$$\frac{2\chi(G)-1}{\chi^2(G)}n(G) \geq \frac{2\chi(G)-1}{2\chi^2(G)}n(GfG') \tag{9}$$

$$\geq \frac{1}{\chi(G)+1}n(GfG') \tag{10}$$

$$= \frac{n(GfG')}{\chi(GfG')}, \tag{11}$$

where (10) follows since $\chi(G) \geq 1$, and (11) follows since $\chi(GfG') = \chi(G)+1$, for all members of the family of graphs in question.

Moreover, Corollary 3.4 implies the following corollary, which generalizes Corollary 3.1.

Corollary 3.5. *For any non-empty graph G and and any function $f : V(G) \to V(G')$,*

$$\alpha(GfG') \geq \frac{3n(G)}{2\chi(G)}$$

Note that when each of the A_i is a color class, and then particularly for planar graphs, the following are consequences of Theorem 3.3. Note that if each A_i is a color class then, $\sum_{i=1}^{r} \alpha(G[A_i]) = \sum_{i=1}^{r} |A_i| = n(G)$, and then the Four Color Theorem implies Corollary 3.6.

Corollary 3.6. *For any planar graph G and any function $f : V(G) \to V(G')$,*

$$\alpha(GfG') \geq \frac{7}{16} n(G),$$

and this bound is sharp.

For sharpness, see the class of graphs described at the end of the proof of Theorem 3.3.

It is interesting to note that Corollary 3.6 also says that if G is planar then

$$\alpha(GfG') \geq \frac{7}{32} n(GfG'),$$

which is only slightly worse than the $\frac{8}{32} = \frac{1}{4}$ that we would get for free if GfG' was planar. So a function graph of a planar graph is almost planar (we lose only $\frac{1}{32}$ from the lower bound). Notice that planarity of function graphs has been characterized in [7].

Now we consider the special case of Theorem 3.3 which follows from considering a bipartition $\{A, B\}$ of the vertex set of G, including tight bounds in the case where G is either bipartite, or has at least two disjoint maximum independent sets.

Corollary 3.7. *For any planar graph G, any partition $\{A, B\}$ of the vertex set $V(G)$, and any function $f : V(G) \to V(G')$,*

$$\alpha(GfG') \geq \frac{3}{4}\Big(\alpha(G[A]) + \alpha(G[B])\Big).$$

Note that Corollary 3.7 would be a poor choice for split graphs since the lower bound of $\frac{3}{4}(\alpha(G) + 1)$ is even worse than the general lower bound of Corollary 2.2.

Notice that if we restrict our attention to bipartite graphs then each of A and B is an independent set. Thus $\alpha(G[A]) + \alpha(G[B]) = n(G)$, where $n(G)$ be the order of G, and so we have the following quick corollary.

Corollary 3.8. *For any bipartite graph G and any function $f : V(G) \to V(G')$,*

$$\alpha(GfG') \geq \frac{3}{4}n(G),$$

and the bound is sharp.

To see the sharpness of the bound, let $G \cong C_n$, for some even n, $f : V(G) \to V(G')$, $f(v_{2j}) = v'_j, f(v_{2j-1}) = v'_j, \forall j, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. Then the set $\{v_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{v'_{2i} : \lfloor \frac{n}{4} \rfloor + 1 \leq i \leq \frac{n}{2}\}$ forms a maximum independent set in $C_n f C_n$.

Corollary 3.9. *For any graph G having two disjoint maximum independent sets, and any function $f : V(G) \to V(G')$,*

$$\alpha(GfG') \geq \frac{3}{2}\alpha(G),$$

and the bound is sharp.

The class in Corollary 3.8 also shows that the bound in this corollary is sharp.

A consequence of Corollary 3.7 gives bounds on the independence number of function graphs for cycles.

Corollary 3.10. *Let C_n be a cycle on n vertices, and f be any function. Then*

$$\left\lceil \frac{3}{4}n(C_n) \right\rceil \leq \alpha(C_n f C_n) \leq 2\left\lfloor \frac{n(C_n)}{2} \right\rfloor,$$

and the bounds are sharp.

The class in Corollary 3.8 also shows that the bound in this corollary is sharp.

4 Special Functions

In this section we find the independence number of the function graph for specialized functions. First we consider the constant function, by characterizing which graphs achieve one of the two values of the independence number.

Theorem 4.1. *For each connected graph G of order n and each constant function $f : V(G) \to \{x\}$, for any vertex $x \in V(G)$,*

$$2\alpha(G) - 1 \leq \alpha(GfG') \leq 2\alpha(G).$$

Moreover, $\alpha(GfG') = 2\alpha(G) - 1$ if and only if G has a unique maximum independent set S and Range$(f) \in S$.

Proof: The upper bound was previously addressed. To obtain the lower bound, let S be a maximum independent set in G, and let Range$(f) = \{w\} \subseteq G'$. Then an independent set will be at least as large as $|S \cup S' - \{w\}|$. Moreover, the lower bound holds with equality if and only if $\{w\} = \text{Range}(f) \subseteq S'$ and every vertex of $G' - S'$ has a neighbor in $S' - \{w\}$, i.e., $S \cup S' - \{w\}$ is the unique maximum independent set in GfG'. ∎

Next we present the independence number of a function graph with identity function, in terms of the order of the graph, n.

Theorem 4.2. *Let G be a connected graph on $n \geq 3$ vertices and id be the identity function. Then*

$$2 \leq \alpha(G\,id\,G') \leq n,$$

and the bounds are sharp. Moreover,

$$\alpha(G\,id\,G') = 2 \text{ if and only if } G \cong K_n,$$

and

$$\alpha(G\,id\,G') = n \text{ if and only if } G \text{ is a bipartite graph}$$

Proof: Choose vertices $u_1 \in V(G)$ and $v_2 \in V(G')$ such that $id(u_1) \neq v_2$. Then $\alpha(G\,id\,G') \geq |\{u_1, v_2\}| = 2$. For the upper bound, let A_1 be a maximum independent set in G. Then $\alpha(G\,id\,G') \leq |A_1 \cup (V(G') - f(A_1))\,| = n$.

For the characterization of the lower bound, notice that $\alpha(K_n, id) = 2$. On the other hand, if $G \not\cong K_n$, say $G \cong K_n - u_1u_2$, for some arbitrary vertices $u_1, u_2 \in V(G)$, then

$$\alpha(G\,id\,G') \geq |\{u_1, u_2, v_3 : id(u_1) \neq v_3, id(u_2) \neq v_3\}| = 3.$$

For the characterization of the upper bound, let G be a bipartite graph, with the partite sets U_1 and U_2, and G' be a copy of G with its partite sets V_1 and V_2, respectively (V_i corresponding to U_i, $i = 1, 2$). Then $\alpha(G\,id\,G') = |U_1 \cup V_2| = |U_2 \cup V_1| = n$. Moreover, if G is not a bipartite graph, then $\alpha(G) < \lceil \frac{n}{2} \rceil$ and similarly $\alpha(G') < \lceil \frac{n}{2} \rceil$, which gives that $\alpha(G\,id\,G') < n$. ∎

All values in between the lower and upper bound of Theorem 4.2 are realizable as we show next.

Proposition 4.3. *Let id be the identity function. For every choice of* $n > 1$ *and for every choice of* i *satisfying* $2 \leq i \leq n$, *there exists a connected graph* G *such that* $\alpha(G\, id\, G') = i$.

Proof: Suppose that $n > 1$ and $2 \leq i \leq n$. Let G be the split graph whose independent set S has size $i-1$ and whose clique K has size $n-i+1$. One maximum independent set in the function graph $G\, id\, G'$ is the set $S \cup \{v'\}$, where v' is an arbitrary vertex in the clique of G'. ■

The illustration of Figure 1 shows the case $n = 5$, $i = 2$, with the dashed edges representing the identity function.

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