Conditions for the Bicolorability of Primitive Hypergraphs

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Abstract

A *primitive hypergraph* is a hypergraph with maximum cardinality three and maximum degree three such that every 3-edge is adjacent only to 2-edges and is incident only to vertices of degree two. Deciding the bicolorability of a primitive hypergraph is NP-complete (a straightforward consequence of results in [14]). We provide sufficient conditions, similar to the Sterboul conditions proved by Défossez [5], for the existence of a bicoloring of a primitive hypergraph, and we provide a polynomial algorithm for bicoloring a primitive hypergraph if those conditions hold. We then draw a connection between this algorithm and the well-known necessary and sufficient conditions given by Berge [1] for maximal matchings in graphs, which leads to a characterization of bicolorability of primitive hypergraphs.

1 Preliminaries

Let $H = (V, E)$ be a *hypergraph* where V is the set of *vertices* and E is the non-empty set of distinct subsets of V referred to as the *edges* of H. For a general hypergraph reference see [2]. A *k-edge* is an edge of cardinality k, noting that if all edges are 2-edges the hypergraph is in fact just a graph. A *bicoloring* or *2-coloring* of a hypergraph is a partitioning of the set of vertices V into two color classes, say red and blue, such that no edge in E is monochromatic. A hypergraph is deemed *bicolorable* if it admits such a bicoloring or assignment of the two colors to the vertices. Much work on the bicolorability of hypergraphs has been done over the

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years [10, 11, 15]; Erdös [7] notes that Miller [17] used the term *property B* when he first investigated bicolorability of hypergraphs with infinite edges. Given the broad interest in and applicability of this problem, bounds and algorithmic results have both been extensively studied; see [18] and the references therein. Recently, Défossez [5] proved a conjecture of Sterboul [6] stating a hypergraph is bicolorable as long as it does not contain a particular type of odd cycle, improving earlier work on this problem [3, 8].

We consider the bicolorability of a particular type of hypergraph with additional restrictions, namely:

(i) each edge is a 2-edge or a 3-edge,

(ii) each vertex has degree at most three, and

(iii) each 3-edge is incident only to vertices of degree at most two, and

(iv) no two 3-edges are adjacent.

We call such a hypergraph a *primitive hypergraph.* Note that the dual of a primitive hypergraph H is itself a primitive hypergraph.

While these requirements may seem quite limiting, our motivation for studying this rather restrictive class of hypergraphs is to better understand the obstructions to finding a polynomial algorithm for general hypergraph bicoloring. A common approach, when faced with a difficult problem to solve, is to study a simpler version of the same problem. If the simpler problem can be solved, ask if that solution can be extended to the more complicated problem, and if not, why not. On the other hand, if the simpler problem is still difficult to solve, then perhaps there is an even simpler version to investigate, and so on. For the hypergraph bicoloring problem, if all edges are restricted to be 2-edges, then the hypergraph becomes a graph, in which case it is well-known how to decide bicolorability in polynomial time. Thus we consider primitive hypergraphs as a means of simplifying the structure to make it as 'graph-like' as possible, while still retaining the NP-completeness of its bicolorability. In recent results in this direction, [14] showed restrictions (i) and (ii) above define a class of hypergraphs for which bicoloring remains NP-complete, while [13] showed that a further strengthening of condition (ii) to "(ii′) each vertex has degree at most two" results in a class of hypergraphs for which bicoloring is in P.

2 Bicoloring Primitive Hypergraphs is NP-Complete

Recall that in a Boolean expression, a literal is a variable or a negation of a variable, a clause is a disjunction of literals, and it is in conjunctive normal form if it is a conjunction of clauses. Satisfiability (SAT), perhaps the canonical NP-Complete problem [4], asks if it is possible to assign true and false values to the variables in the expression so that it evaluates to true. Numerous variants of SAT have been studied over the years (see, for instance, examples in [9]).

We consider a variant of SAT that is equivalent to the bicolorability of primitive hypergraphs, by taking each variable to be a vertex and each clause to be a hyperedge. Reviewing the notation of [14], a $(\leq 3, \leq 3)$-*All+ NAE-SAT* has the following properties:

(i) each clause contains either two or three variables, and
(ii) each variable appears in at most three clauses.

In defining a *Primitive* $(\leq 3, \leq 3)$-*All+ NAE-SAT* instance we add two further restrictions:

(iii) each variable that is in a clause with three disjuncts is in at most one other clause, and

(iv) each variable that is in a clause with three disjuncts is in no other clause with three disjuncts.

Kratochvíl and Tuza show the following result, which we then use to show that *Primitive* $(\leq 3, \leq 3)$-*All+ NAE-SAT* is NP-complete.

Theorem 1 ([14], Theorem 2.1) *The problem* $(\leq 3, \leq 3)$-*All+NAE-SAT is* NP-*complete.*

Theorem 2 *The problem Primitive* $(\leq 3, \leq 3)$-*All+NAE-SAT is* NP-*complete.*

Proof The proof is by reduction from $(\leq 3, \leq 3)$-*All+ NAE-SAT*. Recall that the $(\leq 3, \leq 3)$ condition guarantees that each variable appears in at most three clauses. For every variable a that appears in two clauses (a, b, c) and (a, d, e) of size three (and no other clauses), replace a and the two clauses by three fresh variables a', a_1, and a_2 and four new clauses (a_1, b, c), (a_2, d, e), (a_2, a'), (a_1, a'). The variable a' is the effective complement of both a_1 and a_2, making those two variables logical aliases of the replaced variable a. Similarly, for every variable a that appears in three clauses (a, b, c) and (a, d, e), (a, f, g) of size three, replace a and the three clauses by four fresh variables a', a_1, a_2, and a_3, and six new clauses (a_1, b, c), (a_2, d, e), (a_3, f, g), (a_1, a'), (a_2, a'), and (a_3, a'). As before, a' is the effective complement of each of a_1, a_2, and a_3. Finally, for every vertex a that is in two clauses (a, b, c) and (a, d, e) of size three and another clause (a, f) of size two, the replacements are almost identical to those in the previous case. It is easy to see that the new expression is NAE satisfiable if and only if the original expression is NAE satisfiable. ■

Corollary 2.1 *Bicoloring a primitive hypergraph is* NP*-complete.*

3 Sufficient Conditions

In this section, we focus on sufficient conditions for a primitive graph to be bicolorable. We first provide some definitions and a result of Defossez [5]. We then modify that algorithmic result to provide an alternative set of sufficient conditions.

There are numerous representations of hypergraphs used in the literature; a hyperedge may be drawn as a Venn diagram region that includes all of the incident nodes [12], or as an edge that touches all of the incident nodes [16]. We choose to use a modification of the latter; since all our hyperedges are 2-edges and 3-edges, we depict a 2-edge as a standard graph representation (two vertices joined by a line) and a 3-edge as the following structure:

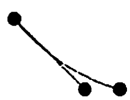

Figure 1: A representation of a 3-edge.

We begin with some definitions, at first closely following the development in [5], which will allow us to provide sufficient conditions for a simple primitive hypergraph to be bicolorable.. A *coloring* C of a hypergraph H is a subset of $V(H)$ for which each edge e of H has at least one vertex not in C, thus partitioning the vertices into two subsets (colors): those in C, and those not in C. A coloring C of a hypergraph H *covers* an edge e of H if and only if at least one vertex of e is in C. A hypergraph H is *bicolorable* if and only if there is a coloring C that covers every edge of H. Note that this definition is equivalent to the definition in section 1.

A sequence $(v_1, e_1, v_2, \ldots, e_n, v_1)$ where the e_is are distinct edges, the v_is are distinct vertices, and $n \geq 3$, is a *cycle* if $v_i \in e_{i-1} \cap e_i$ for $i = 2, \ldots, n$ and $v_1 \in e_1 \cap e_n$. A cycle is considered *odd* if it has an odd number of edges. An odd cycle $(v_1, e_1, v_2, \ldots, e_n, v_1)$ such that any two non-consecutive edges are disjoint and $|e_i \cap e_{i+1}| = 1$ for $i = 1, 2, \ldots, n-1$, is called a *anti-Sterboul cycle* using the terminology of [5]. A hypergraph that does not contain an anti-Sterboul cycle is a *Sterboul hypergraph.* Sterboul's conjecture, proved in [5], is as follows:

Theorem 3 ([5]) *If H is a Sterboul hypergraph, then H is bicolorable.*

Note that this is only a sufficient condition, and thus the presence of an anti-Sterboul cycle does not imply that a hypergraph is not bicolorable. In the case of primitive hypergraphs this sufficient condition can be weakened. Below we define a structure called a theta-flow, and then we show in Theorem 4 that the absence of theta flows together with the absence of odd graph cycles (i.e. odd cycles consisting only of 2-edges) guarantees that a primitive hypergraph is bicolorable.

The theta-flow definition below relies on the concept of a path. A *path* P_i from vertex v to vertex w is a sequence of distinct edges $e_1, e_2, \ldots, e_k$ such that v is a vertex of edge e_1, w is a vertex of e_k, and for any pair of consecutive edges e_i and e_{i+1} on the path, e_i is adjacent to e_{i+1}. Two paths P_1 and P_2 are *disjoint* if every edge of P_1 is disjoint from every edge of P_2.

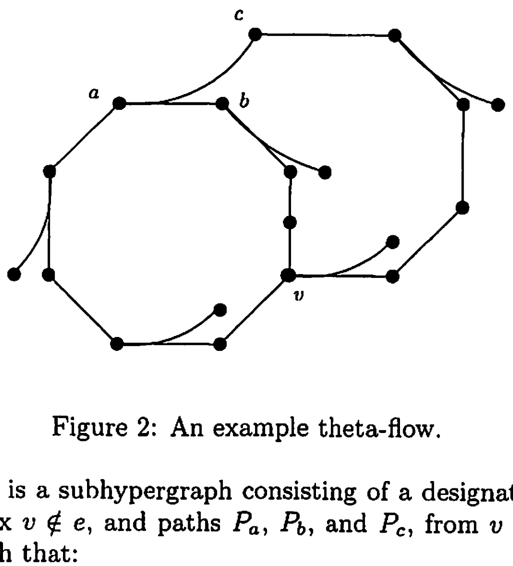

Figure 2: An example theta-flow.

A *theta-flow* is a subhypergraph consisting of a designated 3-edge $e = \{a, b, c\}$, a vertex $v \notin e$, and paths P_a, P_b, and P_c, from v to a, b, and c, respectively, such that:

(i) each intersection of two consecutive edges along a path has size one,
(ii) all pairs of non-consecutive edges along each path are disjoint,
(iii) each pair of paths intersect only at the shared vertex v, and
(iv) all three path lengths have the same parity (all even or all odd).

Figure 2 shows an example theta-flow. P_a is the path from v to a that does not pass through b or c, and similarly for P_b and P_c. $P_a \cup P_b \cup e$, and $P_b \cup P_c \cup e$ and $P_a \cup P_c \cup e$ are all anti-Sterboul cycles, so the absence of anti-Sterboul cycles implies the absence of theta-flows, but not conversely.

Removing any edge from Figure 2 provides an example of a hypergraph that is not a theta-flow.

Theorem 4 *If a primitive hypergraph H has no theta-flow and no odd cycle of 2-edges, then H is bicolorable.*

Proof We prove this by induction on the number of edges in H. If H has no edges the theorem is certainly true. Suppose H has k edges, H contains no odd cycle of 2-edges and no theta-flow, and the implication is true for all primitive hypergraphs with $k-1$ edges. If H has no 3-edges, then H is simply a graph and having no odd cycles implies that H is bicolorable. Otherwise, let e_0 be a 3-edge of H, and observe that the removal of e_0 from H cannot create any new cycles or new theta-flows. So by the inductive assumption, hypergraph $H \setminus \{e\}$ can be bicolored, say by coloring C. If C bicolors e_0, we are done. Otherwise, we perform Algorithm 1, a modification of the Défossez algorithm [5]. The input to this algorithm is the primitive hypergraph H, the bicoloring C and the designated monochrome 3-edge e_0. In the absence of theta-flows, this algorithm modifies the bipartition C of vertices of H to include edge e_0, and thus constructs a bicoloring of H. To verify this, it will suffice to prove the following claim.

Claim 5 *Given a 3-edge $e_0 = \{v_1, v_2, v_3\}$, Algorithm 1 will either find a theta-flow or produce a bicoloring.*

Proof If the return condition never occurs, then the proof of correctness in [5] verifies that Algorithm 1 produces a bicoloring. Otherwise, the only way for the error condition to occur is that $e = e_0$, and in this case it suffices to show that $P \cup Q \cup e$ contains a theta-flow. To verify this, note that $Q \backslash P$ is a path joining a node v of $P \cap Q$ to one of the vertices, say v_2 of e_0, that there is a path P_3 in P from v to v_3, and that there is a path P_1 in P from v to v_1. All three paths have the same parity since the sequence of color reversals along the flow results in reversing the colors of each of the three vertices $\{v_1, v_2, v_3\}$ of e_0, which was originally monochromatic. ■

This completes the proof of Theorem 4. ■

Recall that the absence of anti-Sterboul cycles implies both the absence of theta-flows and the absence of odd cycles of 2-edges. Therefore the sufficient condition of Theorem 4 is weaker than the sufficient condition of Theorem 3 so in the case of primitive hypergraphs, Theorem 4 is a stronger result than Theorem 3. Also, since the absence of odd cycles of 2-edges can be easily checked in polynomial time using breadth first search, the absence of theta-flows is of greater interest algorithmically.

Algorithm 1 : Modified Défossez algorithm applied to a primitive hypergraph

INPUT: A primitive hypergraph $H = (V, E)$ and a bipartition C such that 3-edge $e_0 \in E$ is the only monochromatic edge.
OUTPUT: A bicoloration of H or an Error message only if H has a theta-flow. $G(w)$ represents the predecessor node of w in a directed tree induced by the color reversals.

let $x_0 \in e_0$
$V_0 = \{x_0\}$
$E_0 = \phi$
$G(x_0) = e_0$
switch_color(x_0)
push(x_0, P)
while ($P \neq \phi$)
  $v = top(P)$
  if (there exists $e \in E, |e| \geq 2$, monochromatic such that $v \in e$)
    if ($e \backslash V_0 = \phi$)
      return Error("$P \cup Q \cup e_0$ contains a theta-flow")
    else
      let $w \in e \backslash V_0$
      if ($w \in e_0$)
        $Q = P$
      $V_0 = V_0 \cup w$
      $E_0 = E_0 \cup (vw)$
      $G(w) = e$
      switch_color(w)
      push(w, P)
  else
    pop(P)

4 Necessary and Sufficient Conditions

The Défossez algorithm, when successful in modifying the coloring to increase coverage, does so by a sequence of alternating color reversals along the edges of a connected subhypergraph A of H. When the algorithm fails, it only means that an anti-Sterboul cycle was present (or an odd cycle of 2-edges or a theta-flow in the primitive hypergraph problem). This does not preclude the existence of a connected subhypergraph that can be used to augment the coloring through alternating color reversals. Indeed it is plausible that the Défossez algorithm can be modified to allow backtracking in the event of encountering one of these structures, and proceed to locate an augmenting subhypergraph whenever there is one. Of course, this backtracking search would likely destroy the polynomial performance of the algorithm, which is not surprising since hypergraph bicolorability is known to be NP-complete. This leads to the following natural question: is hypergraph bicolorability equivalent to the ability to always augment a given coloring by alternating color reversals within a connected subhypergraph? This question can be answered in the affirmative for primitive hypergraphs, which we now show in Theorem 6 below, building from the following definitions. The statement and proof of this theorem are closely analogous to the statement and proof of the well-known necessary and sufficient conditions given by Berge [1] for maximal matchings in graphs.

A vertex v of a subhypergraph A of H is *internal to A* if and only if it is incident in H only to edges of A. An *augmenting vertex for a coloring C* in a primitive hypergraph H is a vertex v of $H \backslash C$, for which $C' = C \cup \{v\}$ is a coloring that covers more edges than C. An *augmenting pair v, w of vertices for a coloring C* in a primitive hypergraph H is a pair of adjacent vertices, with $v \notin C$ and $w \in C$ for which $C' = C \cup \{v\} \setminus \{w\}$ is a coloring that covers more edges than C (as defined in section 3).

A subset T of the vertex set of a hypergraph H can be *locally reduced into a coloring C'* if there is a set $S \subset T$ of pairwise non-adjacent vertices for which $T \backslash S = C'$ is a coloring that covers exactly the same edges as T. An *augmenting subhypergraph* in a hypergraph H relative to a coloring C is a connected subhypergraph A of H, for which reversing the color of all the vertices that are internal to A produces a set T which covers more edges than C and which can be locally reduced into a coloring C'.

Theorem 6 *A primitive hypergraph H is bicolorable if and only if for every coloring C of H that does not cover all the edges of H, there is an augmenting subhypergraph A of H relative to C.*

Proof By contraposition, the reverse implication is trivial. That is, if H is not bicolorable, then a maximal (in covering edges) coloring C is a coloring that does not cover all the edges of H and C cannot be augmented. For the forward implication, let C be a coloring of H that does not cover all the edges of H. If there is an augmenting vertex v for C, then the set of edges incident to v (the star of v) forms an augmenting subhypergraph. Similarly if there is an augmenting pair v, w for C, then the set of all edges incident to either v or w (the union of their stars) forms an augmenting subhypergraph. In either of these two situations, we are done, so in the remainder of this proof, we will assume that there is no augmenting vertex and that there is no augmenting pair of vertices for C.

Let C'' be a coloring that covers all the edges of H. Let U be the set of edges incident to at least one vertex in $C \oplus C''$. Let S (for solid) be the edges of U that are incident to any vertex of C and let D (for dashed) be those edges of U that are not incident to any vertex of C. Since C does not cover all edges of H, $|D| > 0$. Therefore there must be some connected component A of U that contains a positive number of dashed edges, that is, $|A \cap D| > 0$. Note that any vertex v of A that is in $C \oplus C''$ must be internal to A, since every edge incident to v is by definition in U, and hence also in A.

Verifying the following two claims completes the proof.

Claim 7 *Reversing the color (relative to C) of every internal vertex of A results in a subset T that can be locally reduced into a coloring C'.*

Proof It will suffice to prove that for every edge e of H, e has a vertex not in T, or there is a local reduction T' of T, such that e has a vertex not in T'. These individual local reductions (if any) will be performed on T to produce the coloring C'. The cases below describe how to handle edges of various different types.

Case 1 *Suppose e is not an edge of A.*

Then its vertex colors will not be changed, since none are internal to A. Hence its vertex colorings will be preserved in T, as will its coverage (or non-coverage).

Case 2 *Suppose e is an edge of A, and all vertices of e are internal to A.*

We consider subcases based on whether e is a 2-edge or a 3-edge.

Case 2a (Figure 3) Suppose e is a 2-edge. If both endpoints of e are in T, then they were both not in C before the changes, and exactly one vertex

v of e must not be in C''. If v has degree one in H, then v is an augmenting vertex for C, which is a contradiction. Therefore, v must also be incident to another edge $e' \in A$. If v is incident only to 2-edges $e_1 = \{v, w_1\}$ and $e_2 = \{v, w_2\}$ that are both in A, then both w_1 and w_2 must be in $C''\backslash C$, in which case v would serve as an augmenting vertex for C, which is a contradiction, so we can also assume that v is only incident to one other edge (besides e) $e' = \{v, w, x\}$ and e' is a 3-edge. Since C'' is a bicoloring, at least one vertex w of e' must be in C''. If $w \notin C$, or $x \notin C$, then v would serve as an augmenting vertex for C, which is a contradiction, so we can also assume that $w \in C$ and $x \in C$. Note that if w had degree one, then v and w would form an augmenting pair for C, which is a contradiction. It follows that w is incident to an edge $e'' = \{w, y\}$, and that $y \notin C$, and $y \notin C''$. Therefore $e'' \notin A$, so w is not internal to A, which implies that $w \in T$. Thus v can be removed from T without uncovering any edges, and giving e an uncolored vertex. It will be seen in the remaining cases that this is the only case that requires the local reduction technique. Also observe that no vertex adjacent to v can create the need for another local reduction since none of these are in $E\backslash(C'' \cup C)$.

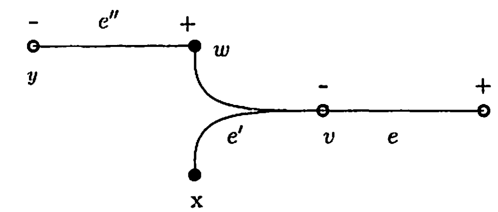

Figure 3: Case 2a. Points with filled interiors represent vertices in C, while others represent vertices not in C. Points labeled + represent vertices in C'', those labeled – represent vertices not in C'', and unlabeled points may or may not be in C''.

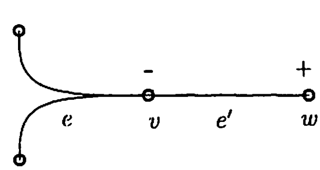

Figure 4: Case 2b

Case 2b (Figure 4) Suppose e is a 3-edge. If reversing the color of all three vertices of e results in all three vertices of e being in T, then they were all not in C before the changes, and at least one vertex v of e must not be in C''. In this event, v is also in another edge $e' = \{v, w\} \in A$, and $w \in C''$ implies that $w \notin C$. It follows that v would serve as an augmenting vertex for C, which is a contradiction. Therefore, we can assume that e has a vertex not in T. Here no local reduction is required for edge e.

Case 3 *Suppose e is an edge of A, and some vertex, say v, of e is not internal to A.*

Again, two subcases are considered.

Case 3a (Figure 5, left) Suppose $e = \{v, w\}$ is a 2-edge. Then $w \in C \oplus C''$, and so if both v and w are in T, then $v \in C$, but $w \notin C$, and so $w \in C''$. However, this would imply that $v \notin C''$, and therefore $v \in C \oplus C''$, which contradicts the assumption that v is not internal to A. Therefore v and w are not both in T, so no local reduction is needed.

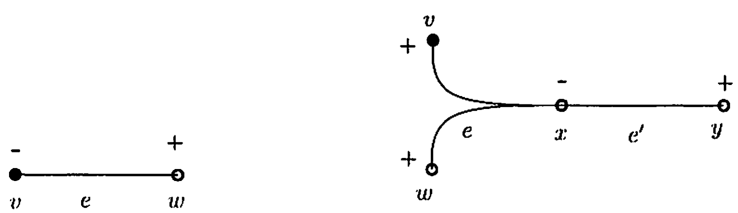

Figure 5: Cases 3a (left) and 3b (right).

Case 3b (Figure 5, right) Suppose $e = \{v, w, x\}$ is a 3-edge, v is not internal, and v, w, and x are all in T. Then $v \in C$, $v \in C''$, and either $w \in C \oplus C''$ or $x \in C \oplus C''$. Assume without loss of generality that $w \in C \oplus C''$, then w is internal to A, so $w \notin C$, and therefore $w \in C''$. From this we conclude that $x \notin C''$. Therefore $x \notin C$, since otherwise x would be internal, which would imply $x \notin T$, contradicting our assumption. But if $x \notin C$ and $x \notin C''$ and $x \in T$, then x must be internal to A, and hence x is in another edge $e' = \{x, y\}$ in A, with $y \in C''$ and $y \notin C$. In this event, vertex x is an augmenting vertex for C, which is a contradiction. Therefore v, w, and x are not all in T. As in cases 2b and 3a, no local reduction is needed.

The three cases above complete the proof of Claim 7. ■

Claim 8 *T (and therefore C') covers more edges than C.*

Proof If E is an edge of $D \cap A$, then, since C'' is a bicoloring, some vertex v of E is in $C'' \backslash C$, and since v is internal to A, v will be in T and E will be covered by T. If E is an edge of $S \cap A$, then some vertex v of E is in C. If v is also in C'' then v is not internal to A since v must be contained in a 2-edge $\{v, w\}$ where w is in neither C nor C'' which implies that $\{v, w\}$ is not in A. In this case v will be in T and so E will be covered by T. Otherwise, if v is not in C'' then some other vertex x of E must be in C''. If x is also in C then, as above, x is not internal to A and therefore x will be in T and E will be covered by T. Otherwise, if x is not in C, then x is internal to A and therefore x will be in T and E will be covered by T. Hence T covers all the edges of A whereas C covers only those of $A \backslash D$ and the claim is proved. ■

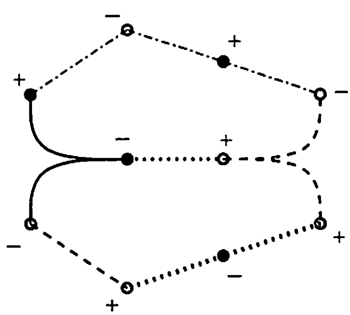

Figure 6: Points with a filled interior represent vertices in C, while others represent vertices not in C. Points labeled $+$ represent vertices in C'', and those labeled $-$ represent vertices not in C''. Solid edges are in S, dashed edges are in D, dotted edges are in $A \backslash (S \cup D)$, and dash-dotted edges are not in A.

Figure 6 illustrates this claim. The six lowest vertices are all internal vertices, so they will be reversed in T, which is shown in Figure 7. Then, as in Case 2a above, one vertex of T will be changed by a local reduction (Figure 8), to obtain coloring C', which covers more edges than T. This new coloring covers both the dashed edges without uncovering the solid edge.

This completes the proof of Theorem 6. ■

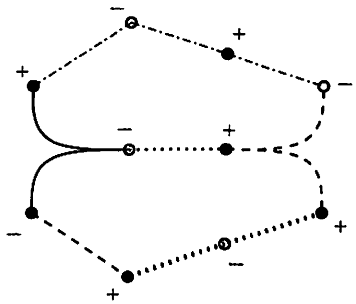

Figure 7: The coloring after reversing the colors of all internal nodes to obtain T.

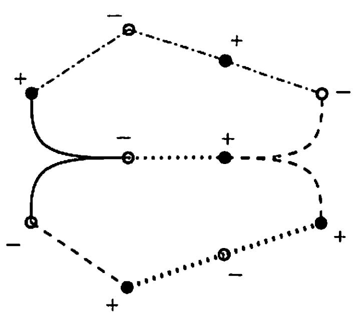

Figure 8: The coloring after locally adjusting T to obtain C'.

5 Conclusion

A simple sufficient condition for the bicolorability of primitive hypergraphs can be checked by a polynomial algorithm (a modification of the Défossez algorithm [5]). This algorithm further motivated the discovery of a necessary and sufficient condition for their bicolorability. Acknowledging the NP-completeness of hypergraph bicolorability, there are numerous interesting questions for exploration, both in determining properties that characterize classes of such hypergraphs, and developing algorithms for obtaining valid colorings. The duality inherent in primitive hypergraphs may also lead to avenues for future study.

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