Constructing Trees with Graceful Labelings Using Caterpillars

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Abstract

Hrnciar and Haviar [3] gave a method to a construct a graceful labeling for all trees of diameter at most five. Based on their method and the methods described in Balbuena et al [1], we shall describe a new construction for gracefully labeled trees by attaching trees to the vertices of a tree admitting a bipartite graceful labeling.

1 Introduction

A tree T with m edges is said to be **graceful** if it admits a labeling $f : V(T) \to \{0, 1, \ldots, m\}$ such that $\{|f(u) - f(v)| \mid uv \in E(T),\ u \neq v\} = \{1, 2, \ldots, m\}$. Such a function f is called a **graceful labeling** of G, and we shall call the pair (G, f) a **graceful tree**. We shall often refer to vertices simply by their labels in (G, f). For an edge $uv \in E(G)$, we shall refer to $|f(u) - f(v)|$ as its **edge label**. The function f is said to be a **bipartite** graceful labeling if it satisfies the additional requirement that , for all $u \in V(T)$ and for all $v \in N_T(u)$, either $f(u) < f(v)$, or $f(u) > f(v)$. That is, there is a bipartition X, Y of T such that the labels of the vertices of X are $0, 1, \ldots, |X| - 1$, and the labels of the vertices of Y are $|X|, |X| + 1, \ldots, m$.

It has been conjectured that all trees admit a graceful labeling, the well-known Graceful Tree Conjecture (see [2]). Among the classes of trees that admit a graceful labeling is the class of **caterpillars**. Trees in this class also admit a bipartite graceful labeling. In this paper, we first introduce the construction of a new class of graceful trees obtained by attaching rooted trees with a special property to vertices of an arbitrary caterpillar. Following this, we show how this construction generalizes to attaching trees to vertices of an arbitrary tree having a bipartite graceful labeling. This construction generalizes the one given by Balbuena et al [1].

Let T be a rooted tree with m edges and having root t with degree μ. Let $X = \{1, \ldots, \lfloor \frac{\mu}{2} \rfloor, m - \lceil \frac{\mu}{2} \rceil + 1, \ldots, m\}$. We say that T is **strongly graceful** if any labeling of the vertices of $N_T(t)$ with labels from X can be extended to a graceful labeling f of T such that $f(t) = 0$. Examples of such trees include **fans** and rooted trees with even degree sequence (as defined by Balbuena et al. [1]) which we describe below.

Let T be a rooted tree with even diameter D. We say that T has **even degree sequence** if every vertex has even degree except for the leaves of T which all belong to level $\frac{D}{2}$, and possibly the root (which could be even or odd). We say that T is **quasi-even** if every vertex of the tree has even degree except for the leaves of T which all occur on level $\frac{D}{2}$, the vertices on level $\frac{D}{2} - 1$, and possibly the root (which can be even or odd). We have the following interesting theorem:

1.1 Theorem (Balbuena et al [1])
If T is a rooted tree having even or quasi-even degree sequence, then T admits a strongly graceful labeling.

We now describe the the main result of this paper which can be interpreted as a generalization of the above theorem. Let (H, f_H) be a graceful tree having n edges where f_H is a bipartite graceful labeling. Let (X, Y) be a bipartition of H where X receives the labels $0, 1, \ldots, k$ and Y receives the labels $k+1, k+2, \ldots, n$. For $i = 0, \ldots, k$, let v_i be the vertex in X for which $f_H(v_i) = k - i$ and for $i = k+1, \ldots, n$, let v_i be the vertex in Y for which $f_H(v_i) = i$. For $i = 0, \ldots, n$ let T_i be a rooted tree with root vertex u_i (where T_i may possibly be trivial), and let μ_i be the degree of u_i in T_i. Let T be the rooted tree with root vertex u obtained by "gluing" all the trees T_i, $i = 0, 1, \ldots, n$ together by identifying each root vertex u_i with u. Let $m = |E(T)|$ and μ be the degree of u in T. The main theorem is:

1.2 Theorem
Let G be the graph obtained from H where for $i = 0, 1, \ldots, n$ the tree T_i is joined to the vertex v_i by identifying u_i with v_i. Suppose that $|X| \leq |Y|$ and the following conditions hold:

(i) $\mu_0 > 0$ *and* μ_i *is positive and even for* $i = 1, \ldots, n$.

(ii) $\sum_{i=0}^{k} \mu_i \geq 2|Y|$.

(iii) T *is a tree having a strongly graceful labeling.*

Then G has a graceful labeling f where $f|X = f_H|X$ and $f|Y = f_H|Y + m$.

The above theorem has the following interesting corollary. A **fan** is a rooted tree where the root vertex is adjacent to all other vertices.

1.3 Corollary
Let (H, f_H) be a graceful tree having n edges where f_H is a bipartite graceful labeling with corresponding bipartition (X,Y). Let G be the graph where we attach a fan F_v at each vertex $v \in V(H)$ having root v. Let $\mu_v = d_{F_v}(v)$, $v \in V(H)$. If $|X| \leq |Y|$, $\sum_{v\in X} \mu_v \geq 2|Y|$, and μ_v is positive and even for all $v \in V(H)$, then G has a graceful labeling.

Proof. We may assume that the labels in X are smaller than the labels in Y; if this is not the case, then redefine f_H to be $n - f_H$. Let T be the tree obtained by "gluing" all the fans F_v, $v \in V(H)$ together, that is, by identifying all the root vertices with a single vertex. Then T is a fan, and thus it is also strongly graceful. It now follows by Theorem 1.2 that G has a graceful labeling. □

2 The Transfer Method

One of the main tools in this paper is a method of Hrnciar and Haviar [3] for transferring a pair of edges incident to one vertex in a graceful tree to another vertex so as to obtain another graceful tree. We shall introduce some notation and definitions.

Let $I = [a,b]$ and $J = [c,d]$ be two disjoint intervals of integers. We say that I and J are **k-combining** if $a + d = b + c = k$; that is, we can partition $I \cup J$ into pairs of integers $\{x,y\}$ where $x + y = a + d$. We call such pairs $\{x,y\}$ **combining pairs.**

Suppose u and v are vertices of a graceful tree (T, f) and the disjoint (integer) intervals $I = [a,b]$ and $J = [c,d]$ form a subset of $N_G(u)$ which does not contain vertices of $N_G(v)$. Suppose these intervals are k-combining. Then we denote by $u \xrightarrow[I,\,J]{} v$ a transfer from u to v where all vertices in $I \cup J$ are made adjacent to v and all edges between u and vertices in $I \cup J$ are deleted. It can be easily seen that such a transfer preserves gracefulness when $k = u + v$ since for each combining pair $\{x,y\}$, $|u - x| = |v - y|$ and $|u - y| = |v - x|$. For $1 \leq i < b - a$ we write $u \xrightarrow[[a,b],\,[c,d]]{2i} v$ to mean the same thing as the transfer $u \xrightarrow[[a,a+i-1],\,[d-i+1,d]]{} v$. When we write $(I,J) \overset{2i}{\to} (I',J')$, we mean that $I' = [a+i-1, b]$ and $J' = [c, d-i+1]$. Furthermore, when we write $(I,J) \overset{+1}{\to} (I',J')$, we mean $I' = [a+1,b]$ and $J' = [c+1,d]$, and when we write $(I,J) \overset{-1}{\to} (I',J')$, we mean $I' = [a,b-1]$ and $J' = [c,d-1]$. Note that if I and J are k-combining and $(I,J) \overset{2i}{\to} (I',J')$, then I' and J' are k-combining. Furthermore, if $(I,J) \overset{+1}{\to} (I',J')$, then I' and J' are $k+1$-combining. If $(I,J) \overset{-1}{\to} (I',J')$, then I' and J' are $k-1$-combining.

3 Attaching Trees to a Caterpillar

A **caterpillar** is a tree consisting of a path P (called the **central path**) where vertices not belonging to P have degree one and are joined to vertices on P. In this section, we describe a construction whereby trees can be attached to the vertices of a caterpillar so that the resulting tree is graceful.

Let H be a caterpillar having n edges and central path $P = v_1 \cdots v_p$ Let $N_H(v_1) = \{v_{11}, v_{12}, \ldots, v_{1k_1}\}$ where $v_{1k_1} = v_2$. For $i = 2, \ldots, p$ let $N_H(v_i)\backslash\{v_{i-1}\} = \{v_{i1}, v_{i2}, \ldots, v_{ik_i}\}$ where $v_{ik_i} = v_{i+1}$, $i = 2, \ldots, p$. For $i = 1, \ldots, p$ let $v_{i0} = v_i$. To each vertex v_{ij} we associate a rooted tree T_{ij} with root u_{ij} (where $T_{i0} = T_{(i-1)k_{i-1}}$, $i = 2, 3, \ldots, p$). It should be noted that some of the trees T_{ij} may possibly be trivial (consisting only of a single vertex). For all i, j, let μ_{ij} be the degree of the root vertex u_{ij} in T_{ij}.

We construct a graceful labeling f_H of H as follows: Let q (resp. q') be the greatest even (resp. odd) integer less than or equal to p. Then $q = 2\lfloor \frac{p}{2} \rfloor$ and $q' = 2\lfloor \frac{p-1}{2} \rfloor + 1$. We define a partition X, Y of $V(G)$ where

$$X = \{v_1, v_{21}, \ldots, v_{2k_2}, v_{41}, \ldots, v_{4k_4}, \ldots, v_{q1}, \ldots, v_{qk_q}\}$$

$$Y = \{v_{11}, \ldots, v_{1k_1}, v_{31}, \ldots, v_{3k_3}, \ldots, v_{q'1}, \ldots, v_{q'k_{q'}}\}.$$

We define a graceful labeling f_H of H where $f_H(v_1) = k, f_H(v_{21}) = k - 1, f_H(v_{22}) = v - 2, \ldots, f_H(v_{qk_q}) = 0$ and $f_H(v_{11}) = k + 1, f_H(v_{12}) = k + 2, \ldots, f_H(v_{q'k_{q'}}) = n$. See Figure 1. More explicitly, let $k_0 = 0$ and $k = 1 + k_0 + k_2 + \cdots + k_q$. Then

$$f_H(v_1) = k,\ f_H(v_{ij}) = k - \sum_{s=0}^{\frac{i}{2}-1} k_{2s} - j, \quad \text{for all } i \in \{2, 4, \ldots, q\},\ j \in \{1, \ldots, k_i\}$$

$$f_H(v_{ij}) = k + \sum_{s=1}^{\frac{i-1}{2}} k_{2s-1} + j, \quad \text{for all } i \in \{1, 3, \ldots, q'\} \text{ and } j \in \{1, \ldots, k_i\}$$

The proof of Theorem 1.2 hinges on the special case for caterpillars.

3.1 Theorem

Let G be the graph obtained from H by joining T_{ij} to v_{ij} by identifying u_{ij} with v_{ij} for each $i = 1, \ldots, p$ and $j = 0, \ldots, k_i$. Let T be the rooted tree obtained by attaching all the trees T_{ij} to a vertex u, that is, by identifying each root vertex u_{ij} with u. Suppose

(i) μ_{ij} is even for all i, j other than possibly μ_{10}

(ii) $\mu_{i0} \geq 2(d_H(v_i) - 1)$ for all $i = 1, \ldots, p$

and

(iii) T is strongly graceful.

Then G has a graceful labeling f where $f|X = f_H|X$ and $f|Y = f_H|Y + m$.

Proof. Without loss of generality, we may assume that p is odd. Furthermore, we may assume that μ_{10} is even; essentially the same arguments apply when it is odd. The tree T is rooted at vertex u which has degree $\mu = \sum_{i,j} \mu_{ij}$. The degree μ must be even by (i). We shall construct a sequence of graceful trees

$$(G_1, f_1),\ (G_2, f_2)\ \dots\ ,(G_{p+1}, f_{p+1}),$$

recursively where $G \simeq G_{p+1}$. Let $m = |E(T)|$. Since T is strongly graceful, there is a graceful labeling τ of T where $\tau(u) = 0$ and the neighbours of u receive the labels $1, 2, \dots, \frac{\mu}{2}, m - \frac{\mu}{2} + 1, \dots, m$. Note that in τ, we may assume any assignment of these labels to the vertices of $N_T(u)$, an important property which we will utilize later on. To construct (G_1, f_1), let G_1 be the graph obtained from H by attaching the tree T to v_1 in H (by identifying u with v_1). Let T_1 be the subgraph of G_1 corresponding to the tree T rooted at v_1. Let $\mu_1 = \mu$ (which is the degree of v_1 in T_1). Let f_1 be a labeling of G_1 where

$$f_1(v) = \begin{cases} \tau(v) + k, & \text{if } v \in V(T_1)\backslash\{v_1\}; \\ f_H(v), & \text{if } v \in X; \\ f_H(v) + m, & \text{if } v \in Y. \end{cases}$$

Note that $f_1(v_1) = k$ and $f_1(v_{11}) = k + m + 1$. Under f_1, the vertices of $V(T_1)\backslash\{v_1\}$ receive labels $k+1, k+2, \dots, k+m$ and edges of T_1 receive the labels $1, \dots, m$. The vertices of X receive the labels $0, \dots, k$, and the vertices of Y receive the labels $m+k+1, m+k+2, \dots, m+n$. The edges of H receive the labels $m+1, m+2, \dots, m+n$. Thus (G_1, f_1) is a graceful tree where $f_1|X = f_H|X$ and $f_1|Y = f_H|Y + m$. To create (G_2, f_2) we transfer some of the vertices of $N_{T_1}(v_1)$ to the vertices $v_{11}, \dots, v_{1k_1}$. We first observe that in G_1 the vertices in $N_{T_1}(v_1)$ have the labels $k+1, k+2, \dots, k+\frac{\mu_1}{2}, k+m-\frac{\mu_1}{2}+1, k+m-\frac{\mu_1}{2}+2, \dots, k+m$. Let $I_1 = [k+1, k+\frac{\mu_1}{2}]$ and $J_1 = [m+k-\frac{\mu_1}{2}+1, k+m]$. Since $f_1(v_1) + f_1(v_{11}) = m + 2k + 1$, the intervals I_1 and J_1 are $f_1(v_1) + f_1(v_{11})$-combining. For $j = 1, \dots, k_1$, we define a pair of intervals I_{1j}, J_{1j} as follows: First, let $I_{11} = I_1$ and $J_{11} = J_1$. For $j = 2, \dots, k_1$ let

$$(I_{1(j-1)}, J_{1(j-1)}) \xrightarrow{\mu_{1(j-1)}} (I'_{1(j-1)}, J'_{1(j-1)}) \xrightarrow{+1} (I_{1j}, J_{1j}).$$

Since I_1 and J_1 are $f_1(v_1)+f_1(v_{11})$-combining and $f_1(v_1)+f_1(v_{1(j+1)}) = f_1(v_1) + f_1(v_{1j}) + 1$, for $j = 1, \dots, k_1 - 1$, one sees that for $j = 1, \dots, k_1$,

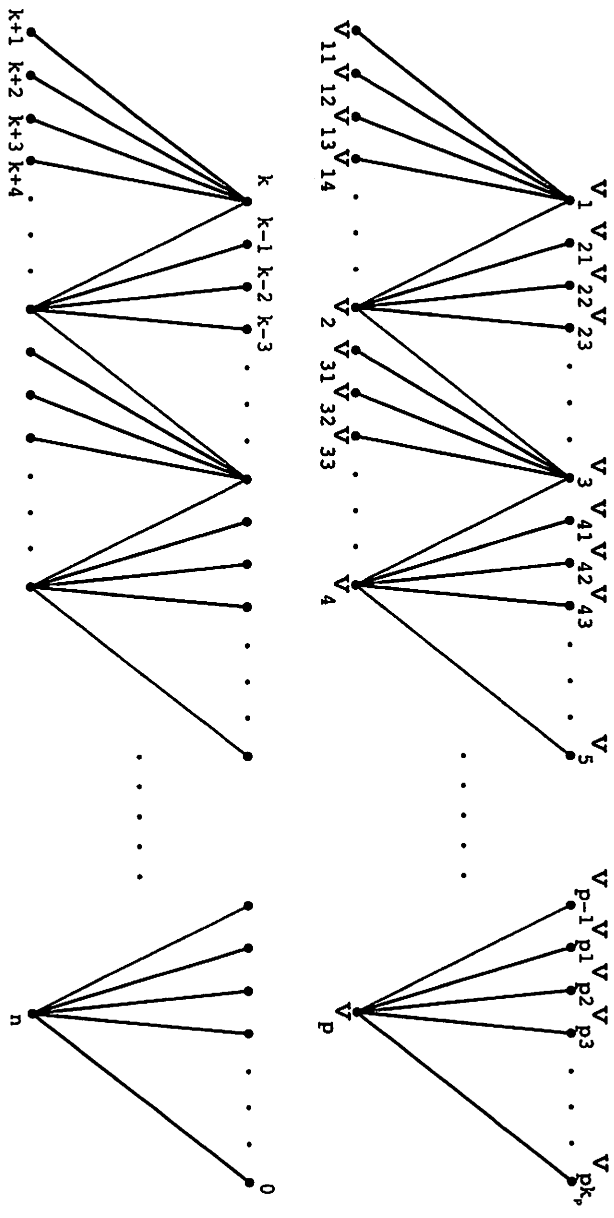

Figure 1: The caterpillar H with graceful labeling

the intervals I_{1j} and J_{1j} are $f_1(v_1) + f_1(v_{1j})$-combining. Let $\mu_2 = \mu_1 - \sum_{j=0}^{k_1-1} \mu_{1j}$ and let (G_2, f_2) be the graceful tree obtained after the successive transfers

$$v_1 \xrightarrow[I_{11},\ J_{11}]{\mu_{11}} v_{11}, \quad v_1 \xrightarrow[I_{12},\ J_{12}]{\mu_{12}} v_{12}, \quad \dots \quad , v_1 \xrightarrow[I_{1(k_1-1)},\ J_{1(k_1-1)}]{\mu_{1(k_1-1)}} v_{1(k_1-1)},$$

$$v_1 \xrightarrow[I_{1k_1},\ J_{1k_1}]{\mu_2} v_2.$$

One sees that $f_2|X = f_1|X = f_H|X$ and $f_2|Y = f_1|Y = f_H|Y + m$. Let T_2 be the tree rooted at v_2 resulting from the last transfer. Let I_2 and J_2 be intervals such that

$$(I_{1k_1}, J_{1k_1}) \xrightarrow{\mu_{10} - 2(k_1 - 1)} (I_2, J_2).$$

Then the union $I_2 \cup J_2$ represents the set of labels of the vertices in $N_{T_2}(v_2)$ in G_2. Observe that I_2, J_2 is $f_2(v_1) + f_2(v_2)$-combining since I_{1k_1}, J_{1k_1} is $f_1(v_1) + f_1(v_2)$-combining. Suppose now that for some $i \geq 2$ we have constructed the graceful tree (G_i, f_i) where $f_i|X = f_H|X$ and $f_i|Y = f_H|Y + m$. Let T_i be the tree rooted at v_i which resulted from the last transfer from v_{i-1} to v_i. Let μ_i be the degree of v_i in T_i. Let I_i and J_i be the disjoint intervals representing the labels of the vertices in $N_{T_i}(v_i)$. We may assume that I_i and J_i are $f_i(v_{i-1}) + f(v_i)$-combining. Let $\mu_{i+1} = \mu_i - \sum_{j=0}^{k_i-1} \mu_{ij}$. To construct (G_{i+1}, f_{i+1}), we define intervals $I_{ij}, J_{ij},\ j = 1, \dots, k_i$ as follows: If $j = 1$, then

$$(I_i, J_i) \xrightarrow{+1} (I_{i1}, J_{i1}), \text{ if i is odd, } \quad \text{and } (I_i, J_i) \xrightarrow{-1} (I_{i1}, J_{i1}), \text{ if i is even.}$$

If $j \geq 2$, let I_{ij} and J_{ij} be such that

$$(I_{i(j-1)}, J_{i(j-1)}) \xrightarrow{\mu_{i(j-1)}} (I'_{i(j-1)}, J'_{i(j-1)}) \xrightarrow{+1} (I_{ij}, J_{ij}), \quad \text{if i is odd}$$

$$(I_{i(j-1)}, J_{i(j-1)}) \xrightarrow{\mu_{i(j-1)}} (I'_{i(j-1)}, J'_{i(j-1)}) \xrightarrow{-1} (I_{ij}, J_{ij}), \quad \text{if i is even.}$$

Assuming that $I_{i(j-1)}$ and $J_{i(j-1)}$ are $f_i(v_i) + f_i(v_{i(j-1)})$-combining, one sees that I_{ij} and J_{ij} are $f_i(v_i) + f_i(v_{ij})$-combining. Thus the transfers

$$v_i \xrightarrow[I_{i1},\ J_{i1}]{\mu_{i1}} v_{i1}, \quad v_i \xrightarrow[I_{i2}, J_{i2}]{\mu_{i2}} v_{i2}, \quad \dots \quad , v_i \xrightarrow[I_{i(k_i-1)},\ J_{i(k_i-1)}]{\mu_{i(k_i-1)}} v_{i(k_i-1)},$$

$$v_i \xrightarrow[I_{ik_i},\ J_{ik_i}]{\mu_{i+1}} v_{i+1}$$

result in a graceful tree (G_{i+1}, f_{i+1}). We say that a vertex in $I_i \cup J_i$ is *untransferred* if it does not get transferred to any of the vertices $v_{i1}, v_{i2}, \dots, v_{ik_i}$ after the above transfers. Let $\varepsilon = \mu_{i1} + \mu_{i2} + \cdots + \mu_{i(k_i-1)}$. There are exactly $\mu_i - \varepsilon - \mu_{i+1} = \mu_{i0}$ untransferred vertices in $I_i \cup J_i$. We observe that by the construction of I_i, J_i, $i = 1, \dots, p$ there are at least $2k_i = 2(d_H(v_i) - 1)$ untransferred vertices, when $i \geq 2$, and at least $2(k_1 - 1) = 2(d_H(v_1) - 1)$ untransferred vertices, when $i = 2$. Hence the condition $\mu_{i0} \geq 2(d_H(v_i) - 1)$ is seen to be necessary. Let H_i be the portion of the rooted tree T_i remaining at v_i. By the above, its root has degree μ_{i0}. We see that $f_{i+1}|X = f_i|X = f_H|X$ and $f_{i+1}|Y = f_i|Y = f_H| + m$.

Continuing, we eventually obtain a graceful tree (G_{p+1}, f_{p+1}) where $f_{p+1}|X = f_H$, $f_{i+1}|Y = f_H|Y + m$, and at each vertex v_{ij} we have a rooted tree H_{ij} whose root has degree μ_{ij}. Note that the trees H_{ij} only depend on the initial labeling of the vertices in $N_T(u)$. Given that T is strongly graceful, we may choose these labels such that $H_{ij} \simeq T_{ij}$ for all i, j. Thus we have that $G \simeq G_{p+1}$. This completes the proof.

□

4 Proof of Theorem 1.2

We have by assumption that $|X| \leq |Y|$ and $\sum_{v_i \in X} \mu_i \geq 2|Y|$. Using this, we can partition Y into $|X| = k + 1$ sets $Y_0, Y_1, \dots, Y_k$ where $1 \leq |Y_i| \leq \frac{\mu_i}{2}$, $i = 0, 1, \dots, k$. For $i = 0, \dots, k$, let $s_i = |Y_i|$ and let $t_i = k + s_0 + \cdots + s_i$. Let $Y_0 = \{v_{k+1}, \dots, v_{k+s_0}\}$ and for $i = 1, 2, \dots, k$, let $Y_i = \{v_{k+t_{i-1}+1}, v_{k+t_{i-1}+2}, \dots, v_{k+t_{i-1}+s_i}\}$. We construct a caterpillar H' having vertices v_i, $i = 0, 1, \dots, n$ as follows: For $i = 0, 1, \dots, k$ join v_i to each of the vertices in Y_i. Next, for $i = 0, \dots, k-1$, join v_i to v_{t_i+1}. Then H' is seen to be a caterpillar which has a central path $v_0 v_{t_0+1} v_1 v_{t_1+1} \cdots v_{k-1} v_{t_{k-1}+1} v_k$. Furthermore, $f_{H'} = f_H$ is seen to be a graceful labeling of H'. For $i = 0, \dots, k-1$ we have that $d_{H'}(v_i) = |Y_i| + 1 \leq \frac{\mu_i}{2} + 1$, and we also have that $d_{H'}(v_k) = |Y_k| \leq \frac{\mu_k}{2}$. Thus $2(d_{H'}(v_i) - 1) \leq \mu_i$, $i = 0, 1, \dots, k$. We also observe that $d_{H'}(v_i) - 1 \leq 1$, $i = k+1, \dots, n$ and hence $2(d_{H'}(v_i) - 1) \leq 2 \leq \mu_i$, $i = k+1, \dots, n$. Thus $2(d_{H'}(v_i) - 1) \leq \mu_i$, $i = 0, \dots, n$. Let G' be the graph obtained from H' where for each $i = 0, 1, \dots, n$ we attach the tree T_i to v_i by identifying u_i with v_i. It now follows by Theorem 3.1 that G' has a graceful labeling f' for which $f'|X = f_{H'}|X$ and $f'|Y = f_{H'}|Y + m$. However, given that f_H is a graceful labeling for H, we also see that $f = f'$ is a graceful labeling for G. This completes the proof.

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