Decomposition of a $2K_{10t+5}$ into H_3 Graphs

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Abstract

An H_3 graph is a multigraph on three vertices with double edges between two pairs of distinct vertices and a single edge between the third pair. To settle the H_3 decomposition problem completely, one needs to complete the decomposition of a $2K_{10t+5}$ into H_3 graphs. In this paper, we present two new construction methods for such decompositions, resulting in previously unknown decompositions for $v = 15, 25, 35, 45$ and two new infinite families.

1 Introduction

A graph can be decomposed into a collection of subgraphs such that every edge of the graph is contained in one of the subgraphs. Decomposing a graph into simple graphs has been well studied in the literature. For a well-written survey on the decomposition of a complete graph into simple graphs with small numbers of points and edges, see [1]. A *multigraph* is a graph where more than one edges between a pair of points is allowed. A complete multigraph λK_v ($\lambda > 1$) is a graph on v points with λ edges between every pair of distinct points. The decomposition of copies of a complete graph into proper multigraphs has not received much attention yet, see [2, 5, 6, 8, 9, 11, 12]. An example of a multigraph is so-called H_3 graph as described below.

1.1 H_3 Graphs

Definition 1 An H_3 graph is a multigraph on three vertices with double edges between two pairs of distinct vertices and a single edge between the third pair.

* The first author thanks the College of Charleston for granting a sabbatical.
† Thanks to The Citadel Foundation for their support.

If $V = \{a, b, c\}$ and there is a double edge between a and b and a double edge between b and c, then we denote the H_3 graph as $\langle a, b, c\rangle_{H_3}$ (see figure 1). An $H_3(v, \lambda)$ is a decomposition of a λK_v into H_3 graphs. In particular, an $H_3(5t, 2)$ is a decomposition of a $2K_{5t}$ graph into $\frac{2\times 5t\times(5t-1)}{2\times 5} = t(5t-1)$ H_3 graphs. In this paper we study the decomposition of a $2K_{5t}$ ($t \geq 1$ with t odd) into H_3 graphs, i.e., an $H_3(10t+5, 2)$.

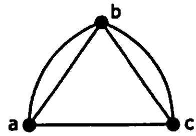

Figure 1: An H_3 Graph

Hurd and Sarvate [9] showed that the necessary condition for existence of an $H_3(v, 2)$ is $v = 5t$ or $v = 5t+1$. They prove that an $H_3(5t+1, 2)$ exists for $t \geq 1$, that there does not exist an $H_3(5, 2)$, and that an $H_3(10, 2)$ and an $H_3(15, 2)$ exist. The general case for an $H_3(5t, 2)$ where $t > 3$ was left open. Sarvate and Zhang [13] prove that an $H_3(10t, 2)$ exists for $t \geq 1$ (i.e. an $H_3(5t, 2)$ exists for t even). In this paper, we continue to work on this problem of $H_3(10t+5, 2)$s. We provide examples of such decompositions for $v = 15, 25, 35, 45$ and of two infinite families, but to obtain a complete solution is still an open problem.

The following definition and results from combinatorial designs are well-known (for example, see [3, 10, 15]).

Definition 2 A *1-factor* of a graph G is a set of pairwise disjoint edges which partition the vertex set. A *1-factorization* of a graph G is the set of 1-factors which partition the edge set of the graph.

A 1-factorization of K_{2n} contains $(2n-1)$ 1-factors as elements [14].

Definition 3 A *group divisible design* GDD$(g, u, k; \lambda_1, \lambda_2)$ is a collection of k-subsets (called blocks) of a set V of v points such that each point appears in r (called the *replication number*) blocks. The points of V are partitioned into u subsets (called *groups*) of size g each. Any two points within the same group are called *first associates* and appear together in λ_1 blocks; any two points not in the same group are called the *second associates* and appear together in λ_2 blocks.

If the blocks in a design can be partitioned into *resolution (or parallel) classes* such that the blocks of each class partition the set V, then the design is called *resolvable*. A resolvable GDD$(g, u, k; 0, \lambda)$ is denoted by RGDD$(g, u, k; 0, \lambda)$. Note that the number of resolution classes is equal to the replication number $r = \frac{\lambda(v-g)}{k-1}$.

Theorem 1 (Theorem 19.33 in [4]) A 3-RGDD (i.e., RGDD$(g, u, 3; 0, 1)$) of type g^u exists if and only if $u \geq 3$ and (1) $g \equiv 1, 5 \pmod 6$ and $u \equiv 3 \pmod 6$; (2)

$g \equiv 3(\text{mod } 6)$ and $u \equiv 1(\text{mod } 2)$; (3) $g \equiv 2, 4(\text{mod } 6)$ and $u \equiv 0(\text{mod } 3)$, except for $g^u \in \{2^3, 2^6\}$; (4) $g \equiv 0(\text{mod } 6)$, except for $g^u = 6^3$.

Let P_k denote a path with k vertices and $k-1$ edges, and let the notation (G, P_k)-design denote a decomposition of a graph G into P_ks.

Theorem 2 *[7] (Horton) There exists a resolvable $(\lambda K_n, P_3)$-design if and only if $n \equiv 0(\text{mod } 3)$ and $\lambda(n-1) \equiv 0(\text{mod } 4)$.*

Theorem 3 *[7] (Ushio) There exists a resolvable $(K_{m,n}, P_3)$-design if and only if $m + n \equiv 0(\text{mod } 3)$, $m \leq 2n \leq 4m$ and $3mn \equiv 0(\text{mod } 2(m+n))$.*

1.2 New Families of H_3 Decompositions

We begin with an observation.

Theorem 4 *In any $H_3(v, \lambda)$, at most one vertex can occur in the H_3 graphs having only degree four.*

Proof: If there are two (or more) vertices in the H_3 graphs having only degree four then they can not appear together in any of the H_3 graphs. □

Note that there can be at most one vertex in the H_3 graphs of an $H_3(5t, 2)$ having only degree four. The first construction of an infinite sequence of $H_3(10t + 5, 2)$ is given below. It produces, for example, an $H_3(15, 2)$ where no vertex occurs as a degree-four vertex in contrast to the known example of an $H_3(15, 2)$ (see next section) where exactly one vertex occurs as a degree-four vertex.

The first construction uses the following procedure.

Procedure PATHS-TO-H3-GRAPHS($R, n' = r$): Given $R = \{R_1, \ldots, R_r\}$, a collection of r resolution classes from a resolvable (K_v, P_3)-design. For each block $\{a, b, c\}$ in $R_i (i = 1, \ldots, r)$, construct H_3 graphs $\langle \infty_i, a, b \rangle_{H_3}$ and $\langle \infty_i, c, b \rangle_{H_3}$ where $\infty_i \notin V$. In the resulting H_3 graphs, there are two edges between ∞_i and any point in V and two edges between any pair of distinct points in V. The number of new points is $n' = r$.

Theorem 5 *If $v \equiv 9(\text{mod } 60)$, then an $H_3(\frac{7v-3}{4}, 2)$ exists. In other words, an $H_3(105t + 15, 2)$ exists for $t \geq 0$.*

Proof: If $v \equiv 9(\text{mod } 60)$, then $v \equiv 0(\text{mod } 3)$ and $v - 1 \equiv 0(\text{mod } 4)$. By Theorem 2, there exists a resolvable (K_v, P_3)-design. The number of resolution classes is $r = \frac{3(v-1)}{4} \equiv 1(\text{mod } 5)$. Perform PATHS-TO-H3-GRAPHS($R, n' = r$) and obtain an $H_3(n', 2)$ on the n' new points (note that $n' \equiv 1(\text{mod } 5)$). We have an $H_3(v + n', 2) = H_3(\frac{7v-3}{4}, 2)$. In other words, if we let $v = 60t + 9 (t \geq 0)$, an $H_3(\frac{7v-3}{4}, 2) = H_3(105t + 15, 2)$ exists. □

Theorem 6 *If $m+n \equiv 0(mod\ 3)$, $m \le 2n \le 4m$, $3mn \equiv 0(mod\ 2(m+n))$, and an $H_3(\frac{3mn}{2(m+n)}, 2)$ exists, then an $H_3(m+n+\frac{3mn}{2(m+n)}, 2)$ exists.*

Proof: By Theorem 3, there exists a resolvable $(K_{m,n}, P_3)$-design. The number of resolution classes is $r = \frac{3mn}{2(m+n)}$. Perform PATHS-TO-H3-GRAPHS $(R, n' = r)$ and obtain an $H_3(n', 2)$ on the n' new points. We have an $H_3(v + n', 2) = H_3(m+n+\frac{3mn}{2(m+n)}, 2)$. □

Corollary 1 *If $m = 5t (t \ge 2)$, $n = 10t$ and an $H_3(5t, 2)$ exists, then an $H_3(20t, 2)$ exists; If $m = 40t (t \ge 1)$, $n = 120t$ and an $H_3(45t, 2)$ exists, then an $H_3(205t, 2)$ exists; If $m = 25t(t \ge 1)$, $n = 100t$ and an $H_3(30t, 2)$ exists, then an $H_3(155t, 2)$ exists.*

Proof: If $m = 5t(t \ge 2)$ and $n = 10t$, then $r = \frac{3mn}{2(m+n)} = 5t$. If $m = 40t(t \ge 1)$ and $n = 120t$, then $r = 45t$. If $m = 25t(t \ge 1)$ and $n = 100t$, then $r = 30t$. Apply Theorem 6 to each of the three cases, we have an $H_3(20t, 2)$, an $H_3(205t, 2)$, and an $H_3(155t, 2)$. □

Here is another new procedure which also gives an infinite sequence of $H_3(10t+5, 2)$ and previously unknown $H_3(45, 2)$.

Procedure THREE-FOR-TWO-CLASSES$(P_1, \ldots, P_r, n' = \frac{3r}{2})$: Given r resolution classes $P_1, \ldots, P_r$ of a RGDD$(g, 3, 3; 0, 1)$ where $r = g$ is an even number and $n' = \frac{3r}{2}$ new points that are not in V. Note that there are g blocks in each class. Let G_1, G_2 and G_3 be the three groups, and $P_i = \{B_{i1}, \ldots, B_{ig}\}$ $(i = 1, \ldots, r)$ where $B_{ij} = \{b^1_{ij} \in G_1, b^2_{ij} \in G_2, b^3_{ij} \in G_3\}$ $(j = 1, \ldots, g)$ represents a block in P_i containing three points from each of the three groups (note that $\lambda_1 = 0$). Divide the resolution classes into $\frac{r}{2}$ pairs and perform the following procedures for the ith pair P_x and P_y $(i = 1, \ldots \frac{r}{2})$: For $j = 1, \ldots, g$, construct $\langle b^1_{xj}, b^2_{xj}, \infty^1_i\rangle_{H_3}$ and $\langle b^1_{xj}, b^3_{xj}, \infty^1_i\rangle_{H_3}$. Construct $\langle b^2_{xj}, b^3_{xj}, \infty^2_i\rangle_{H_3}$ and $\langle b^2_{yj}, b^1_{yj}, \infty^2_i\rangle_{H_3}$. Also, construct $\langle b^3_{yj}, b^1_{yj}, \infty^3_i\rangle_{H_3}$ and $\langle b^3_{yj}, b^2_{yj}, \infty^3_i\rangle_{H_3}$. Since each pair of resolution classes use 3 new points, the total number of new points is $\frac{3r}{2}$. In the resulting H_3 graphs from the procedure, there are two edges between any of the n' new point and any point in V. Since $\lambda_2 = 1$, there are two edges between any pair of points in V which are second associates.

Theorem 7 *If an $H_3(15t, 2)$ exists for $t \ge 1$, then an $H_3(45t, 2)$ exists.*

Proof: By Theorem 1 cases (3) and (4), a RGDD$(10t, 3, 3; 0, 1))$ exists. Note that $r = g = 10t$ and $v = 3g = 30t$. Perform THREE-FOR-TWO-CLASSES $(P_1, \ldots, P_r, n' = \frac{3r}{2})$. The total number of new points is $n' = 15t$. Since an $H_3(10t, 2)$ exists [13], obtain an $H_3(10t, 2)$ for the $10t$ points in each of the three groups. Also, obtain an $H_3(15t, 2)$ for the $15t$ new points. In the resulting H_3 graphs, there are two edges between any pair of distinct points in V, two edges between any point in V and any new point, and two edges between any pair of distinct new points. Thus, an $H_3(30t + 15t, 2) = H_3(45t, 2)$ exists. □

2 Small Examples

Now we provide previously unknown $H_3(10t+5,2)$ for $v = 25$ and 35. First we reproduce an example for $v = 15$, as it has exactly one vertex which occurs in the decomposition as degree four only. Recall that an $H_3(45,2)$ is produced in the previous section.

2.1 Difference Sets for $H_3(10t+5,2)$

These examples are based on *difference sets* and *difference families*, for example, see Stinson [15] for details.

Definition 4 Suppose $(G,+)$ is a finite group of order v with the identity element "0". Let $2 \le k < v$ be positive integers. A (v,k,λ) difference set in $(G,+)$ is a subset $D \subseteq G$ that satisfies the following properties: 1. $|D| = k$, 2. the multiset $[x-y : x,y \in D, x \neq y]$ contains every element in $G\backslash\{0\}$ exactly λ times. A difference family $[D_1,\ldots,D_l]$ is a collection of subsets of G satisfying properties 1 and 2 for some integer $l \ge 1$.

In many cases G is taken as $(Z_v,+)$, the group of integers modulo v.

Example 1 *A* $(7,3,1)$*-difference set in* $(Z_7,+)$ *is* $D = \{3,0,2\}$ *since* $0-3 = 4, 2-3 = 6, 3-0 = 3, 2-0 = 2, 3-2 = 1$ *and* $0-2 = 5$. *We get every element of* $Z_7\backslash\{0\}$ *exactly once as a difference of two distinct elements in D.*

We interpret the difference sets for an $H_3(5t,2)$ as follows: let $\langle a,b,c\rangle$ be a difference set corresponding to the H_3 graph $\langle a,b,c\rangle_{H_3}$. Then it gives the difference $|a-b|$ twice and the difference $|b-c|$ twice and the difference $|a-c|$ once. For example, for $v = 2(10s+7)+1 = 5(4s+3)$, the number of H_3 graphs is $2(4s+3)(10s+7)$. For all $s \ge 0$, let $V = \{\infty\} \cup (Z_2 \times Z_{10s+7})$.

To obtain $2(4s+3)(10s+7)$ H_3 graphs of an $H_3(5(4s+3),2)$, we can construct a difference family of $2(4s+3)$ difference sets and then expand the difference sets in the difference family modulo $(*, 10s+7)$. The difference sets in the difference family must provide each of the differences $(0,1)_i,\ldots,(0,5s+3)_i$ twice for $i = 1$ and 2 (i.e., differences from each of the two sets appear twice), respectively, and each of the differences $(1,i)$ twice, for $i = 0,\ldots,10s+6$, respectively.

Example 2 *A difference family solution for an* $H_3(15,2)$ *is given in the table below. Expand the* 6 *difference sets modulo* $(*,7)$*, respectively, we can obtain a total of* 42 H_3 *graphs for an* $H_3(15,2)$.

Difference sets	*Differences*
$\langle(0,1),(1,1),(1,4)\rangle$	$(1,0)$ *twice,* $(0,3)_2$ *twice,* $(1,3)$ *once*
$\langle(0,1),(1,3),(1,5)\rangle$	$(1,2)$ *twice,* $(0,2)_2$ *twice,* $(1,4)$ *once*
$\langle(0,1),(1,2),(0,3)\rangle$	$(1,1)$ *twice,* $(1,-1)=(1,6)$ *twice,* $(0,2)_1$ *once*
$\langle(0,1),(0,0),(0,3)\rangle$	$(0,1)_1$ *twice,* $(0,3)_1$ *twice,* $(0,2)_1$ *once*
$\langle(0,1),(1,6),(1,5)\rangle$	$(1,5)$ *twice,* $(0,1)_2$ *twice,* $(1,4)$ *once*
$\langle(0,1),\infty,(1,4)\rangle$	$(1,3)$ *once*

This example also shows that there can be exactly one vertex (the vertex ∞) in the H_3 graphs of an $H_3(15,2)$ having only degree four.

Example 3 *There are* 120 H_3 *blocks in an* $H_3(25,2)$. *We obtain a difference family solution of* 10 *difference sets and expand them modulo* $(*,12)$. *Let* $V = \{\infty\} \cup (Z_2 \times Z_{12}) == \{\infty\} \cup \{(0,0),\ldots,(0,11)\} \cup \{(1,0),\ldots,(1,11)\}$. *Note that we count the difference* $(0,6)_1$ *twice in the first difference set* $\langle(0,0),(0,2),(0,6)\rangle$ *because when we expand the difference set modulo* 12, *the* 7^{th} *block is* $\langle(0,6),(0,8),(0,0)\rangle$. *Similarly, we count the difference* $(0,6)_2$ *twice in the second difference set* $\langle(1,0),(1,1),(1,6)\rangle$.

Difference sets	*Differences*
$\langle(0,0),(0,2),(0,6)\rangle$	$(0,2)_1$ *twice*, $(0,4)_1$ *twice*, $(0,6)_1$ *twice*
$\langle(1,0),(1,1),(1,6)\rangle$	$(0,1)_2$ *twice*, $(0,5)_2$ *twice*, $(0,6)_2$ *twice*
$\langle(0,0),(0,5),(1,0)\rangle$	$(0,5)_1$ *twice*, $(1,-5)=(1,7)$ *twice*, $(1,0)$ *once*
$\langle(0,0),(0,1),(1,10)\rangle$	$(0,1)_1$ *twice*, $(1,9)$ *twice*, $(1,10)$ *once*
$\langle(1,0),(1,2),(0,9)\rangle$	$(0,2)_2$ *twice*, $(1,-7)=(1,5)$ *twice*, $(1,-9)=(1,3)$ *once*
$\langle(0,0),(1,11),(0,3)\rangle$	$(1,11)$ *twice*, $(1,8)$ *twice*, $(0,3)_1$ *once*
$\langle(0,0),(1,4),(0,3)\rangle$	$(1,4)$ *twice*, $(1,1)$ *twice*, $(0,3)_1$ *once*
$\langle(1,0),(1,3),(0,9)\rangle$	$(0,3)_2$ *twice*, $(1,-6)=(1,6)$ *twice*, $(1,-9)=(1,3)$ *once*
$\langle(1,0),(1,4),(0,2)\rangle$	$(0,4)_2$ *twice*, $(1,2)$ *twice*, $(1,-2)=(1,10)$ *once*
$\langle(0,0),\infty,(1,0)\rangle$	$(1,0)$ *once*

Example 4 *There are* 238 H_3 *blocks in an* $H_3(35,2)$. *We obtain a difference family solution of* 14 *difference sets and expand them modulo* $(*,17)$. *Let* $V = \{\infty\} \cup (Z_2 \times Z_{17}) == \{\infty\} \cup \{(0,0),\ldots,(0,16)\} \cup \{(1,0),\ldots,(1,16)\}$.

Difference sets	*Differences*
$\langle(0,0),(0,1),(1,2)\rangle$	$(0,1)_1$ *twice*, $(1,1)$ *twice*, $(1,2)$ *once*
$\langle(0,0),(0,2),(1,0)\rangle$	$(0,2)_1$ *twice*, $(1,-2)=(1,15)$ *twice*, $(1,0)$ *once*
$\langle(0,0),(0,3),(1,7)\rangle$	$(0,3)_1$ *twice*, $(1,4)$ *twice*, $(1,7)$ *once*
$\langle(0,0),(0,4),(1,13)\rangle$	$(0,4)_1$ *twice*, $(1,9)$ *twice*, $(1,13)$ *once*
$\langle(0,0),(0,5),(1,13)\rangle$	$(0,5)_1$ *twice*, $(1,8)$ *twice*, $(1,13)$ *once*
$\langle(0,0),(0,7),(1,0)\rangle$	$(0,7)_1$ *twice*, $(1,-7)=(1,10)$ *twice*, $(1,0)$ *once*
$\langle(0,0),(0,8),(1,7)\rangle$	$(0,8)_1$ *twice*, $(1,-1)=(1,16)$ *twice*, $(1,7)$ *once*
$\langle(0,0),(1,12),(0,6)\rangle$	$(1,12)$ *twice*, $(1,6)$ *twice*, $(0,6)_1$ *once*
$\langle(0,0),(1,3),(0,6)\rangle$	$(1,3)$ *twice*, $(1,-3)=(1,14)$ *twice*, $(0,6)_1$ *once*
$\langle(1,0),(1,3),(1,7)\rangle$	$(0,3)_2$ *twice*, $(0,4)_2$ *twice*, $(0,7)_2$ *once*
$\langle(1,0),(1,2),(1,10)\rangle$	$(0,2)_2$ *twice*, $(0,8)_2$ *twice*, $(0,10)_2=(0,7)_2$ *once*
$\langle(1,0),(1,1),(1,6)\rangle$	$(0,1)_2$ *twice*, $(0,5)_2$ *twice*, $(0,6)_2$ *once*
$\langle(1,0),(0,12),(1,6)\rangle$	$(1,-12)=(1,5)$ *twice*, $(1,-6)=(1,11)$ *twice*, $(0,6)_2$ *once*
$\langle(0,0),\infty,(1,2)\rangle$	$(1,2)$ *once*

3 Summary

We discussed the decomposition of a $2K_{10t+5}$ into H_3 graphs in this paper. Specifically, we provided difference family solutions for $v = 15, 25, 35$, and decomposition constructions for two infinite families. These show the existence of an $H_3(10t+5,2)$ for $v = 15, 25, 35, 45$ (by Theorem 7), 135 (by Theorem 7), 155 (by Corollary 1), 205 (by Corollary 1), 225 (by Theorem 5) and $10t+5$ for $t \equiv 1 (\mathrm{mod}\ 21)$ (by Theorem 5). To obtain a complete solution is still an open problem.

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