On Eulerian Regular Complete 5-Partite Graphs and a Cycle Decomposition Problem

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Abstract

An Eulerian graph G of size m is said to satisfy the Eulerian Cycle Decomposition Conjecture if the minimum number of odd cycles in a cycle decomposition of G is a, the maximum number of odd cycles in a cycle decomposition is b and ℓ is an integer such that $a \leq \ell \leq b$ where ℓ and m are of the same parity, then there is a cycle decomposition of G with exactly ℓ odd cycles. Several regular complete 5-partite graphs are shown to have this property.

Key Words: Eulerian graph, Eulerian Cycle Decomposition Conjecture, regular complete multipartite graph.

AMS Subject Classification: 05C38, 05C45.

1 Introduction

Eulerian graphs, namely those graphs containing an Eulerian circuit, were essentially characterized by Euler [7] in 1736 as connected graphs in which every vertex has even degree. In 1912 Veblen [9] presented his own characterization of Eulerian graphs as connected graphs possessing a decomposition into cycles. Consequently, every complete graph K_n of odd order $n \geq 3$ has a cycle decomposition. Furthermore, for every even integer $n \geq 4$ and a perfect matching M of K_n, the graph $K_n - M$ has a cycle decomposition. In 1981, Alspach [2] made a conjecture about the lengths of cycles that can exist in a cycle decomposition of these two classes of graphs. This conjecture was verified in 2012 by Bryant, Horsley and Pettersson [3].

Theorem 1.1 **(Bryant, Horsley and Pettersson)** *For an odd integer $n \geq 3$ and integers $m_1, m_2, \ldots, m_t$ such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$, the graph K_n can be decomposed into the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$. Also, for an even integer $n \geq 4$ and integers $m_1, m_2, \ldots, m_t$ such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ with $m_1 + m_2 + \cdots + m_t = \binom{n}{2} - \frac{n}{2}$, the graph K_n can be decomposed into a 1-factor and the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$.*

More recently, a problem involving cycle decompositions of Eulerian graphs was introduced in [4].

The Eulerian Cycle Decomposition Conjecture (ECDC)

An Eulerian graph G of size m is said to satisfy the Eulerian Cycle Decomposition Conjecture if a is the minimum number of odd cycles in a cycle decomposition of G, b is the maximum number of odd cycles in a cycle decomposition of G and for every integer ℓ such that $a \leq \ell \leq b$ where ℓ and m are of the same parity, there exists a cycle decomposition of G containing exactly ℓ odd cycles.

The major problem here is then: *Which Eulerian graphs satisfy the ECDC?* Not all Eulerian graphs satisfy the ECDC as Meszka [8] showed by giving an example of an Eulerian graph with maximum degree 4 and minimum degree 2 not having this property. A problem in this connection is determining an expression $f(n)$ such that if G is an Eulerian graph of order n with minimum degree $\delta(G) \geq f(n)$, then G satisfies the ECDC.

The complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ of order $n = \sum_{i=1}^{k} n_i$ has k partite sets $V_1, V_2, \ldots, V_k$ containing $n_1, n_2, \ldots, n_k$ vertices, respectively. If $n_i = r$ for each i $(1 \leq i \leq k)$, then this graph is denoted by $K_{k(r)}$. The graph $K_{k(r)}$ is therefore a $(k-1)r$-regular complete k-partite graph of order kr and size $\binom{k}{2}r^2$. Furthermore, $K_{k(r)}$ is the complete graph K_k if $r = 1$ and $K_{k(r)}$ is Eulerian if and only if k is odd or r is even.

It is an immediate consequence of Theorem 1.1 that the complete graph K_n satisfies the ECDC for every odd integer $n \geq 3$ and the graph $K_{k(2)}$ satisfies the ECDC for every integer $k \geq 2$. In [4] it was shown that every Eulerian complete 3-partite graph satisfies the ECDC. For each integer $k \geq 4$, it was shown in [1] that every regular complete k-partite graph $K_{k(r)}$, for which $k \equiv 1, 3 \pmod 6$ as well as those graphs $K_{k(r)}$, for which $k \equiv 0, 4 \pmod 6$ and $r \geq 2$ even, satisfy the ECDC. Therefore, the regular complete k-partite graph of smallest order for which the ECDC has not been verified is the 12-regular complete 5-partite graph, $K_{5(3)} = K_{3,3,3,3,3}$ of order 15 and size 90. We establish this result here. In addition, we show that the regular complete 5-partite graphs $K_{5(4)}$ and $K_{5(5)}$, as well as all graphs $K_{5(r)}$ for which $r \geq 6$ and $r \equiv 0 \pmod 3$, satisfy the ECDC. We begin with the graph $K_{5(3)}$. We refer to the book [5] for graph theoretic notation and terminology not described in this paper.

2 The Graph $K_{5(3)}$ and the ECDC

In order to show that $K_{5(3)}$ satisfies the ECDC, we need to know the maximum and minimum number of odd cycles in a cycle decomposition of $K_{5(3)}$. For the purpose of doing this, the following result will be useful. This result is a special case of a more general theorem of Colbourn, Hoffman and Rees [6].

Theorem 2.1 **(Colbourn, Hoffman and Rees)** *For integers* $k \geq 3$ *and* $r \geq 1$, *the graph* $K_{k(r)}$ *is* C_3*-decomposable if and only if* $(k-1)r$ *is even and* $\binom{k}{2}r^2$ *is a multiple of* 3.

For the graph $K_{5(3)}$, it is convenient here to denote its five partite sets by $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$, $D = \{d_1, d_2, d_3\}$ and $E = \{e_1, e_2, e_3\}$. (See Figure 1.) The following theorem determines the maximum and minimum numbers of odd cycles in a cycle decomposition of $K_{5(3)}$. For simplicity, we express a cycle $(u_1, u_2, \ldots, u_k, u_1)$, $k \geq 3$, as $(u_1, u_2, \ldots, u_k)$.

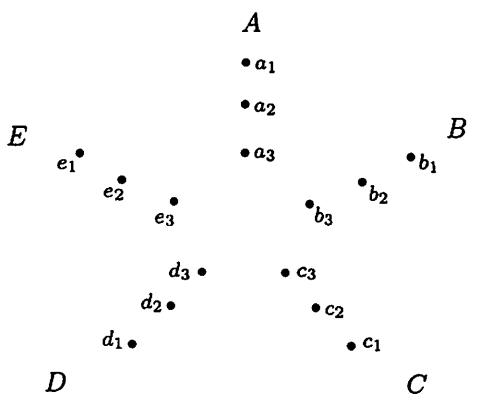

Figure 1: The five partite sets of $K_{5(3)}$

Proposition 2.2 *The maximum number of odd cycles in a cycle decomposition of* $K_{5(3)}$ *is* 30 *and the minimum number of odd cycles in a cycle decomposition of* $K_{5(3)}$ *is* 0.

Proof. Let $G = K_{5(3)}$. By Theorem 2.1, the graph G is C_3-decomposable. Since the size of G is 90, there exists a cycle decomposition of G into 30 triangles (odd cycles) and so the maximum number is 30. It remains to show that G can be decomposed into even cycles only.

Let $H_1 \cong K_{4(2)}$ be the subgraph of G with partite sets $\{b_2, b_3\}$, $\{c_2, c_3\}$, $\{d_2, d_3\}$ and $\{e_2, e_3\}$, let $H_2 \cong K_{1,1,1,1,3}$ be the subgraph of G induced by the set $\{b_1, c_1, d_1, e_1\} \cup \{a_1, a_2, a_3\}$ and let H_3 be the spanning subgraph of G with edge set $E(G) - (E(H_1) \cup E(H_2))$. Thus the graph G is decomposed into H_1, H_2 and H_3. Observe that H_3 is a bipartite graph with partite sets $V(H_1)$ and $V(H_2)$. Since $\deg_{H_3} v = 6$ if $v \in V(H_1) \cup \{b_1, c_1, d_1, e_1\}$ and $\deg_{H_3} v = 8$ if $v \in \{a_1, a_2, a_3\}$, it follows that H_3 is Eulerian.

As a consequence of Theorem 1.1, the graph $H_1 \cong K_{4(2)}$ has a cycle decomposition $\mathcal{D}$ into even cycles (or with exactly 0 odd cycles). The graph H_2 has a cycle decomposition $\mathcal{D}'$ into four even cycles, namely $(b_1, d_1, a_1, c_1, e_1, a_3) \cong C_6$, $(b_1, c_1, d_1, e_1) \cong C_4$, $(b_1, a_1, e_1, a_2) \cong C_4$ and $(c_1, a_2, d_1, a_3) \cong C_4$. Since H_3 is an Eulerian bipartite graph, H_3 has a cycle decomposition $\mathcal{D}''$ into even cycles. Then $\mathcal{D} \cup \mathcal{D}' \cup \mathcal{D}''$ is a cycle decomposition of G into even cycles. Hence the minimum number of odd cycles in a cycle decomposition of G is 0.

As a consequence of Proposition 2.2, to show that $K_{5(3)}$ satisfies the ECDC, it is required to show for every even integer ℓ with $2 \le \ell \le 28$ that there is a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(3)}$ with exactly ℓ odd cycles. First, we show the existence of a cycle decomposition $\mathcal{D}_{24}$. Figure 2 describes such a cycle decomposition of $K_{5(3)}$ into 26 cycles, exactly 24 of which are triangles T_i $(1 \le i \le 24)$, one an 8-cycle C_8 and the other a 10-cycle C_{10}.

$T_1 = (a_1, b_1, c_1)$	$T_2 = (b_1, c_2, d_3)$	$T_3 = (a_1, d_1, e_1)$	$T_4 = (a_2, d_2, e_2)$
$T_5 = (b_3, c_2, d_2)$	$T_6 = (a_3, d_3, e_3)$	$T_7 = (b_1, d_1, e_2)$	$T_8 = (b_1, d_2, e_1)$
$T_9 = (b_2, d_1, e_3)$	$T_{10} = (a_2, c_1, d_1)$	$T_{11} = (b_3, d_3, e_2)$	$T_{12} = (c_1, d_2, e_3)$
$T_{13} = (a_1, c_2, e_2)$	$T_{14} = (b_1, c_3, e_3)$	$T_{15} = (a_3, c_2, d_1)$	$T_{16} = (a_3, b_2, e_2)$
$T_{17} = (a_1, b_3, e_3)$	$T_{18} = (b_2, c_1, e_1)$	$T_{19} = (a_1, c_3, d_2)$	$T_{20} = (a_1, b_2, d_3)$
$T_{21} = (a_2, c_3, d_3)$	$T_{22} = (a_2, b_3, e_1)$	$T_{23} = (a_2, b_2, c_2)$	$T_{24} = (a_3, b_3, c_3)$

$$\begin{aligned} C_8 &= (a_3, d_2, b_2, c_3, e_2, c_1, d_3, e_1) \\ C_{10} &= (a_2, b_1, a_3, c_1, b_3, d_1, c_3, e_1, c_2, e_3). \end{aligned}$$

Figure 2: A cycle decomposition $\mathcal{D}_{24}$ of $K_{5(3)}$ having exactly 24 odd cycles

From the cycle decomposition $\mathcal{D}_{24}$, we construct a cycle decomposition $\mathcal{D}_{26}$ of $K_{5(3)}$ having exactly 26 odd cycles. In order to do this, we introduce some useful notation, For two edge-disjoint graphs F and G, let $F \cup G$ denote the graph induced by $E(F) \cup E(G)$. Figure 3 shows the subgraph $H = C_8 \cup C_{10}$ of $K_{5(3)}$ induced by $E(C_8) \cup E(C_{10})$, where C_8 (indicated with dashed lines) and C_{10} (indicated with solid lines) are the two even cycles in the cycle decomposition $\mathcal{D}_{24}$ of $K_{5(3)}$ in Figure 2. The Eulerian graph H can be decomposed into a 5-cycle $Q_1 = (c_1, e_2, c_3, e_1, d_3)$, a 7-cycle $Q_2 = (a_3, c_1, b_3, d_1, c_3, b_2, d_2)$ and a 6-cycle $Q_3 = (a_2, b_1, a_3, e_1, c_2, e_3)$. Then

$$\mathcal{D}_{26} = \{T_1, T_2, \ldots, T_{24}, Q_1, Q_2, Q_3\}$$

is a cycle decomposition of $K_{5(3)}$ having exactly 26 odd cycles.

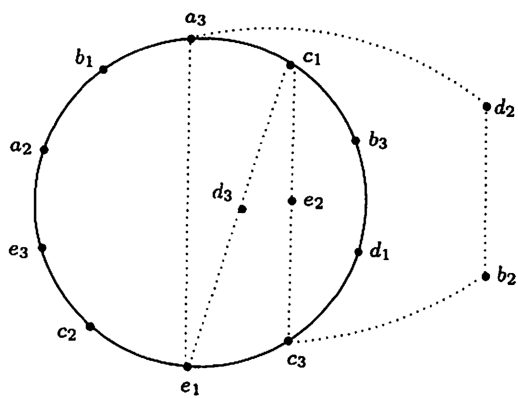

Figure 3: The subgraph $H = C_8 \cup C_{10}$ in $K_{5(3)}$

Next, from the cycle decomposition $\mathcal{D}_{24}$ in Figure 2, we construct a cycle decomposition $\mathcal{D}_{28}$ of $K_{5(3)}$ into 28 odd cycles. Consider the subgraph $F = T_{23} \cup T_{24} \cup C_8 \cup C_{10}$ of $K_{5(3)}$ in Figure 4, which is induced by $E(T_{23}) \cup E(T_{24}) \cup E(C_8) \cup E(C_{10})$, where T_{23}, T_{24}, C_8 and C_{10} are four cycles in the cycle decomposition $\mathcal{D}_{24}$ of $K_{5(3)}$ described in Figure 2. Then F can be decomposed into four triangles $Q_1 = (a_2, c_2, e_3)$, $Q_2 = (a_3, b_3, c_1)$, $Q_3 = (b_3, c_3, d_1)$, $Q_4 = (a_3, c_3, e_1)$, a 5-cycle $Q_5 = (a_2, b_1, a_3, d_2, b_2)$ and a 7-cycle $Q_6 = (b_2, c_2, e_1, d_3, c_1, e_2, c_3)$. Then

$$\mathcal{D}_{28} = \{T_1, T_2, \ldots, T_{22}, Q_1, Q_2, \ldots, Q_6\}$$

is a cycle decomposition of $K_{5(3)}$ into exactly 28 odd cycles and no even cycles.

Therefore, we have the following.

Proposition 2.3 *For each integer* $\ell = 24, 26, 28$, *there is a cycle decomposition of* $K_{5(3)}$ *with exactly* ℓ *odd cycles.*

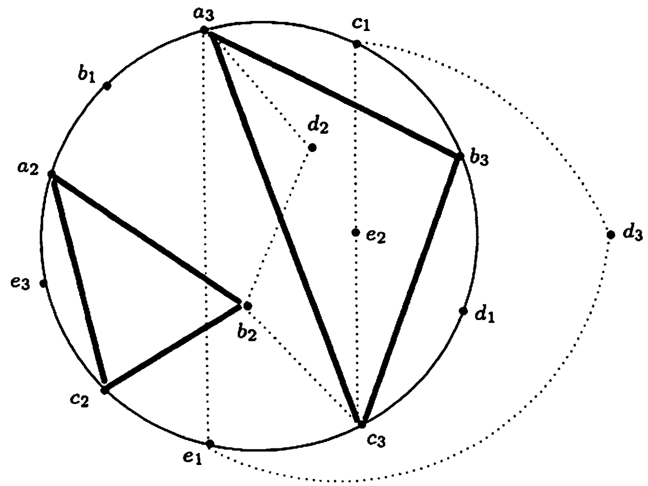

Figure 4: The subgraph $F = T_{23} \cup T_{24} \cup C_8 \cup C_{10}$ in $K_{5(3)}$

To show that $K_{5(3)}$ satisfies the ECDC, it therefore remains to show that for every even integer ℓ with $2 \le \ell \le 22$, there is a cycle decomposition of $K_{5(3)}$ with exactly ℓ odd cycles. Next, we establish this fact when $2 \le \ell \le 14$.

Proposition 2.4 *For each even integer ℓ with $2 \le \ell \le 14$, there is a cycle decomposition of $K_{5(3)}$ with exactly ℓ odd cycles.*

Proof. For $G = K_{5(3)}$, let $H_1 \cong K_{5(2)}$ be the the subgraph of G with partite sets $\{a_2, a_3\}$, $\{b_2, b_3\}$, $\{c_2, c_3\}$, $\{d_2, d_3\}$ and $\{e_2, e_3\}$, let $H_2 \cong K_5$ be the subgraph of G induced by the set $\{a_1, b_1, c_1, d_1, e_1\}$ and let H_3 be the spanning subgraph of G with edge set $E(G) - (E(H_1) \cup E(H_2))$. Observe that H_3 is a bipartite graph with partite sets $V(H_1)$ and $V(H_2)$. Since $\deg_{H_3} v = 4$ if $v \in V(H_1)$ and $\deg_{H_3} v = 8$ if $v \in V(H_2)$, it follows that H_3 is Eulerian. Thus the graph G is decomposed into H_1, H_2 and H_3. Since $K_{5(2)}$ satisfies the ECDC, for each even integer ℓ' with $0 \le \ell' \le 12$, there is a cycle decomposition $\mathcal{D}$ of $K_{5(2)}$ with exactly ℓ' odd cycles. The graph K_5 has a cycle decomposition $\mathcal{D}'$ into two 5-cycles. The Eulerian bipartite graph H_3 has a cycle decomposition $\mathcal{D}''$ into even cycles. For each even integer ℓ with $2 \le \ell \le 14$, write $\ell = 2 + \ell'$ for some even integer ℓ' with $0 \le \ell' \le 12$. Then $\mathcal{D} \cup \mathcal{D}' \cup \mathcal{D}''$ is a cycle decomposition of G with exactly ℓ odd cycles.

We now show the existence of a cycle decompositions $\mathcal{D}_\ell$ of $K_{5(3)}$ with exactly ℓ odd cycles for the remaining values of ℓ.

Proposition 2.5 *For each even integer ℓ with $10 \leq \ell \leq 22$, there is a cycle decomposition of $K_{5(3)}$ with exactly ℓ odd cycles.*

Proof. For $G = K_{5(3)}$, let $H_1 \cong K_{5(2)}$ be the the subgraph of G with partite sets $\{a_2, a_3\}$, $\{b_2, b_3\}$, $\{c_2, c_3\}$, $\{d_2, d_3\}$ and $\{e_2, e_3\}$ and let T_1, T_2, ..., T_{10} be the ten edge-disjoint triangles of G described below:

$$T_1 = (a_1, b_1, d_3) \quad T_2 = (e_1, c_1, d_3) \quad T_3 = (b_1, c_1, e_3)$$
$$T_4 = (a_1, d_1, e_3) \quad T_5 = (c_1, d_1, a_3) \quad T_6 = (b_1, e_1, a_3)$$
$$T_7 = (d_1, e_1, b_3) \quad T_8 = (c_1, a_1, b_3) \quad T_9 = (e_1, a_1, c_3)$$
$$T_{10} = (d_1, b_1, c_3).$$

Finally, let H_2 be the bipartite 4-regular subgraph G whose partite sets are $\{a_1,\ b_1,\ c_1,\ d_1,\ e_1\}$ and $\{a_2,\ b_2,\ c_2,\ d_2,\ e_2\}$ and whose edge set is $E(G) - [E(H_1) \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{10})]$. Then $\{H_1, H_2, T_1, T_2, \ldots, T_{10}\}$ is a decomposition of G. Since $K_{5(2)}$ satisfies the ECDC, it follows that, for each even integer ℓ' with $0 \leq \ell' \leq 12$, there is a cycle decomposition $\mathcal{D}$ of $K_{5(2)}$ with exactly ℓ' odd cycles. The Eulerian bipartite graph H_2 has a cycle decomposition $\mathcal{D}'$ into even cycles. For each even integer ℓ with $10 \leq \ell \leq 22$, write $\ell = 10 + \ell'$ for some even integer ℓ' with $0 \leq \ell' \leq 12$. Then $\mathcal{D}$, $\mathcal{D}'$ and $T_1, T_2, \ldots, T_{10}$ give rise to a cycle decomposition of G with exactly ℓ odd cycles.

As an example, we illustrate a cycle decomposition $\mathcal{D}_{22}$ of $K_{5(3)}$ into 25 cycles, exactly 22 of which are triangles T_i ($1 \leq i \leq 22$), one 4-cycle and two 10-cycles. We begin with the triangles T_1, T_2, T_3, T_4 shown in Figure 5(a). By rotating T_1, T_2, T_3, T_4 clockwise through an angle of $2\pi/5$ radians, we obtain T_5, T_6, T_7, T_8, which are edge-disjoint from T_1, T_2, T_3, T_4. We continue this three more times and obtain 20 edge-disjoint triangles $T_1, T_2, \ldots, T_{20}$ as follows:

$$T_1 = (e_2, a_1, b_2) \quad T_2 = (e_3, a_1, b_3) \quad T_3 = (d_3, a_1, c_3) \quad T_4 = (d_2, a_1, c_2)$$
$$T_5 = (a_2, b_1, c_2) \quad T_6 = (a_3, b_1, c_3) \quad T_7 = (e_3, b_1, d_3) \quad T_8 = (e_2, b_1, d_2)$$
$$T_9 = (b_2, c_1, d_2) \quad T_{10} = (b_3, c_1, d_3) \quad T_{11} = (a_3, c_1, e_3) \quad T_{12} = (a_2, c_1, e_2)$$
$$T_{13} = (c_2, d_1, e_2) \quad T_{14} = (c_3, d_1, e_3) \quad T_{15} = (b_3, d_1, a_3) \quad T_{16} = (b_2, d_1, a_2)$$
$$T_{17} = (d_2, e_1, a_2) \quad T_{18} = (d_3, e_1, a_3) \quad T_{19} = (c_3, e_1, b_3) \quad T_{20} = (c_2, e_1, b_2).$$

Figure 5(b) shows the two triangles T_{21}, T_{22} (whose edges are drawn with dashed lines) and the 4-cycle (whose edges are drawn with solid lines) in the cycle decomposition $\mathcal{D}_{22}$, where $T_{21} = (a_1, b_1, c_1)$, $T_{22} = (a_1, d_1, e_1)$ and $Q_1 = (b_1, d_1, c_1, e_1)$. Figures 5(c) and 5(d) show the two copies Q_2 and Q_3 of C_{10}, respectively, in the cycle decomposition $\mathcal{D}_{22}$, namely

$$Q_2 = (a_2, b_3, c_2, d_3, e_2, a_3, b_2, c_3, d_2, e_3)$$
$$Q_3 = (a_2, c_3, e_2, b_3, d_2, a_3, c_2, e_3, b_2, d_3).$$

Then $\mathcal{D}_{22} = \{T_1, T_2, \ldots, T_{22}, Q_1, Q_2, Q_3\}$ is cycle decomposition of $K_{5(3)}$ having exactly 22 odd cycles.

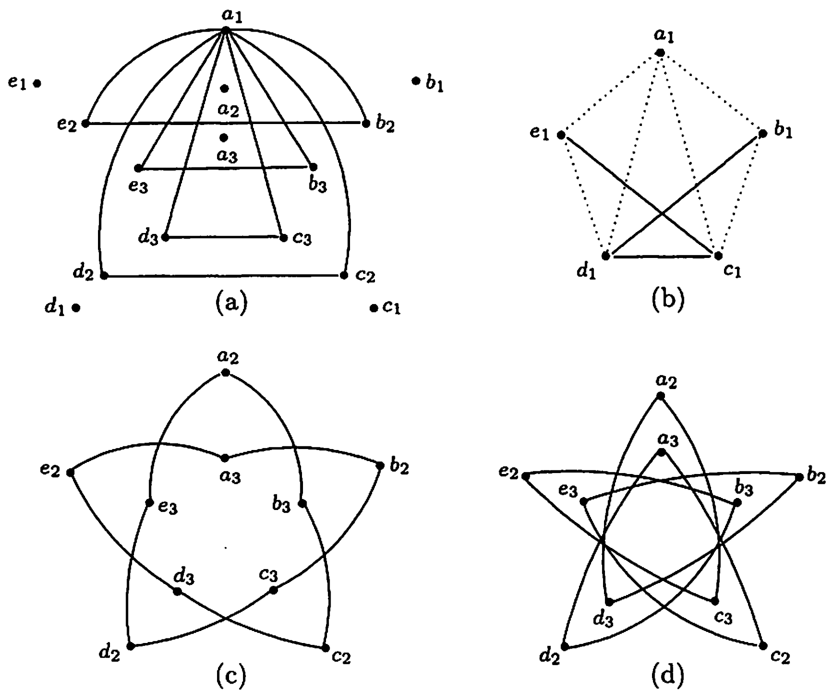

Figure 5: Illustrating a cycle decomposition $\mathcal{D}_{22}$ of $K_{5(3)}$

Combining Propositions 2.3, 2.4 and 2.5, we have the following theorem.

Theorem 2.6 *The graph $K_{5(3)}$ satisfies the ECDC.*

3 The Graphs $K_{5(4)}$ and $K_{5(5)}$ and the ECDC

In this section, we show that the two graphs $K_{5(4)}$ and $K_{5(5)}$ also satisfy the ECDC. We begin with $K_{5(4)}$. The graph $K_{5(4)}$ is a 16-regular complete 5-partite graph of order 20 and size $\binom{5}{2}4^2 = 160$. Therefore, the maximum number of odd cycles in a cycle decomposition of $K_{5(4)}$ is at most 52. We first show that the minimum number of odd cycles in a cycle decomposition of $K_{5(4)}$ is 0 and, in fact, $K_{5(4)}$ has a cycle decomposition with exactly ℓ odd cycles for every even integer ℓ with $0 \leq \ell \leq 40$.

Proposition 3.1 *For each even integer ℓ with $0 \leq \ell \leq 40$, the graph $K_{5(4)}$ has a cycle decomposition with exactly ℓ odd cycles.*

Proof. Let $G = K_{5(4)}$ with partite sets V_1, V_2, V_3, V_4, V_5, where

$$V_i = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}\} \text{ for } 1 \le i \le 5.$$

For $\ell = 0$, let $H_1 \cong K_{5(2)}$ be the subgraph of G with partite sets $\{v_{i,1}, v_{i,2}\}$ for $i = 1, 2, \ldots, 5$, let $H_2 \cong K_{5(2)}$ be the subgraph of G induced by $\{v_{i,3}, v_{i,4}\}$ for $i = 1, 2, \ldots, 5$, and let H_3 be the spanning subgraph of G with edge set $E(G) - (E(H_1) \cup E(H_2))$. The graph G can therefore be decomposed into H_1, H_2 and H_3, where H_3 is an Eulerian bipartite graph with partite sets $V(H_1)$ and $V(H_2)$. Since $K_{5(2)}$ satisfies the ECDC, there is a cycle decomposition of $K_{5(2)}$ into even cycles. Because H_3 is an Eulerian bipartite graph, H_3 has a cycle decomposition into even cycles. Let $\mathcal{D}_i$ be a cycle decomposition of H_i into even cycles for $i = 1, 2, 3$. Then $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ is a cycle decomposition of G into even cycles.

For each even integer ℓ with $2 \le \ell \le 32$, we can write $\ell = 2 + \ell'$ where ℓ' is an even integer with $0 \le \ell' \le 30$. Let $H_1 \cong K_{5(3)}$ be the subgraph of G with partite sets $V_i - \{v_{i,1}\}$ for $i = 1, 2, \ldots, 5$, let $H_2 \cong K_5$ be the subgraph of G induced by $\{v_{1,1}, v_{2,1}, \ldots, v_{5,1}\}$ and let H_3 be the spanning subgraph of G with edge set $E(G) - (E(H_1) \cup E(H_2))$. The graph G can be decomposed into H_1, H_2 and H_3, where H_3 is an Eulerian bipartite graph with partite sets $V(H_1)$ and $V(H_2)$. By Theorem 2.6, for each even integer ℓ' with $0 \le \ell' \le 30$, there is a cycle decomposition $\mathcal{D}'_{\ell'}$ of H_1 with exactly ℓ' odd cycles. The graph $H_2 \cong K_5$ has a cycle decomposition $\mathcal{D}''_2$ with exactly two odd cycles. Since H_3 is an Eulerian bipartite graph, H_3 has a cycle decomposition $\mathcal{D}_0$ into even cycles. Then $\mathcal{D}'_{\ell'} \cup \mathcal{D}''_2 \cup \mathcal{D}_0$ is a cycle decomposition $\mathcal{D}_\ell$ of G with exactly ℓ odd cycles.

For each even integer ℓ with $10 \le \ell \le 40$, we can write $\ell = 10 + \ell'$ where ℓ' is an even integer with $0 \le \ell' \le 30$. Let $H_1 \cong K_{5(3)}$ be the subgraph of G with partite sets $V_i - \{v_{i,1}\}$ for $i = 1, 2, \ldots, 5$. Next we construct 10 edge-disjoint triangles $T_1, T_2, \ldots, T_{10}$ as follows. We begin with the triangles $T_1 = (v_{1,4}, v_{2,1}, v_{5,1})$ and $T_2 = (v_{1,4}, v_{3,1}, v_{4,1})$ shown in Figure 6. By rotating T_1 and T_2 clockwise through an angle of $2\pi/5$ radians, we obtain T_3 and T_4 (drawn with dashed lines). We continue this three more times and obtain 10 edge-disjoint triangles $T_1, T_2, \ldots, T_{10}$

Finally, let H_2 be the bipartite Eulerian subgraph G whose partite sets are $\{v_{i,1} : 1 \le i \le 5\}$ and and $\{v_{i,2}, v_{i,3} : 1 \le i \le 5\}$ and whose edge set is

$$E(G) - [E(H_1) \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{10})].$$

Then $\{H_1, H_2, T_1, T_2, \ldots, T_{10}\}$ is a decomposition of G. Since $K_{5(3)}$ satisfies the ECDC, for each even integer ℓ' with $0 \le \ell' \le 30$, there is a cycle decomposition $\mathcal{D}$ of H_1 with exactly ℓ' odd cycles. The Eulerian bipartite graph H_2 has a cycle decomposition $\mathcal{D}'$ into even cycles. For each even integer ℓ with $10 \le \ell \le 40$, write $\ell = 10 + \ell'$ for some even integer ℓ'

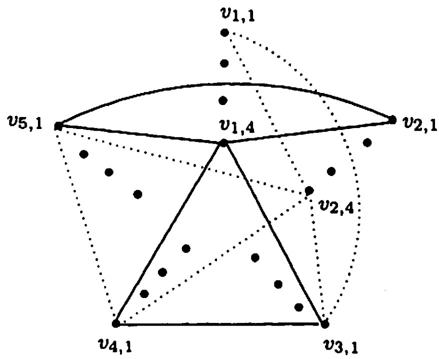

Figure 6: Constructing 10 edge-disjoint triangles $T_1, T_2, \ldots, T_{10}$ in $K_{5(4)}$

with $0 \leq \ell' \leq 30$. Then $\mathcal{D}$, $\mathcal{D}'$ and $T_1, T_2, \ldots, T_{10}$ give rise to a cycle decomposition of G with exactly ℓ odd cycles.

To show the existence of a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(4)}$ with exactly ℓ odd cycles for the remaining values of ℓ, it is convenient here to denote the five partite sets of $K_{5(4)}$ by

$$A = \{a_1, a_2, a_3, a_4\},\ B = \{b_1, b_2, b_3, b_4\},\ C = \{c_1, c_2, c_3, c_4\},$$
$$D = \{d_1, d_2, d_3, d_4\} \text{ and } E = \{e_1, e_2, e_3, e_4\}.$$

First, we show that $G = K_{5(4)}$ has a cycle decomposition $\mathcal{D}_{52}$ into 50 triangles and two 5-cycles. We begin by constructing 30 triangles T_1, T_2, …, T_{30}. Let $T_1, T_2, \ldots, T_6$ be the six triangles shown in Figure 7(a).

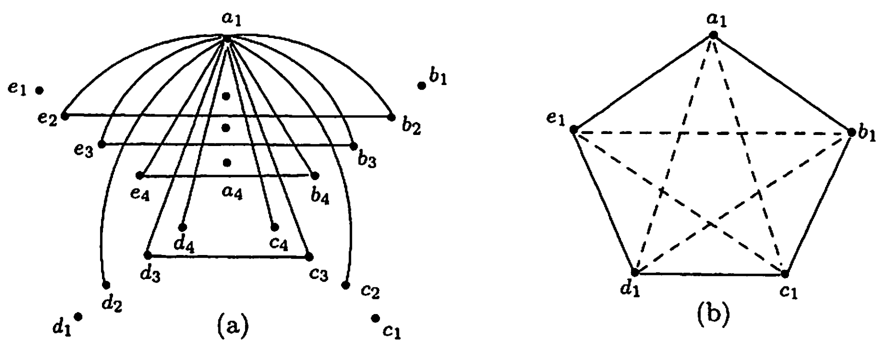

Figure 7: Constructing 32 edge-disjoint odd cycles in $K_{5(4)}$

By rotating $T_1, T_2, \ldots, T_6$ clockwise through an angle of $2\pi/5$ radians, we obtain $T_7, T_8, \ldots, T_{12}$, which are edge-disjoint from $T_1, T_2, \ldots, T_6$. We continue this three more times and obtain the 30 edge-disjoint triangles $T_1, T_2, \ldots, T_{30}$. The subgraph K_5, induced by $\{a_1, b_1, c_1, d_1, e_1\}$, can be decomposed into the two 5-cycles

$$F_1 = (a_1, b_1, c_1, d_1, e_1) \text{ and } F_2 = (a_1, c_1, e_1, b_1, d_1)$$

shown in Figure 7(b). Let

$$H = G - [E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{30}) \cup E(F_1) \cup E(F_2)].$$

Thus H is a 3-partite graph obtained from the complete 3-partite graph $K_{5,5,5}$ with partite sets $\{a_i, b_i, c_i, d_i, e_i\}$, $i = 2, 3, 4$, by removing the five triangles (a_2, a_3, a_4), (b_2, b_3, b_4), (c_2, c_3, c_4), (d_2, d_3, d_4) and (e_2, e_3, e_4). The graph H is illustrated in Figure 8, where the five triangles are drawn in dashed lines and the edges of H are not drawn. The graph H can be decomposed into 20 triangles $Q_1, Q_2, \ldots, Q_{20}$ as follows:

$$(a_4, b_2, e_3),\ (a_4, e_2, b_3),\ (a_4, c_2, d_3),\ (a_4, d_2, c_3),$$
$$(b_4, a_2, c_3),\ (b_4, c_2, a_3),\ (b_4, d_2, e_3),\ (b_4, e_2, d_3),$$
$$(c_4, b_2, d_3),\ (c_4, d_2, b_3),\ (c_4, a_2, e_3),\ (c_4, e_2, a_3),$$
$$(d_4, c_2, e_3),\ (d_4, e_2, c_3),\ (d_4, b_2, a_3),\ (d_4, a_2, b_3),$$
$$(e_4, a_2, d_3),\ (e_4, d_2, a_3),\ (e_4, b_2, c_3),\ (e_4, c_2, b_3).$$

Then $\mathcal{D}_{52} = \{F_1, F_2, T_1, T_2, \ldots, T_{30}, Q_1, Q_2, \ldots, Q_{20}\}$ is a cycle decomposition of G into 52 odd cycles.

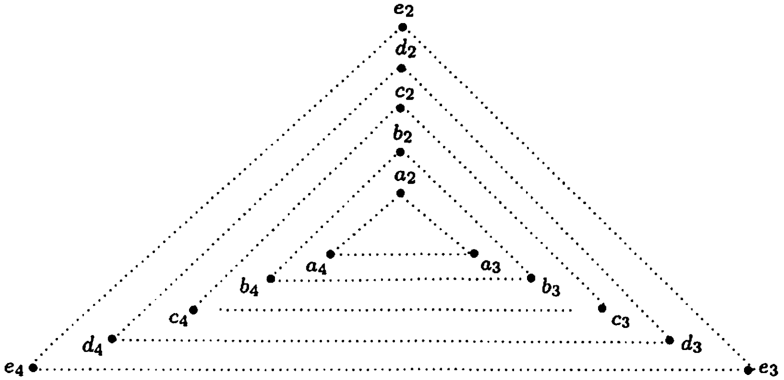

Figure 8: The graph H in $K_{5(4)}$

From the cycle decomposition $\mathcal{D}_{52}$, we construct a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(4)}$ having exactly ℓ odd cycles for each even integer ℓ with $42 \leq \ell \leq 50$. For each $i = 2, 3, 4$, consider four 3-cycles (a_i, b_i, d_1), (b_i, c_i, e_1),

(c_i, d_i, a_1) and (a_i, d_i, e_1) in $\mathcal{D}_{52}$. These four 3-cycles form the graph F shown in Figure 9. Then F can be decomposed into

(1) a 5-cycle $(a_i, b_i, c_i, d_i, e_1)$ and a 7-cycle $(a_i, d_1, b_i, e_1, c_i, a_1, d_i)$ or

(2) three 4-cycles (a_i, b_i, c_i, d_i), (a_i, d_1, b_i, e_1) and (e_1, c_i, a_1, d_i).

By replacing four triangles in $\mathcal{D}_{52}$ by either a 5-cycle and a 7-cycle or three 4-cycles, we can construct a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(4)}$ having exactly ℓ odd cycles for each even integer ℓ with $42 \le \ell \le 50$.

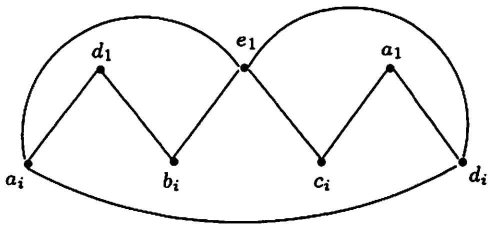

Figure 9: The graph F in $K_{5(4)}$

We now show that $K_{5(5)}$ satisfies the ECDC. The graph $K_{5(5)}$ is a 20-regular complete 5-partite graph of order 25 and size $\binom{5}{2}5^2 = 250$. Therefore, the maximum number of odd cycles in a cycle decomposition of $K_{5(5)}$ is at most 82. We first show that the minimum number of odd cycles in a cycle decomposition of $K_{5(5)}$ is 0 and, in fact, $K_{5(5)}$ has a cycle decomposition with exactly ℓ odd cycles for every even integer ℓ with $0 \le \ell \le 70$.

Proposition 3.2 *For each even integer ℓ with $0 \le \ell \le 70$, the graph $K_{5(5)}$ has a cycle decomposition with exactly ℓ odd cycles.*

Proof. Let $G = K_{5(5)}$ with partite sets V_1, V_2, V_3, V_4, V_5, where

$$V_i = \{v_{i,j} : 1 \le j \le 5\} \text{ for } 1 \le i \le 5.$$

For each even integer ℓ with $0 \le \ell \le 42$, we can write $\ell = \ell' + \ell''$ where ℓ' and ℓ' are even integers with $0 \le \ell' \le 30$ and $0 \le \ell'' \le 12$. Let $H_1 \cong K_{5(3)}$ be the subgraph of G with partite sets $\{v_{i,1}, v_{i,2}, v_{i,3}\}$ for $1 \le i \le 5$, let $H_2 \cong K_{5(2)}$ be the subgraph of G induced by $\{v_{i,4}, v_{i,5}\}$ for $1 \le i \le 5$ and let H_3 be the spanning subgraph of G with edge set $E(G) - (E(H_1) \cup E(H_2))$. The graph G is therefore decomposed into H_1, H_2 and H_3, where H_3 is an Eulerian bipartite graph with partite sets $V(H_1)$ and $V(H_2)$. By Theorem 2.6, for each even integer ℓ' with $0 \le \ell' \le 30$, there is a cycle decomposition $\mathcal{D}'_{\ell'}$ of H_1 with exactly ℓ' odd cycles. Since $K_{5(2)}$ satisfies the ECDC, for each even integer ℓ' with $0 \le \ell'' \le 12$, there is a cycle decomposition $\mathcal{D}''_{\ell''}$ of H_2 having exactly ℓ'' odd cycles. Because H_3 is an Eulerian bipartite graph, H_3 has a cycle decomposition $\mathcal{D}_0$ into even cycles. Then $\mathcal{D}'_{\ell'} \cup \mathcal{D}''_{\ell''} \cup \mathcal{D}_0$ is a cycle decomposition of G having exactly ℓ odd cycles.

For each even integer ℓ with $40 \le \ell \le 70$, we can write $\ell = 40 + \ell'$ where ℓ' is an even integer with $0 \le \ell' \le 30$. Let $H_1 \cong K_{5(3)}$ be the subgraph of G with partite sets $\{v_{i,3}, v_{i,4}, v_{i,5}\}$ for $1 \le i \le 5$. Next, we construct 40 edge-disjoint triangles $T_1, T_2, \ldots, T_{40}$ as follows.

We begin with the eight triangles $T_1, T_2, \ldots, T_8$ of Figure 10, namely

$$T_1 = (v_{1,5}, v_{2,1}, v_{5,1}),\ T_2 = (v_{1,5}, v_{2,2}, v_{5,2}),$$
$$T_3 = (v_{1,5}, v_{3,1}, v_{4,1}),\ T_4 = (v_{1,5}, v_{3,2}, v_{4,2}),$$
$$T_5 = (v_{1,4}, v_{2,1}, v_{5,2}),\ T_6 = (v_{1,4}, v_{2,2}, v_{5,1}),$$
$$T_7 = (v_{1,4}, v_{3,1}, v_{4,2}),\ T_8 = (v_{1,4}, v_{3,2}, v_{4,1}).$$

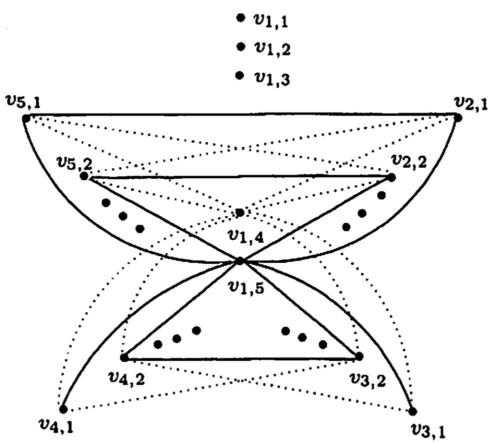

Figure 10: The eight edge-disjoint triangles $T_1, T_2, \ldots, T_8$ in $K_{5(5)}$

By rotating these eight triangles $T_1, T_2, \ldots, T_8$ clockwise through an angle of $2\pi/5$ radians, we obtain another eight edge-disjoint triangles T_9, $T_{10}, \ldots, T_{16}$. We continue this three more times and obtain 40 edge-disjoint triangles T_1, T_2, $\ldots$, T_{40} Next, let H_2 be the bipartite Eulerian subgraph G whose partite sets are

$$\{v_{j,1}, v_{j,2} : 1 \le j \le 5\} \text{ and } \{v_{j,3}, v_{j,4}, v_{j,5} : 1 \le j \le 5\}$$

and whose edge set is

$$E(G) - [E(H_1) \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{40})].$$

Then $\{H_1, H_2, T_1, T_2, \ldots, T_{40}\}$ is a decomposition of G. Since $K_{5(3)}$ satisfies the ECDC, for each even integer ℓ' with $0 \leq \ell' \leq 30$, there is a cycle decomposition $\mathcal{D}'_{\ell'}$ of H_1 with exactly ℓ' odd cycles. The Eulerian bipartite graph H_2 has a cycle decomposition $\mathcal{D}_0$ into even cycles. Then $\mathcal{D}'_{\ell'}$, $\mathcal{D}_0$ and $T_1, T_2, \ldots, T_{40}$ give rise to a cycle decomposition of G with exactly $\ell = 40 + \ell'$ odd cycles. ■

To show the existence of a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(5)}$ with exactly ℓ odd cycles for the remaining values of ℓ, it is again convenient here to denote the five partite sets of $K_{5(5)}$ by $A = \{a_1, a_2, \ldots, a_5\}$, $B = \{b_1, b_2, \ldots, b_5\}$, $C = \{c_1, c_2, \ldots, c_5\}$, $D = \{d_1, d_2, \ldots, d_5\}$ and $E = \{e_1, e_2, \ldots, e_5\}$.

First, we show that $G = K_{5(5)}$ has a cycle decomposition $\mathcal{D}_{82}$ into 80 triangles and two 5-cycles. We begin by constructing 40 triangles T_1, T_2, …, T_{40}. Let $T_1, T_2, \ldots, T_8$ be the eight triangles obtained by joining the vertex a_1 to the two incident edges of each edge in the set

$$\{b_ie_i, c_id_i : i = 2, 3, 4, 5\}.$$

In a manner similar to that described in the case of $K_{5(4)}$ (see Figure 7(a)), we then rotate $T_1, T_2, \ldots, T_8$ clockwise through an angle of $2\pi/5$ radians, producing the eight triangles $T_9, T_9, \ldots, T_{16}$ that are edge-disjoint from $T_1, T_2, \ldots, T_8$. We continue this three more times and obtain the 40 edge-disjoint triangles $T_1, T_2, \ldots, T_{40}$. The subgraph K_5, induced by $\{a_1, b_1, c_1, d_1, e_1\}$, can be decomposed into the two 5-cycles

$$F_1 = (a_1, b_1, c_1, d_1, e_1) \text{ and } F_2 = (a_1, c_1, e_1, b_1, d_1).$$

Let $H = G - [E(T_1) \cup E(T_2) \cup \cdots \cup E(T_{40}) \cup E(F_1) \cup E(F_2)]$. Thus H is a 4-partite graph obtained from the complete 4-partite graph $K_{5,5,5,5} = K_{4(5)}$ with partite sets $\{a_i, b_i, c_i, d_i, e_i\}$, $i = 2, 3, 4, 5$, by removing the five copies of K_4 induced by $A - \{a_1\}$, $B - \{b_1\}$, $C - \{c_1\}$, $D - \{d_1\}$ and $E - \{e_1\}$, respectively. The graph H can be decomposed into 40 triangles $Q_1, Q_2, \ldots, Q_{40}$ as follows:

$$(a_3, b_4, c_5),\ (b_3, c_4, d_5),\ (c_3, d_4, e_5),\ (d_3, e_4, a_5),\ (e_3, a_4, b_5),$$
$$(a_3, e_4, d_5),\ (b_3, a_4, e_5),\ (c_3, b_4, a_5),\ (d_3, c_4, b_5),\ (e_3, d_4, c_5),$$
$$(a_2, c_3, e_4),\ (a_2, d_3, b_4),\ (a_2, e_3, d_5),\ (a_2, b_3, c_5),\ (a_2, c_4, e_5),\ (a_2, d_4, b_5),$$
$$(b_2, d_3, a_4),\ (b_2, e_3, c_4),\ (b_2, a_3, e_5),\ (b_2, c_3, d_5),\ (b_2, d_4, a_5),\ (b_2, e_4, c_5),$$
$$(c_2, e_3, b_4),\ (c_2, a_3, d_4),\ (c_2, b_3, a_5),\ (c_2, d_3, e_5),\ (c_2, e_4, b_5),\ (c_2, a_4, d_5),$$
$$(d_2, a_3, c_4),\ (d_2, b_3, e_4),\ (d_2, c_3, b_5),\ (d_2, e_3, a_5),\ (d_2, a_4, c_5),\ (d_2, b_4, e_5),$$
$$(e_2, b_3, d_4),\ (e_2, c_3, a_4),\ (e_2, d_3, c_5),\ (e_2, a_3, b_5),\ (e_2, b_4, d_5),\ (e_2, c_4, a_5).$$

Then $\mathcal{D}_{82} = \{F_1, F_2, T_1, T_2, \ldots, T_{40}, Q_1, Q_2, \ldots, Q_{40}\}$ is a cycle decomposition of G into 82 odd cycles.

As in the case of $K_{5(4)}$, we can construct (from the cycle decomposition $\mathcal{D}_{82}$) a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(5)}$ having exactly ℓ odd cycles for each even integer ℓ with $72 \le \ell \le 80$. For each $i = 2, 3, 4$, consider four 3-cycles (a_i, b_i, d_1), (b_i, c_i, e_1), (c_i, d_i, a_1) and (a_i, d_i, e_1) in $\mathcal{D}_{82}$. These four 3-cycles form the graph F shown in Figure 9. Then F can be decomposed into (1) a 5-cycle $(a_i, b_i, c_i, d_i, e_1)$ and a 7-cycle $(a_i, d_1, b_i, e_1, c_i, a_1, d_i)$ or (2) three 4-cycles (a_i, b_i, c_i, d_i), (a_i, d_1, b_i, e_1) and (e_1, c_i, a_1, d_i). By replacing four triangles in $\mathcal{D}_{82}$ by either a 5-cycle and a 7-cycle or three 4-cycles, we can construct a cycle decomposition $\mathcal{D}_\ell$ of $K_{5(5)}$ having exactly ℓ odd cycles for each even integer ℓ with $72 \le \ell \le 80$.

In summary, we have the following result.

Theorem 3.3 *The graphs $K_{5(4)}$ and $K_{5(5)}$ satisfy the ECDC.*

4 The Graphs $K_{5(r)}$ when $r \equiv 0 \pmod 3$ and the ECDC

Now that it has been shown that $K_{5(3)}$, $K_{5(4)}$ and $K_{5(5)}$ satisfy the ECDC, the next graph to consider is $K_{5(6)}$. Here we show, in fact, that for every integer $r \ge 6$ and $r \equiv 0 \pmod 3$, the graph $K_{5(r)}$ satisfies the ECDC. For an integer $r \ge 3$ and $r \equiv 0 \pmod 3$, let $r = 3t$ for some positive integer t. The graph $K_{5(3t)}$ is a $(12t)$-regular complete 5-partite graph of order $15t$ and size $\binom{5}{2}(3t)^2 = 90t^2$.

Theorem 4.1 *For each integer $r \ge 3$ with $r \equiv 0 \pmod 3$, the graph $K_{5(r)}$ satisfies the ECDC.*

Proof. By Theorem 2.6, we may assume that $r \ge 6$. Let $G = K_{5(r)}$ with partite sets V_1, V_2, V_3, V_4, V_5, where $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,r}\}$ for $1 \le i \le 5$. Since $r \equiv 0 \pmod 3$ and $r \ge 6$, it follows that $r = 3t$ for some integer $t \ge 2$. By Theorem 2.1, the graph G is C_3-decomposable. Since the size of G is $\binom{5}{2}(3t)^2 = 90t^2$, there exists a cycle decomposition of G into $30t^2$ triangles (odd cycles). Hence the maximum number of odd cycles in a cycle decomposition of G is $30t^2$. Next, we show that G has a cycle decomposition with exactly ℓ odd cycles for each even integer ℓ with $0 \le \ell \le 30t^2$.

By Theorem 2.1, the graph $K_{5(3)}$ is C_3-decomposable. Since the size of $K_{5(3)}$ is 90, there exists a cycle decomposition of $K_{5(3)}$ into 30 triangles. Replacing each triangle in this decomposition by $K_{t,t,t} = K_{3(t)}$ results in a $K_{3(t)}$-decomposition of G into 30 copies of $K_{3(t)}$, which we denote by $H_1, H_2, \ldots, H_{30}$. By Theorem 2.6, each graph H_i satisfies the ECDC and so

- if t is even, then for each even integer $\ell_i \in \{0, 2, \dots, t^2\}$, there exists a cycle decomposition $\mathcal{D}_i$ of H_i with exactly ℓ_i odd cycles and
- if t is odd, then for each odd integer $\ell_i \in \{1, 3, \dots, t^2\}$, there exists a cycle decomposition $\mathcal{D}_i$ of H_i with exactly ℓ_i odd cycles.

We consider two cases, according to whether t is even or t is odd.

Case 1. *$t \geq 2$ is even.* Since ℓ is an even integer with $0 \leq \ell \leq 30t^2$, we can write $\ell = \ell_1 + \ell_2 + \cdots + \ell_{30}$ where $\ell_i \in \{0, 2, \dots, t^2\}$ and $i = 1, 2, \dots, 30$. Let $\mathcal{D}_i$ be a cycle decomposition of H_i with exactly ℓ_i odd cycles for $1 \leq i \leq 30$. Then $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_{30}$ is a cycle decomposition of G with exactly ℓ odd cycles.

Case 2. *$t \geq 3$ is odd.* First, assume that ℓ is an even integer with $0 \leq \ell \leq 30$. Let $H_1 \cong K_{5(3(t-1))}$ be the subgraph of G with partite sets $V_i - \{v_{i,1}, v_{i,2}, v_{i,3}\}$ for $1 \leq i \leq 5$, let $H_2 \cong K_{5(3)}$ be the subgraph of G with partite sets $\{v_{i,1}, v_{i,2}, v_{i,3}\}$ for $1 \leq i \leq 5$ and let H_3 be the spanning subgraph of G with edge set $E(G) - (E(H_1) \cup E(H_2))$. The graph G can be decomposed into H_1, H_2 and H_3, where H_3 is a bipartite graph with partite sets $V(H_1)$ and $V(H_2)$. Since $\deg_{H_3} v = 12$ if $v \in V(H_1)$ and $\deg_{H_3} v = 12(t-1)$ if $v \in V(H_2)$, it follows that H_3 is Eulerian. Since $t - 1 \geq 2$ is even, it follows by Case 1 that H_1 has a cycle decomposition $\mathcal{D}'_0$ of G into even cycles. By Theorem 2.6, for each even integer ℓ with $0 \leq \ell \leq 30$, there is a cycle decomposition $\mathcal{D}''_\ell$ of H_2 with exactly ℓ odd cycles. Since H_3 is an Eulerian bipartite graph, it follows that H_3 has a cycle decomposition $\mathcal{D}_0$ into even cycles. Then $\mathcal{D}'_0 \cup \mathcal{D}''_\ell \cup \mathcal{D}_0$ is a cycle decomposition of G having exactly ℓ odd cycles.

Next, assume that ℓ is an even integer with $30 \leq \ell \leq 30t^2$. Then we can write $\ell = \ell_1 + \ell_2 + \cdots + \ell_{30}$ where $\ell_i \in \{1, 3, \dots, t^2\}$ and $i = 1, 2, \dots, 30$. Let $\mathcal{D}_i$ be a cycle decomposition of H_i with exactly ℓ_i odd cycles for $1 \leq i \leq 30$. Then $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_{30}$ is a cycle decomposition of G with exactly ℓ odd cycles.

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