On a Graph Theoretic Division Algorithm and Maximal Decompositions of Graphs

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Abstract

For two graphs H and G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called an H-maximal k-decomposition if $H_i \cong H$ for $1 \leq i \leq k$ and R contains no subgraph isomorphic to H. Let $\text{Min}(G, H)$ and $\text{Max}(G, H)$ be the minimum and maximum k, respectively, for which G has an H-maximal k-decomposition. A graph H without isolated vertices is said to possess the intermediate decomposition property if for each connected graph G and each integer k with $\text{Min}(G, H) \leq k \leq \text{Max}(G, H)$, there exists an H-maximal k-decomposition of G. For a set S of graphs and a graph G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called an S-maximal k-decomposition if $H_i \cong H$ for some $H \in S$ for each integer i with $1 \leq i \leq k$ and R contains no subgraph isomorphic to any subgraph in S. Let $\text{Min}(G, S)$ and $\text{Max}(G, S)$ be the minimum and maximum k , respectively, for which G has an S-maximal k-decomposition. A set S of graphs without isolated vertices is said to possess the intermediate decomposition property if for every connected graph G and each integer k with $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. While all those graphs of size 3 have been determined that possess the intermediate decomposition property as have all sets consisting of two such graphs, here all remaining sets of graphs having size 3 that possess the intermediate decomposition property are determined.

Key Words: maximal decompositions, remainder subgraph, intermediate decomposition property.

AMS Subject Classification: 05C70.

1 Introduction

A graph H is said to *divide* a graph G, often expressed by writing $H \mid G$, if G is H-decomposable, that is, if G has a decomposition $\{H_1, H_2, \ldots, H_k\}$, where $H_i \cong H$ for $i = 1, 2, \ldots, k$. If G has size m, H has size m' and $H \mid G$, then certainly $m' \mid m$. On the other hand, if $H \nmid G$, then either G does not contain a subgraph isomorphic to H or G contains a decomposition of $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ where $H_i \cong H$ for each i $(1 \leq i \leq k)$ and R is a nonempty subgraph of G containing no subgraph isomorphic to H. The subgraph R may be referred to as the *remainder subgraph* for this decomposition. As described in [4], this observation may be considered a graph theory analogue of the famous Division Algorithm for integers, where if the positive integer b is divided by the positive integer a, then there exist integers q and r with $0 \leq r < a$ such that $b = aq + r$. Unlike the Division Algorithm for integers where q and r are unique, in this so-called Division Algorithm for graphs G and H, resulting in a decomposition $\mathcal{D}$ (above) of G in terms of H, the integer k and remainder graph R need not be unique. This observation suggests the problem of determining all graphs H such that for every graph G the integers k in such decompositions constitute a set of consecutive integers.

As described in [4], one of the major topics in graph theory concerns graph decompositions. A problem of primary interest in this case has been to determine for graphs G and H whether it is possible to decompose G into subgraphs, each isomorphic to H, that is, whether G is H-decomposable. A classic historical problem in this context is the determination of those integers $n \geq 3$ for which the complete graph K_n is K_3-decomposable. This is equivalent to the problem of determining those integers $n \geq 3$ for which there is a Steiner triple system S_n, a problem initiated and solved in 1847 by the famous combinatorialist Thomas Kirkman [8], who showed that this occurred if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. Another familiar result of this type is that K_n can be decomposed (actually factored in this case) into Hamiltonian cycles if and only if n is odd, a result attributed to Walecki [2]. Not all decomposition problems have dealt with decomposing a graph into subgraphs, each isomorphic to the same graph. The following theorem, due to Bryant, Horsley and Pettersson [5], verified a conjecture on cycle decompositions made by Alspach [1] in 1981.

Theorem 1.1 *Suppose that $n \geq 3$ is an odd integer and that m_1, m_2, $\ldots$, m_t are integers such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ and $m_1 + m_2 + \cdots + m_t = \binom{n}{2}$. Then K_n can be decomposed into the cycles C_{m_1}, C_{m_2}, $\ldots$, C_{m_t}. Furthermore, for every even integer $m \geq 4$ and integers $m_1, m_2, \ldots, m_t$ such that $3 \leq m_i \leq n$ for each i $(1 \leq i \leq t)$ with $m_1 + m_2 + \cdots + m_t = (n^2 - 2n)/2$, there is a decomposition of K_n into a*

1-factor and the cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_t}$.

The famous topologist Oswald Veblen [9] proved that every Eulerian graph can be decomposed into cycles. A conjecture involving cycle decompositions of Eulerian graphs was introduced in [6].

The Eulerian Cycle Decomposition Conjecture (ECDC)

Let G be an Eulerian graph of size m, where a is the minimum number of odd cycles in a cycle decomposition of G and b is the maximum number of odd cycles in a cycle decomposition of G. For every integer ℓ such that $a \leq \ell \leq b$ and ℓ and m are of the same parity, there exists a cycle decomposition of G containing exactly ℓ odd cycles.

It is therefore a consequence of the theorem by Bryant, Horsley and Pettersson that the ECDC is true for all complete graphs of odd order. This conjecture was verified for several classes of graphs in [3, 6] but remains open in general.

In [4] we investigated, for graphs G and H, decompositions of G into $k+1 \geq 1$ subgraphs, k of which are isomorphic to H and where the remaining subgraph contains no subgraph isomorphic to H. For two graphs H and G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called *H-maximal* or an *H-maximal k-decomposition* if $H_i \cong H$ for $1 \leq i \leq k$ and R contains no subgraph isomorphic to H. If G contains no subgraph isomorphic to H, then $k = 0$ and $R = G$. For graphs H and G, let

$$\begin{aligned} \text{Min}(G,H) &= \min\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\} \\ \text{Max}(G,H) &= \max\{k : G \text{ has an } H\text{-maximal } k\text{-decomposition}\}. \end{aligned}$$

Obviously, $\text{Min}(G,H) \leq \text{Max}(G,H)$. Throughout this work, we assume that H is a graph without isolated vertices. A graph H is said to possess the *intermediate decomposition property* (IDP) and H is called an *ID-graph* if for each graph G and each integer k with

$$\text{Min}(G,H) \leq k \leq \text{Max}(G,H),$$

there exists an H-maximal k-decomposition of G. Trivially, the graph K_2 is an ID-graph. On the other hand, neither the claw $K_{1,3}$ nor the triangle K_3 is an ID-graph. For example, the graph G of Figure 1 has a $K_{1,3}$-maximal 1-decomposition and a $K_{1,3}$-maximal 3-decomposition but has no $K_{1,3}$-maximal 2-decomposition. Similarly, the graph F of Figure 1 has a K_3-maximal 1-decomposition and a K_3-maximal 3-decomposition but has no K_3-maximal 2-decomposition.

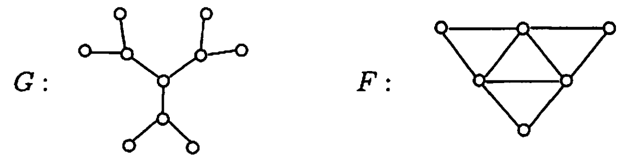

Figure 1: Illustrating that $K_{1,3}$ and K_3 are not ID-graphs

These observations lead to the following problem that initially appeared in [4].

The Intermediate Value Problem for H-Maximal Decompositions

Which graphs (without isolated vertices) are ID-graphs?

In [4] all ID-graphs of size 2 or 3 are determined. For a graph F and a positive integer k, the graph kF is the union of k disjoint copies of the graph F.

Theorem 1.2 [4] *A graph H of size 2 or 3 is an ID-graph unless*

$$H \in \{K_3, K_{1,3}, 3K_2\}.$$

For a set S of graphs and a graph G, a decomposition $\mathcal{D} = \{H_1, H_2, \ldots, H_k, R\}$ of G is called *S-maximal* or an *S-maximal k-decomposition* if $H_i \cong H$ for some $H \in S$ for each integer i with $1 \le i \le k$ and R contains no subgraph isomorphic to any subgraph in S. For a set S of graphs without isolated vertices and a graph G, let

$$\begin{aligned} \mathrm{Min}(G,S) &= \min\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\} \\ \mathrm{Max}(G,S) &= \max\{k : G \text{ has an } S\text{-maximal } k\text{-decomposition}\}. \end{aligned}$$

A set S of graphs without isolated vertices is said to possess the *intermediate decomposition property* (IDP) and S is called an *ID-set* if for every graph G and each integer k with $\mathrm{Min}(G,S) \le k \le \mathrm{Max}(G,S)$, there exists an S-maximal k-decomposition of G. For example, if $S = \{P_3\}$ or $S = \{2K_2\}$, then S is an ID-set by Theorem 1.2. On the other hand, the set $S = \{K_{1,3}, K_3\}$ is not an ID-set. For example, the graph G of Figure 1 has an S-maximal 1-decomposition and an S-maximal 3-decomposition but has no S-maximal 2-decomposition. (On the other hand, the graph F of Figure 1 has an S-maximal k-decomposition for $k = 1, 2, 3$.) As another illustration, the set $S = \{K_3, C_4\}$ is not an ID-set. For example, the graph G of Figure 2 has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $H_1 \cong C_4$ and $R_1 \cong C_8$ and an S-maximal 4-decomposition $\mathcal{D}_4 = \{L_1, L_2, L_3, L_4, R_4\}$

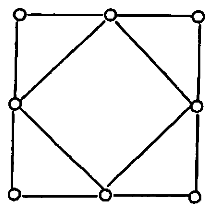

Figure 2: Illustrating that $\{C_3, C_4\}$ is not an ID-set

where $L_i \cong K_3$ for $i = 1, 2, 3, 4$ and R_4 is an empty graph. However, G has neither an S-maximal 2-decomposition nor an S-maximal 3-decomposition.

The following problem also appeared in [4].

The Intermediate Value Problem for S-Maximal Decompositions

Which sets of graphs (without isolated vertices) are ID-sets?

In [4] all ID-sets consisting of two graphs of size 3 are determined. For two graphs F and H, the graph $F + H$ denotes the union of F and H.

Theorem 1.3 [4] *Every 2-element subset S of*

$$\{P_4,\ K_3,\ K_{1,3},\ P_3 + K_2,\ 3K_2\}$$

is an ID-set unless S is a 2-element subset of $\{K_3, K_{1,3}, 3K_2\}$.

By Theorems 1.2 and 1.3, all ID-sets consisting of one or two graphs of size 3 without isolated vertices were determined in [4]. In this paper, we determine all ID-sets consisting of three or more graphs of size 3. We refer to the book [7] for graph theoretic notation and terminology not described in this paper.

2 Preliminary Results

In this section, we present some information which will be useful in determining graphs or sets of graphs possessing an intermediate decomposition property. For a set S of graphs, a graph G is said to have the *intermediate decomposition property with respect to S* (IDP-S) if for each integer k with $\text{Min}(G, S) \leq k \leq \text{Max}(G, S)$, there exists an S-maximal k-decomposition of G. In this case, the graph G is referred to as an *IDP-S* graph; otherwise, G is a *non-IDP-S* graph. Therefore, if every graph is an IDP-S graph, then S is an ID-set.

Theorem 2.1 [4] *Let S be a set of graphs without isolated vertices that is not an ID-set and let $\mathcal{F}_S$ be the set of all non-IDP-S graphs, where G is a graph of minimum size in $\mathcal{F}_S$. Moreover, let a and b be the smallest integers with $1 \le a < b-1$ such that (i) G has an S-maximal a-decomposition $\mathcal{D}_a = \{H_1, H_2, \ldots, H_a, R_a\}$ and an S-maximal b-decomposition $\mathcal{D}_b = \{L_1, L_2, \ldots, L_b, R_b\}$ but (ii) G has no S-maximal k-decomposition for every integer k with $a < k < b$.*

(I) *If $\mathcal{D}_c$ is an S-maximal c-decomposition of G where $c \ge b$, then $H_i \notin \mathcal{D}_c$ for all i with $1 \le i \le a$.*

(II) *For all pairs i, j where $i \in \{1, 2, \ldots, a\}$ and $j \in \{1, 2, \ldots, b\}$, it follows that $E(H_i) \cap E(L_j) \ne \emptyset$.*

(III) *The number b satisfies $b \le \min\{|E(H_i)| : 1 \le i \le a\}$.*

By Theorem 2.1, every graph of size 2 is an ID-graph. For a set S of graphs without isolated vertices that is not an ID-set, a graph G of minimum size that is not an IDP-S graph (as described in Theorem 2.1) is referred to as a *minimum non-IDP-S graph.* If $S = \{H\}$ consists of a single graph H, then a minimum non-IDP-S graph is also referred to as a *minimum non-IDP-H graph.* We now apply Theorem 2.1 to prove that $\{3K_2, P_4\}$ is an ID-set. This result was stated in [4] without a proof and so we provide a complete proof here.

Theorem 2.2 *The set $\{3K_2, P_4\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{3K_2, P_4\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 2.1 then, G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where H_1, $L_i \in S$ ($i = 1, 2, 3$) and neither R_1 nor R_3 contains an subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$ and we may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1, 2, 3$ by Theorem 2.1(II). Since L_i and L_j are edge-disjoint for $i \ne j$ and $i, j \in \{1, 2, 3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| = t \ge 6$. We claim the following:

$$H_1 = 3K_2 \text{ and } L_i = P_4 \text{ for } i = 1, 2, 3. \tag{1}$$

We first show that $H_1 = 3K_2$. Assume, to the contrary, that $H_1 = P_4 = (v_1, v_2, v_3, v_4)$ where $e_i = v_i v_{i+1}$ for $i = 1, 2, 3$ (see Figure 3). We now show that $L_i \cong P_4$ for $i = 1, 2, 3$. If this is not the case, then we may assume, without loss of generality, that $L_1 \cong 3K_2$ or $L_2 \cong 3K_2$. We consider these two cases.

H_1 :

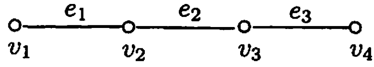

Figure 3: The graph H_1 in the proof of Theorem 2.2

Case 1. $L_1 \cong 3K_2$. Let $E(L_1) = \{e_1, f_1, f_2\}$ where $e_1 = v_1v_2$. Thus each f_i $(i = 1, 2)$ is incident with neither v_1 nor v_2. We show that f_i $(i = 1, 2)$ is not incident with v_3. If this were not the case, then we may assume that f_1 is incident with v_3. Let $F_1 = G[\{e_1, e_2, f_1\}] \cong P_4$, $F_2 = L_3$ and $R_2 = G-(E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Since f_1 and f_2 are nonadjacent, at most one of f_1 and f_2 can be incident with v_4. We may assume that f_1 is not incident with v_4. Let $F_1 = G[\{e_1, e_3, f_1\}] \cong 3K_2$, $F_2 = L_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Thus, $L_1 \cong P_4$. Similarly, $L_3 \cong P_4$.

Case 2. $L_2 \cong 3K_2$. Let $E(L_2) = \{e_2, g_1, g_2\}$ where $e_2 = v_2v_3$. Thus each g_i $(i = 1, 2)$ is incident with neither v_2 nor v_3. We show that each g_i $(i = 1, 2)$ is incident with neither v_1 nor v_4. If this were not the case, then we may assume that g_1 is incident with v_4. Let $F_1 = L_1$, $F_2 = G[\{g_1, e_2, e_3\}] \cong P_4$ and $R_2 = G-(E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Hence neither g_1 nor g_2 in L_2 is adjacent to any edge in $\{e_1, e_2, e_3\}$. Since $L_1 \cong P_4$ (by Case 1), there is an edge $f \in L_1 - \{e_1\}$ that is adjacent to $e_1 = v_1v_2$ and so f is incident with exactly one of v_1 and v_2. Thus G contains a subgraph F isomorphic to one of the graphs in Figure 4(a)–(e).

- If F is the graph in Figure 4(a), let $F_1 = G[\{f, e_1, e_2\}] \cong P_4$ and $F_2 = L_3$.
- If F is the graph in Figure 4(b)–(d), let $F_1 = G[\{e_1, f, e_3\}] \cong P_4$ and $F_2 = L_2$.
- If F is the graph in Figure 4(e), let $F_1 = G[\{f, e_2, e_3\}] \cong P_4$ and $F_2 = \{e_1, g_1, g_2\} \cong 3K_2$.

In each case, let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Thus, $L_2 \cong P_4$.

Therefore, if $H_1 \cong P_4$, then $L_i \cong P_4$ for $i = 1, 2, 3$. Hence $\mathcal{D}_1$ is a P_4-maximal 1-decomposition and $\mathcal{D}_3$ is a P_4-maximal 3-decomposition. However then, since P_4 is an ID-graph, G has a P_4-maximal 2-decomposition

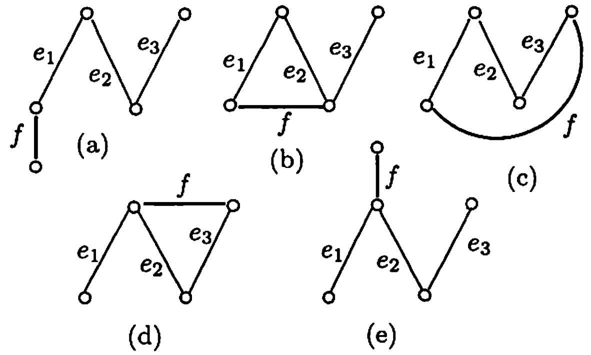

Figure 4: A step in the proof of Theorem 2.2

(and so an S-maximal 2-decomposition), which is impossible. Therefore, as claimed, $H_1 = 3K_2$.

Next, we show that $L_i = P_4$ for $i = 1, 2, 3$. We consider two cases.

Case (i). At least two of L_1, L_2 and L_3 are isomorphic to $3K_2$, say $L_1 \cong L_2 \cong 3K_2$. Let $E(L_1) = \{e_1, f_1, f_2\}$ and $E(L_2) = \{e_2, g_1, g_2\}$. We show that each f_i $(i = 1, 2)$ is adjacent to both e_2 and e_3 and each g_i $(i = 1, 2)$ is adjacent to both e_1 and e_3. If this is not the case, we may assume that f_1 is not adjacent to e_2. Then let $F_1 = G[\{e_1, e_2, f_1\}] \cong 3K_2$, let $F_2 = L_3$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Therefore, we may assume that G contains a subgraph isomorphic to the graph of Figure 5. Let $F_1 = G[\{g_1, e_1, g_2\}] \cong P_4$, $F_2 = G[\{e_2, f_1, e_3\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible.

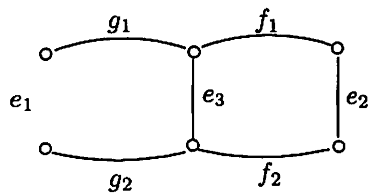

Figure 5: A step in Case (i) of the proof of Theorem 2.2

Case (ii). Exactly one of L_1, L_2 and L_3 is isomorphic to $3K_2$, say $L_1 \cong 3K_2$ and $L_i = P_4$ for $i = 2, 3$. Again, let $E(L_1) = \{e_1, f_1, f_2\}$. By the argument employed in Case (i), each of f_1 and f_2 is adjacent to e_2 and e_3. Thus, G contains the graph of Figure 6(a) as a subgraph.

We first show that no edge in $L_2 - e_2$ can be adjacent to both e_1 and e_2 and no edge in $L_3 - e_3$ can be adjacent to both e_1 and e_3. If this is not case, we may assume that $g \in E(L_2 - e_2)$ and g is adjacent to both e_1 and e_2. Let $F_1 = G[\{e_1, g, e_2\}] \cong P_4$, $F_2 = G[\{f_1, e_3, f_2\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Suppose that $E(L_2) = \{e_2, g_1, g_2\}$ and $E(L_3) = \{e_3, h_1, h_2\}$ where g_1 is adjacent to e_2 and h_1 is adjacent to e_3. Note that neither g_1 nor h_1 can be adjacent to both e_2 and e_3; for otherwise, we may assume that g_1 is adjacent to both e_2 and e_3. Then let $F_1 = L_1$, $F_2 = G[\{e_2, g_1, e_3\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Hence G contains one of the graphs of Figure 6(b)–(d) as a subgraph.

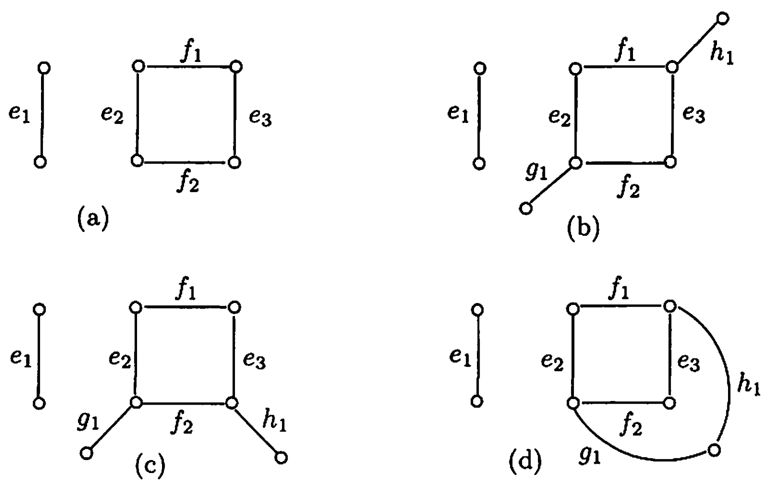

Figure 6: A step in Case (ii) of the proof of Theorem 2.2

First, suppose that G contains a subgraph isomorphic to the graph in Figure 6(b). Now let $F_1 = G[\{e_1, g_1, h_1\}] \cong 3K_2$, $F_2 = G[\{e_2, f_2, e_3\}] \cong P_4$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Since R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Next, suppose that G contains a subgraph isomorphic to the graph in Figure 6(c) or in Figure 6(d). However then, R_1 contains the subgraph $G[\{g_1, f_2, h_1\}] \cong P_4$, which is a contradiction.

Therefore, $H_1 = 3K_2$ and $L_i = P_4$ for $i = 1, 2, 3$, as we claimed in (1). We now show that if an edge in $L_i - e_i$ that is adjacent to e_i, then this edge is not adjacent to any edges in $E(H_1) - \{e_i\}$ for $i = 1, 2, 3$. If this is not the case, we may assume that $f_1 \in E(L_1)$ is adjacent to e_1 and e_2 (see Figure 7).

Let $F_1 = G[\{e_1, f_1, e_2\}] \cong P_4$, $F_2 = L_3$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Then $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible.

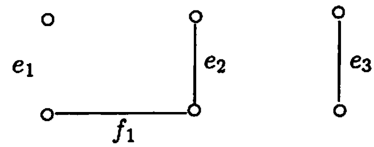

Figure 7: A step in Case (ii) the proof of Theorem 2.2

Next, we show that each e_i is the interior edge of L_i for $i = 1, 2, 3$. If this is not the case, we may assume that $L_1 = (e_1, f_1, f_2)$. If f_2 is not adjacent to e_2, then let $F_1 = G[\{e_1, f_2, e_2\}] \cong 3K_2$, $F_2 = L_3$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Then $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. Similarly, if f_2 is not adjacent to e_3, then there is an S-maximal 2-decomposition, which is impossible. Thus f_2 is adjacent to both e_2 and e_3. However then, since f_1 is adjacent to f_2, either f_1 and e_2 are adjacent or f_1 and e_3 are adjacent, which is impossible. Therefore, e_i is the interior edge of L_i for $i = 1, 2, 3$, as claimed.

Let $L_1 = (f_1, e_1, f_2)$, $L_2 = (g_1, e_2, g_2)$ and $L_3 = (h_1, e_3, h_2)$. It then follows from the argument above that L_1, L_2 and L_3 have the following properties:

(a) No edge in L_i $(i = 1, 2, 3)$ is adjacent to any edges in $\{e_1, e_2, e_3\} - \{e_i\}$.

(b) Since R_1 contains no subgraph isomorphic to P_4, it follows that $\{f_1, f_2\}$, $\{g_1, g_2\}$ and $\{h_1, h_2\}$ are sets of two independent edges.

Since R_1 contains no subgraph isomorphic to $3K_2$, there are adjacent edges in $\{f_1, f_2, g_1, g_2, h_1, h_2\}$. By (a) and (b), we may assume that f_1 and g_1 are adjacent. Let $F_1 = G[\{e_1, f_1, g_1\}] \cong P_3$, $F_2 = \{f_2, e_2, e_3\} \cong 3K_2$ and let $R_2 = G - (E(F_1) \cup E(F_2))$. Then $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition, which is impossible. ∎

3 ID-Sets and Non-ID Sets of Graphs Size 3

In this section, all ID-sets consisting of graphs of size 3 are described. First, observe that if S is the set of all graphs (connected or disconnected) of the same size m, then S is an ID-set. To see this, let G be a graph, $a = \mathrm{Min}(G, S)$ and let $\mathcal{D} = \{H_1, H_2, \ldots, H_a, R\}$ be any S-maximal a-decomposition of G. Since R contains no subgraph that is isomorphic to any graph in S, it follows that $0 \le |E(R)| \le m - 1$. Thus $\mathrm{Min}(G, S) = \mathrm{Max}(G, S) = a$. We state this observation below.

Observation 3.1 [4] *For each positive integer m, the set S_m of all graphs (connected or disconnected) of size m is an ID-set.*

By Observation 3.1, the set $\{P_4, K_3, K_{1,3}, P_3+K_2, 3K_2\}$ consisting of all graphs of size 3 is an ID-set. By Theorems 1.2, 1.3 and Observation 3.1, it remains to determine the ID-sets and non-ID sets that are subsets S of $\{P_4, K_3, K_{1,3}, P_3+K_2, 3K_2\}$ with $|S| = 3$ or $|S| = 4$.

We first show that neither of the sets $\{K_3, 3K_2, P_4\}$ and $\{K_{1,3}, 3K_2, P_4\}$ is an ID set. For $\{K_3, 3K_2, P_4\}$, let $G = K_3 + 2K_{1,3}$ be the union of K_3 and two copies of $K_{1,3}$. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$, where $H_1 \cong K_3$ and $R_1 \cong 2K_{1,3}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ where $L_1 \cong L_2 \cong L_3 \cong 3K_2$ and R_3 is an empty graph. However, G has no S-maximal 2-decomposition. For $\{K_{1,3}, 3K_2, P_4\}$, let $G = 2K_3 + K_{1,3}$ be the union of two copies of K_3 and $K_{1,3}$. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$, where $H_1 \cong K_{1,3}$ and $R_1 \cong 2K_3$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ where $L_1 \cong L_2 \cong L_3 \cong 3K_2$ and R_3 is an empty graph. However, G has no S-maximal 2-decomposition. Hence neither set is an ID-set.

Next, we show that if $S \subseteq \{P_4, K_3, K_{1,3}, P_3+K_2, 3K_2\}$ with $|S| \in \{3, 4\}$ such that S is neither $\{K_3, 3K_2, P_4\}$ nor $\{K_{1,3}, 3K_2, P_4\}$, then S is an ID-set. In order to do this, we first present three useful observations, the first of which is a consequence of Theorem 2.1(I) and (II).

Observation 3.2 *Let S be a non-ID-set of graphs of size 3. If G is a minimum non-IDP-S graph having an S-maximal 1-decomposition $\{H_1, R_1\}$, then the size of R_1 is at least 6.*

Observation 3.3 *Suppose that R is a graph without isolated vertices having size $t \geq 6$.*

(*a*) *If R does not contain P_3+K_2 as a subgraph, then $R = tK_2$, $R = K_{1,t}$ or $R = K_4$.*

(*b*) *If R does not contain $3K_2$ as a subgraph, then R has at most two components and $R = K_{1,t}$, $R = K_{1,r} + K_{1,s}$ where $1 \leq r \leq s$ and $r+s=t$, $R = 2K_3$ or $R = K_3 + K_{1,t-3}$.*

(*c*) *If R does not contain P_4 as a subgraph, then each component of R is K_3 or stars.*

Observation 3.4 *Suppose that S is a non-ID-set of graphs and S contains an ID-subset S_0. If G is a non-IDP-S graph such that G has an S-maximal a-decomposition $\mathcal{D}_a = \{H_1, H_2, \ldots, H_a, R_a\}$ and an S-maximal b-decomposition $\mathcal{D}_b = \{L_1, L_2, \ldots, L_b, R_b\}$ but no S-maximal k-decomposition* *for every integer k with $a < k < b$, then either $H_i \in S - S_0$ for some $i \in \{1,2,\ldots,a\}$ or $L_j \in S - S_0$ for some $j \in \{1,2,\ldots,b\}$.*

Proposition 3.5 *The set $\{K_{1,3}, K_3, P_4\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{K_{1,3}, K_3, P_4\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. Since each graph in S has size 3, it follows by Theorem 2.1(III) that G has an S-maximal 1-decomposition $\mathcal{D}_1$ and an S-maximal 3-decomposition $\mathcal{D}_3$ but G has no S-maximal 2-decomposition. Let $\mathcal{D}_1 = \{H_1, R_1\}$ and $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$, where each of H_1, L_1, L_2, L_3 is isomorphic to some graph in S and R_1 and R_3 contain no subgraph isomorphic to any graph in S. Let $E(H_1) = \{e_1, e_2, e_3\}$. We may assume, without loss of generality, that $e_i \in E(L_i)$ for $i = 1,2,3$ by Theorem 2.1(II). Since L_i and L_j are edge-disjoint for $i \neq j$ and $i,j \in \{1,2,3\}$, it follows that $L_i - e_i$ is a subgraph of R_1 and so $|E(R_1)| \geq 6$. Furthermore, each component of R_1 has size at most 2 (since R_1 contains no subgraph isomorphic to any graph in S).

We now construct an S-maximal 2-decomposition $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ of G as follows. Let $F_1 = L_1 \in S$. Now, let $e \in E(L_2) - \{e_2\}$ that is adjacent to e_2. Then the subgraph $F_2 = G[\{e_2, e_3, e\}]$ induced by $\{e_2, e_3, e\}$ is a connected subgraph of size 3 and so $F_2 \in S$. Furthermore, $E(F_1) \cap E(F_2) = \emptyset$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$, it follows that R_2 is a subgraph of R_1 and so R_2 contains no subgraph isomorphic to any graph in S. Therefore, $\mathcal{D}_2 = \{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

Proposition 3.6 *Each of the following sets is an ID-set:*

$$\{K_{1,3}, K_3, 3K_2\},\ \{K_{1,3}, K_3, 3K_2, P_4\} \text{ and } \{K_{1,3}, K_3, 3K_2, P_3 + K_2\}. \quad (2)$$

Proof. Let S be one of the sets in (2). Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 2.1 then, G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where $|E(R_1)| = t \geq 6$ by Observation 3.2. Since R_1 does not contain $3K_2$ as a subgraph, $R_1 = K_{1,t}$, $R_1 = K_{1,r} + K_{1,s}$ where $1 \leq r \leq s$ and $r + s = t$, $R_1 = 2K_3$ or $R_1 = K_3 + K_{1,t-3}$ by Observation 3.3(b). Since R_1 contains neither $K_{1,3}$ nor K_3 and $|E(R_1)| = t \geq 6$, this is impossible.

Proposition 3.7 *Each of the following sets is an ID-set:*

$$\{K_{1,3}, 3K_2, P_3 + K_2\} \text{ and } \{K_{1,3}, 3K_2, P_3 + K_2, P_4\}. \quad (3)$$

Proof. Let S be one of the sets in (3). Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. By Theorem 2.1 then, G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ where

$|E(R_1)| = t \geq 6$ by Observation 3.2. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 3.3(a). Since R_1 contains neither $K_{1,3}$ nor $3K_2$ as a subgraph and $t \geq 6$, this is impossible.

Proposition 3.8 *The set $\{K_{1,3}, K_3, P_3 + K_2\}$ is an ID-set.*

Proof. Assume, to the contrary, that $S = \{K_{1,3}, K_3, P_3 + K_2\}$ is not an ID-set. Let G be a minimum non-IDP-S graph. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Then $|E(R_1)| = t \geq 6$ by Observation 3.2. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 3.3(a). Since R_1 contains neither $K_{1,3}$ nor K_3 as a subgraph and $t \geq 6$, it follows that $R_1 = tK_2$. Let $E(H_1) = \{e_1, e_2, e_3\}$, where say $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. Since $R_1 = tK_2$, it follows that $L_i - e_i = 2K_2$ and so $L_i = P_3 + K_2$ for $i = 1, 2, 3$.

- If $H_1 = K_{1,3}$, then let $S' = \{K_{1,3}, P_3 + K_2\}$.
- If $H_1 = K_3$, then let $S' = \{K_3, P_3 + K_2\}$.
- If $H_1 = P_3 + K_2$, then let $S' = \{P_3 + K_2\}$.

In each case, S' is an ID-set. Since $\mathcal{D}_1$ is an S'-maximal 1-decomposition and $\mathcal{D}_3$ is an S'-maximal 3-decomposition, it follows that G has an S'-maximal 2-decomposition, which is a contradiction. ∎

Proposition 3.9 *Each of the following sets is an ID-set:*

$$\{K_{1,3}, P_4, P_3 + K_2\} \text{ and } \{K_{1,3}, K_3, P_4, P_3 + K_2\}. \quad (4)$$

Proof. Let S be one of the sets in (4). Assume, to the contrary, that S is not an ID-set. Let G be a minimum non-IDP-S graph. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Then $|E(R_1)| = t \geq 6$ by Observation 3.2. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 3.3(a). Since R_1 contains neither $K_{1,3}$ nor P_4 as a subgraph and $t \geq 6$, it follows that $R_1 = tK_2$. Let $E(H_1) = \{e_1, e_2, e_3\}$, where say $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. We consider two cases.

Case 1. $S = \{K_{1,3}, P_4, P_3 + K_2\}$. Then $L_i \in \{P_4, P_3 + K_2\}$. Since $\{P_4, P_3 + K_2\}$ is an ID-set, it follows that $H_1 \cong K_{1,3}$. Because $\{K_{1,3}, P_4\}$ and $\{K_{1,3}, P_3 + K_2\}$ are both ID-sets, at least one of L_i $(1 \leq i \leq 3)$ is P_4 and at least one of L_i $(1 \leq i \leq 3)$ is $P_3 + K_2$. We may assume that

$L_1 \cong P_4 = (f_1, e_1, f_2)$ where e_1 is the middle edge of L_1 and $L_2 \cong P_3 + K_2$ where e_2 is adjacent an edge in L_2, say e_2 is adjacent to g in L_2. This implies that G contains a subgraph isomorphic to one of the graphs in Figures 8(a) and 8(b), where the edges in L_1 are drawn in bold. In each case, let $F_1 = L_1$ and $F_2 = G[\{e_2, e_3, g\}]$. Thus $F_2 \cong K_{1,3}$ or $F_2 \cong P_4$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$, $E(F_1) \cap E(F_2) = \emptyset$, and R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

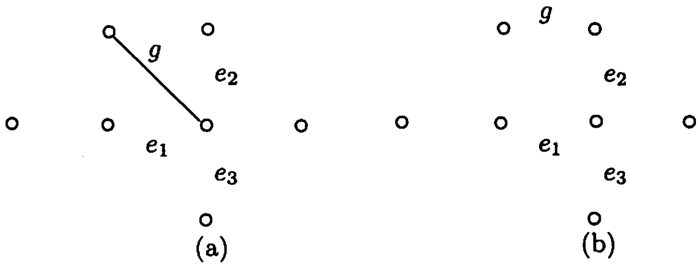

Figure 8: A step in the proof of Proposition 3.9

Case 2. $S = \{K_{1,3}, K_3, P_4, P_3 + K_2\}$. Then $L_i \in \{P_4, P_3 + K_2\}$, where $i = 1, 2, 3$. In particular, $L_i \ncong K_3$ for $i = 1, 2, 3$. Furthermore, we claim that $H_1 \ncong K_3$. If this were not the case, then observe that at least one edge of R_1 is adjacent to some edge of H_1; for otherwise, $G = K_3 + tK_2$ and G cannot have an S-maximal 3-decomposition. On the other hand, since $t \geq 6$, at least two edges of R_1 are not adjacent to any edge of H_1. Let $f_1, f_2, f_3 \in E(R_1)$ such that f_1 is adjacent to some edge of H_1, say f_1 is adjacent to e_1, while neither f_2 nor f_3 is adjacent to any edge of H_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$, $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$, $E(F_1) \cap E(F_2) = \emptyset$ and R_2 is a subgraph of R_1, it follows that $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. Thus, $H_1 \ncong K_3$, as claimed. Hence $H_1, L_i \in S' = \{K_{1,3}, P_4, P_3 + K_2\}$ for $i = 1, 2, 3$. Since S' is an ID-set by Case 1, it follows that G has an S-maximal 2-decomposition, which, again, is impossible.

In order to show that the remaining sets of graphs of size 3 are ID-sets, we first present a lemma.

Lemma 3.10 *Let* $S = \{K_3, P_4, P_3 + K_2\}$. *If* G *is a minimum non-IDP-S graph and* $\mathcal{D}_1 = \{H_1, R_1\}$ *is an S-maximal* 1*-decomposition, then* $R_1 \neq tK_2$ *where* $t \geq 6$.

Proof. Assume, to the contrary, that $R_1 = tK_2$ where $t \geq 6$. Since G is a minimum non-IDP-S graph, G also has an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Let $E(H_1) = \{e_1, e_2, e_3\}$, where say $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. Since $R_1 = tK_2$, each e_i $(i = 1, 2, 3)$ must be adjacent to some edge of R_1; for otherwise, at least one of L_1, L_2, L_3 is $3K_2$, which is impossible.

First, suppose that $H_1 = K_3$ or $H_1 = P_4$. Then at least one edge of R_1 is adjacent to some edge of H_1. Since $R_1 = tK_2$ and $t \geq 6$, at least two edges of R_1 are not adjacent to any edge of H_1. Let $f_1, f_2, f_3 \in E(R_1)$ such that f_1 is adjacent to some edge of H_1, say f_1 is adjacent to e_1, while neither f_2 nor f_3 is adjacent to any edge of H_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2$, $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Since $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$, it follows that R_2 is a subgraph of R_1 and $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

Next, suppose that $H_1 = P_3 + K_2$. We may assume that e_1 and e_2 are adjacent to edges in H_1. Since $t \geq 6$, there is an edge in R_1 that is not adjacent to any edge of H_1. Furthermore, since G is connected, at least one of e_1 and e_2 is adjacent to some edge of R_1 and e_3 is adjacent to some edge of R_1. Let $f_1, f_2, f_3 \in E(R_1)$ such that f_1 is not adjacent to any edge of H_1, f_2 is adjacent to e_2 and f_3 is adjacent to e_3. First, suppose that $f_2 \neq f_3$. Let $F_1 = G[\{e_2, f_1, f_2\}] \cong P_3 + K_2$, $F_2 = G[\{e_1, e_3, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. Next, suppose that $f_2 = f_3$. Since $t \geq 6$, there is $f_4 \in E(R_1) - \{f_1\}$ such that f_4 is not adjacent to any edge of H_1. Let $F_1 = G[\{e_1, e_2, f_4\}] \cong P_3 + K_2$, $F_2 = G[\{e_3, f_1, f_3\}] \cong P_3 + K_2$ and $R_2 = G - (E(F_1) \cup E(F_2))$. In either case, $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$. Therefore, R_2 is a subgraph of R_1 and so $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction.

Proposition 3.11 *Each of the following sets is an ID-set:*

$$\{K_3, 3K_2, P_3 + K_2\},\ \{K_3, P_4, P_3 + K_2\},$$

$$\{3K_2, P_4, P_3 + K_2\},\ \{K_3, 3K_2, P_4, P_3 + K_2\}.$$

Proof. Let S be one of the sets described above. Assume, to the contrary, that S is not an ID-set. Then G has an S-maximal 1-decomposition $\mathcal{D}_1 = \{H_1, R_1\}$ and an S-maximal 3-decomposition $\mathcal{D}_3 = \{L_1, L_2, L_3, R_3\}$ but no S-maximal 2-decomposition. Then $|E(R_1)| = t \geq 6$ by Observation 3.2. Since R_1 does not contain $P_3 + K_2$ as a subgraph, $R_1 = tK_2$, $R_1 = K_{1,t}$ or $R_1 = K_4$ by Observation 3.3(a). For each set S under consideration, it follows that (i) either $3K_2 \in S$ or $S = \{K_3, P_4, P_3 + K_2\}$ and (ii) either $P_4 \in S$ or $K_3 \in S$. Hence $R_1 \neq tK_2$ (by Lemma 3.10) and $R_1 \neq K_4$. Therefore, $R_1 = K_{1,t}$. Let $E(R_1) = \{f_1, f_2, \ldots, f_t\}$ where $t \geq 6$ and let

$E(H_1) = \{e_1, e_2, e_3\}$ where, say, $e_i \in E(L_i)$ for $i = 1, 2, 3$, and so $L_i - e_i$ is a subgraph of R_1. First, we make an observation. For $i = 1, 2, 3$, since (a) $L_i \in S$ and $K_{1,3} \notin S$ and (b) $L_i - e_i$ is a subgraph of R_1 and $R_1 = K_{1,t}$, it follows that e_i is not incident with the center vertex of R_1 (or e_i cannot be adjacent to all edges in R_1). Since $H_1 \in \{K_3, 3K_2, P_4, P_3 + K_2\}$, we consider four cases.

Case 1. $H_1 = K_3$. Then at least three edges in R_1 that are not adjacent to any edges in H_1, say f_1, f_2 and f_3 are three such edges in R_1. Let $F_1 = G[\{e_1, f_1, f_2\}] \cong P_3 + K_2 \in S$, $F_2 = G[\{e_2, e_3, f_3\}] \cong P_3 + K_2 \in S$ and $R_2 = G - (E(F_1) \cup E(F_2))$.

Case 2. $H_1 = 3K_2$. Then $\{3K_2, P_3 + K_2\} \subseteq S$. We may assume that e_1 is adjacent to f_1 (and possibly to f_2), e_2 is adjacent to f_3 (and possibly to f_4) and e_3 is adjacent to f_5 (and possibly to f_6). More precisely, neither e_1 nor e_3 is adjacent to f_3 and e_2 is not adjacent to f_1 or f_2. Let $F_1 = G[\{e_1, e_3, f_3\}] \cong 3K_2 \in S$, $F_2 = G[\{e_2, f_1, f_2\}] \cong P_3 + K_2 \in S$ and $R_2 = G - (E(F_1) \cup E(F_2))$.

Case 3. $H_1 = P_4$. As we observed earlier, no edge in H_1 is incident with the center vertex of R_1. Thus at least two edges in R_1 are not adjacent to any edge in H_1, say f_1 and f_2 are two such edges in R_1. We may assume that $H_1 = (e_1, e_2, e_3)$. Then there is $f_3 \in E(R_1) - \{f_1, f_2\}$ such that f_3 is not adjacent to e_1. (It is possible that $t = 6$ and G contains the graph of Figure 9 as a subgraph). Let $F_1 = G[\{e_1, f_1, f_3\}] \cong P_3 + K_2 \in S$ where $E(P_3) = \{f_1, f_3\}$, $F_2 = G[\{e_2, e_3, f_2\}] \cong P_3 + K_2 \in S$, where $E(P_3) = \{e_2, e_3\}$, and $R_2 = G - (E(F_1) \cup E(F_2))$.

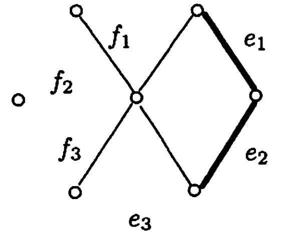

Figure 9: A step in the proof of Proposition 3.11

Case 4. $H_1 = P_3 + K_2$. Let e_1 and e_2 be the two adjacent edges in H_1. We may assume (i) f_1 is not adjacent to e_1 or e_2 and (ii) f_2 and f_3 are not adjacent to e_3. Let $F_1 = G[\{e_1, e_2, f_1\}] \cong P_3 + K_2 \in S$, $F_2 = G[\{e_3, f_2, f_3\}] \cong P_3 + K_2 \in S$ and $R_2 = G - (E(F_1) \cup E(F_2))$.

In each case, $e_1, e_2, e_3 \in E(F_1) \cup E(F_2)$ and $E(F_1) \cap E(F_2) = \emptyset$. Hence R_2 is a subgraph of R_1 and so $\{F_1, F_2, R_2\}$ is an S-maximal 2-decomposition of G, which is a contradiction. ■

In summary, we have the following.

Theorem 3.12 *A subset S of the set $\{P_4, K_3, K_{1,3}, P_3 + K_2, 3K_2\}$ of all graphs of size 3 without isolated vertices is an ID-set if and only if S is not one of the following eight sets:*

$$\{3K_2\},\ \{K_3\},\ \{K_{1,3}\},\ \{3K_2, K_3\},\ \{3K_2, K_{1,3}\},$$
$$\{K_3, K_{1,3}\},\ \{3K_2, K_3, P_4\},\ \{3K_2, K_{1,3}, P_4\}.$$

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