On Monochromatic Spectra in Graphs

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Abstract

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_2$ be a vertex coloring of G where $c(v) \neq 0$ for at least one vertex v of G. Then the coloring c induces a new coloring $\sigma : V(G) \to \mathbb{Z}_2$ of G defined by $\sigma(v) = \sum_{u \in N[v]} c(u)$ where $N[v]$ is the closed neighborhood of v and addition is performed in $\mathbb{Z}_2$. If $\sigma(v) = 0 \in \mathbb{Z}_2$ for every vertex v in G, then the coloring c is called a (modular) monochromatic $(2,0)$-coloring of G. A graph G having a monochromatic $(2,0)$-coloring is a (monochromatic) $(2,0)$-colorable graph. The minimum number of vertices colored 1 in a monochromatic $(2,0)$-coloring of G is the $(2,0)$-chromatic number of G and is denoted by $\chi_{(2,0)}(G)$. For a $(2,0)$-colorable graph G, the monochromatic $(2,0)$-spectrum $S_{(2,0)}(G)$ of G is the set of all positive integers k for which exactly k vertices of G can be colored 1 in a monochromatic $(2,0)$-coloring of G. Monochromatic $(2,0)$-spectra are determined for several well-known classes of graphs. If G is a connected graph of order $n \geq 2$ and $a \in S_{(2,0)}(G)$, then a is even and $1 \leq |S_{(2,0)}(G)| \leq \lfloor n/2 \rfloor$. It is shown that for every pair k, n of integers with $1 \leq k \leq \lfloor n/2 \rfloor$, there is a connected graph G of order n such that $|S_{(2,0)}(G)| = k$. A set S of positive even integers is $(2,0)$-realizable if S is the monochromatic $(2,0)$-spectrum of some connected graph. Although there are infinitely many non-$(2,0)$-realizable sets, it is shown that every set of positive even integers is a subset of some $(2,0)$-realizable set. Other results and questions are also presented on $(2,0)$-realizable sets in graphs.

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1 Introduction

For an integer $k \geq 2$ and a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ be a coloring where adjacent vertices may be assigned the same color. Then the coloring c induces a new coloring $\sigma_c : V(G) \to \mathbb{Z}_k$ of the graph G defined by

$$\sigma_c(v) = \sum_{u \in N[v]} c(u) \tag{1}$$

where $N[v]$ is the closed neighborhood of v (consisting of v and the vertices in the open neighborhood $N(v)$ of v) and addition is performed in $\mathbb{Z}_k$. The number $\sigma_c(v)$ is called the *color sum* of a vertex v with respect to the coloring c. (We also write $\sigma(v)$ for $\sigma_c(v)$ if the coloring c under consideration is clear.) If $\sigma_c(u) = \sigma_c(v)$ for every two vertices u and v in G, then the coloring c is called a *modular monochromatic k-coloring* or simply a *monochromatic k-coloring*. For any connected graph G, there is always a (trivial) monochromatic k-coloring c where $c(v) = 0$ for each vertex v of G. Our interest, however, lies with nontrivial monochromatic k-colorings c, where then $c(v) \neq 0$ for at least one vertex v of G. These concepts were introduced and studied in [1] and were inspired by the well-known combinatorial problem called the Lights Out Puzzle (also see [5]) and studied further in [2, 3].

For a given integer t with $0 \leq t \leq k-1$, a monochromatic k-coloring c of G is said to be of *type* t if the induced vertex coloring σ has the property that $\sigma(v) = t$ for each vertex v of G. Such a coloring is also referred to as a *(modular) monochromatic (k,t)-coloring*. A graph G is *monochromatic (k,t)-colorable* or *(k,t)-colorable* if G has a monochromatic (k,t)-coloring for some integers k and t with $0 \leq t \leq k-1$. We are particularly interested in monochromatic $(2,1)$-colorings and $(2,0)$-colorings. These two colorings are not only closely related to the Lights Out Puzzle but also related to some well-known studied domination parameters, namely odd and even dominations in graphs (see [1, 7, 8]).

A vertex v of a graph G *dominates* a vertex u if u is in the closed neighborhood $N[v]$ of v. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. A dominating set S in G is an *odd dominating set* if every vertex of G is dominated by an odd number of vertices of S. In [9] Sutner showed that every graph has an odd dominating set. As a consequence of Sutner's Theorem, it was observed in [1] that every connected graph G is $(2,1)$-colorable. On the other hand, not every graph is $(2,0)$-colorable. A dominating set S in a graph G is an *even dominating set* if every vertex of G is dominated by an even number of vertices of S and the minimum cardinality of an even dominating set in G is the *even domination number* of G and is denoted by $\gamma_e(G)$. It is known that not every graph has an even dominating set and not every graph has a monochromatic $(2,0)$-coloring. For example, the cycle of order 5 has neither an even dominating set nor a monochromatic $(2,0)$-coloring. The minimum number of vertices colored 1 in a modular monochromatic $(2,0)$-coloring of G is defined in [1] the $(2,0)$-*chromatic number* of G and is denoted by $\chi_{(2,0)}(G)$. A monochromatic $(2,0)$-coloring of G that assigns the color 1 to exactly $\chi_{(2,0)}(G)$ vertices of G is a *minimum monochromatic* $(2,0)$-*coloring* of G. If G is a connected graph of order n such that $\chi_{(2,0)}(G)$ and $\gamma_e(G)$ both exist, then

$$2 \le \chi_{(2,0)}(G) \le \gamma_e(G) \le n \tag{2}$$

and $\chi_{(2,0)}(G)$ and $\gamma_e(G)$ are even. In [2], the relationship between $(2,0)$-chromatic numbers and even domination numbers of graphs was studied. More precisely, it was shown that (i) for each pair a, b of even integers with $2 \le a \le b$, there is a connected graph G such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$ and (ii) there is a connected graph G of order n such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$ if and only if $a = b$, or $a \le b/2$ and $(a, b, n) \ne (2, 4, 4)$ or $b/2 < a < b$ and $n \ge (2a + b)/2$ (see [2]). The $(2,0)$-chromatic numbers have been determined for several well-known classes of graphs. Furthermore, all trees of order n whose $(2,0)$-chromatic number belongs to the set $\{n, n-1, n-2, n-3\}$ have been characterized (see [1, 2, 3]).

In this work, we investigate the concept of the monochromatic $(2,0)$-spectrum of a graph. For a $(2,0)$-colorable graph G, the *monochromatic* $(2,0)$-*spectrum* $S_{(2,0)}(G)$ (or simply $(2,0)$-*spectrum*) of G is the set of all positive integers k for which exactly k vertices of G can be colored 1 in a monochromatic $(2,0)$-coloring of G. It then follows by (2) that if G is a $(2,0)$-colorable graph of order n and $a \in S_{(2,0)}(G)$, then $\chi_{(2,0)}(G) \le a \le n$. The following result describes another property of elements in the $(2,0)$-spectrum of a graph. A graph G is called an *odd-degree graph* if every vertex of G has odd degree.

Proposition 1.1 [1] *If c is a monochromatic $(2,0)$-coloring of a connected $(2,0)$-colorable graph G, then the subgraph of G induced by the vertices colored 1 by c is an odd-degree graph and so the number of vertices colored 1 by c is even.*

By Proposition 1.1, if G is a connected $(2,0)$-colorable graph of order n, then

$$S_{(2,0)}(G) \subseteq \{\chi_{(2,0)}(G), \chi_{(2,0)}(G)+2, \ldots, 2\lfloor n/2 \rfloor\} \subseteq \{2, 4, \ldots, 2\lfloor n/2 \rfloor\}. \tag{3}$$

We refer to the books [4, 6] for graph theory notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

2 On $(2,0)$-Spectra of Some Well-Known Graphs

In this section, we first determine the $(2,0)$-spectra for some well-known classes of graphs. In order to do this, we first describe several well-known graphs that are $(2,0)$-colorable (see [1]).

Proposition 2.1 (*a*) *Every nontrivial complete graph is* $(2,0)$*-colorable.*

(*b*) *For positive integers* r *and* s*, the complete bipartite graph* $K_{r,s}$ *of order* $r+s$ *is* $(2,0)$*-colorable if and only if* r *and* s *are odd.*

(*c*) *A path of order* $n \geq 2$ *is* $(2,0)$*-colorable if and only if* $n \equiv 2 \pmod 3$.

(*d*) *A cycle of order* $n \geq 3$ *is* $(2,0)$*-colorable if and only if* $n \equiv 0 \pmod 3$.

For each integer $n \geq 2$, let $[[n]] = \{2, 4, \ldots, 2\lfloor n/2 \rfloor\}$ be the set of all even integers between 2 and n. We begin with graphs with the largest possible $(2,0)$-spectrum. For a complete graph K_n with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and for each i with $1 \leq i \leq \lfloor n/2 \rfloor$, let c_i be the coloring that assigns the color 1 to each vertex in $\{v_1, v_2, \ldots, v_{2i}\}$ and the color 0 to the remaining vertices of K_n. Then each c_i $(1 \leq i \leq \lfloor n/2 \rfloor)$ is a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $2i$ vertices of K_n. Therefore, $S_{(2,0)}(K_n) = [[n]]$ for each $n \geq 2$. However, the complete graph K_n is not the only graph of order n whose $(2,0)$-spectrum is $[[n]]$. In fact, more can be said. First, we present some additional definitions. For two vertex-disjoint graphs G and H, let $G + H$ and $G \vee H$ denote the union and join of G and H, respectively. For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and n pairwise vertex-disjoint graphs $H_1, H_2, \ldots, H_n$, the *composition* $G[H_1, H_2, \ldots, H_n]$ of G and H_i $(1 \leq i \leq n)$ is the graph $H_1 + H_2 + \cdots + H_n$ (the union of $H_1, H_2, \ldots, H_n$) together with the edges in the set $\{xy : x \in V(H_i), y \in V(H_j), v_iv_j \in E(G)\}$. If there is a graph H such that $H_i \cong H$ for $1 \leq i \leq n$, then we write $G[H] = G[H_1, H_2, \ldots, H_n]$.

Proposition 2.2 *For each integer* $n \geq 5$*, there is a connected graph* G *that is not complete such that* $S_{(2,0)}(G) = [[n]]$.

Proof. For each even integer $n \geq 6$, let $G = P_{\frac{n}{2}}[K_2]$ be the composition of the path $P_{n/2}$ of order $n/2$ and K_2 where $(u_1, u_2, \ldots, u_{n/2})$ and $(v_1, v_2, \ldots, v_{n/2})$ are two copies of $P_{\frac{n}{2}}$ in G. For each i with $1 \leq i \leq n/2$, let $V_i = \{u_1, v_1, u_2, v_2, \ldots, u_i, v_i\}$ and let c_i be the coloring that assigns the color 1 to each vertex in V_i and the color 0 to the remaining vertices of G. Since c_i is a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $2i$ vertices of G for $1 \leq i \leq n/2$, it follows that $S_{(2,0)}(G) = [[n]]$.

For each odd integer $n \geq 5$, let $G = \left(\frac{n-1}{2}K_2\right) \vee K_1$, where $\frac{n-1}{2}K_2$ is the union of $\frac{n-1}{2}$ vertex-disjoint copies of K_2. For each $1 \leq i \leq (n-1)/2$, let c_i be the coloring that assigns the color 1 to each vertex in exactly i copies of K_2 in G and the color 0 to the remaining vertices of G. Since each c_i is a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $2i$ vertices of G, it follows that $S_{(2,0)}(G) = [[n]]$.

At the other extreme, there are $(2,0)$-colorable graphs having the smallest possible $(2,0)$-spectrum, namely a singleton. A $(2,0)$-colorable graph G of order n is defined in [1] to be a $(2,0)$-*extremal graph* if $\chi_{(2,0)}(G) = n$. Thus if G is a $(2,0)$-extremal graph of order n, then $S_{(2,0)}(G) = \{\chi_{(2,0)}(G)\} = \{n\}$. It is not surprising that $(2,0)$-extremal graphs are not the only $(2,0)$-colorable graphs having a singleton $(2,0)$-spectrum. In fact, if G is a $(2,0)$-colorable graph such that G has a unique monochromatic $(2,0)$-coloring, then $S_{(2,0)}(G)$ is a singleton. This is the case for $(2,0)$-colorable complete bipartite graphs, paths and cycles, as we show next. First, we state a useful observation.

Observation 2.3 [1] *Let u and v be two nonadjacent vertices of a connected $(2,0)$-colorable graph G such that $N(u) = N(v)$. If c is a monochromatic k-coloring of G for some integer $k \geq 2$, then $c(u) = c(v)$.*

Proposition 2.4 *Let G be a $(2,0)$-colorable graph of order $n \geq 3$.*

(a) If G is a complete bipartite graph, then $S_{(2,0)}(G) = \{n\}$.

(b) If G is a path, then $S_{(2,0)}(G) = \{2(n+1)/3\}$.

(c) If G is a cycle, then $S_{(2,0)}(G) = \{2n/3\}$.

Proof. First, we consider the complete bipartite graph $K_{r,s}$, where $1 \leq r \leq s$ and $n = r+s$. By Proposition 2.1, if $K_{r,s}$ is $(2,0)$-colorable, then r and s are both odd. Let U and V be the two partite sets of $K_{r,s}$ and let c be a monochromatic $(2,0)$-coloring of $K_{r,s}$. By Observation 2.3, $c(x) = c(y)$ if x and y belong to the same partite set of $K_{r,s}$. Since c assigns the color 1 to at least one vertex of $K_{r,s}$, we may assume that $c(x) = 1$ for each $x \in U$. On the other hand, if $y \in V$, then $\sigma(y) = |U| + c(y) = 0$ in $\mathbb{Z}_2$. Since $|U|$ is odd, it follows that $c(y) = 1$ and so $c(v) = 1$ for each $v \in V$ by Observation 2.3. Therefore, the only monochromatic $(2,0)$-coloring of $K_{r,s}$ is the coloring that assigns the color 1 to every vertex of $K_{r,s}$; that is, $K_{r,s}$ is $(2,0)$-extremal. Therefore, $S_{(2,0)}(K_{r,s}) = \{\chi_{(2,0)}(K_{r,s})\} = \{r+s\}$.

Next, let $P_n = (v_1, v_2, \ldots, v_n)$ be a $(2,0)$-colorable path of order n. Thus $n \equiv 2 \pmod 3$ by Proposition 2.1. Let c be a monochromatic $(2,0)$-coloring of P_n. We claim that $c(v_1) = 1$; for otherwise, assume that $i_0 \in \{1, 2, \ldots, n\}$ is the smallest integer such that $c(v_{i_0}) = 1$ and so $i_0 \geq 2$.

If $i_0 = 2$, then $\sigma(v_{i_0-1}) = c(v_{i_0-1}) + c(v_{i_0}) = 0 + 1$; while if $i_0 \geq 3$, then $\sigma(v_{i_0-1}) = c(v_{i_0-2}) + c(v_{i_0-1}) + c(v_{i_0}) = 0 + 0 + 1$. In either case, $\sigma(v_{i_0-1}) = 1$, a contradiction. Thus, as claimed, $c(v_1) = 1$. Since $\sigma(v_1) = 0$, it follows that $c(v_2) = 1$. Next, since $\sigma(v_2) = 0$, it forces that $c(v_3) = 0$; while since $\sigma(v_3) = 0$, it follows that $c(v_4) = 1$ and so on. Continuing this procedure, we obtain

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \pmod 3 \\ 0 & \text{if } i \equiv 0 \pmod 3. \end{cases} \tag{4}$$

Therefore, the coloring c defined in (4) is the unique monochromatic $(2,0)$-coloring of P_n. Thus $S_{(2,0)}(P_n) = \{\chi_{(2,0)}(P_n)\} = \{2(n+1)/3\}$.

A similar argument shows that C_n has a unique monochromatic $(2,0)$-coloring (up to isomorphism in this case) and so $S_{(2,0)}(C_n) = \{\chi_{(2,0)}(C_n)\} = \{2n/3\}$ for each $n \geq 3$.

We have seen that if G has a unique $(2,0)$-coloring (up to isomorphism), then $S_{(2,0)}(G)$ is a singleton. However, the converse of this statement is not true in general. For example, let G be the graph obtained from $P_8 = (v_1, v_2, \ldots, v_8)$ and three pairwise nonadjacent vertices x_1, x_2, x_3 by joining each x_i $(1 \leq i \leq 3)$ to both v_4 and v_5 (see Figure 1).

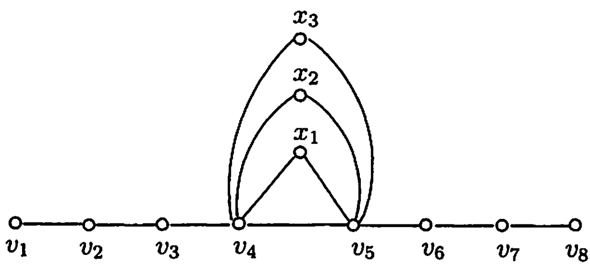

Figure 1: The graph with two different monochromatic $(2,0)$-colorings

Then the coloring c' of G that assigns the color 1 to each vertex in the set $\{v_1, v_2, v_4, v_5, v_7, v_8\}$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G. On the other hand, the coloring c'' that assigns the color 1 to each vertex in the set $\{v_1, v_2, v_4, x_1, x_2, x_3\}$ and the color 0 to the remaining vertices of G is also a monochromatic $(2,0)$-coloring of G. The subgraph induced by the vertices colored 1 in c' is $3K_2$; while the subgraph induced by the vertices colored 1 in c'' is $K_2 + K_{1,3}$ and so c' and c'' are different. We claim that $S_{(2,0)}(G) = \{6\}$; that is, every monochromatic $(2,0)$-coloring of G must assign the color 1 to exactly 6 vertices of G. Let c be any monochromatic $(2,0)$-coloring of

G. It then follows by Observation 2.3 that $c(x_1) = c(x_2) = c(x_3) \in \{0, 1\}$. It can be shown that if $c(x_1) = c(x_2) = c(x_3) = 0$, then $c = c'$; while $c(x_1) = c(x_2) = c(x_3) = 1$, then $c = c''$. Therefore, $S_{(2,0)}(G) = \{6\}$ and G has exactly two different monochromatic $(2,0)$-colorings. In general, using a similar graph structure along with a similar argument to the one just described, we can establish the following.

Theorem 2.5 *For each even integer $k \geq 4$, there exists a $(2,0)$-colorable graph G such that (i) $S_{(2,0)}(G) = \{k\}$ and (ii) G has two monochromatic $(2,0)$-colorings c' and c'' for which the subgraphs induced by the vertices colored 1 by c' and c'', respectively, are non-isomorphic.*

For each graph G of order n we have considered thus far, either

$$|S_{(2,0)}(G)| = 1 \text{ or } |S_{(2,0)}(G)| = \lfloor n/2 \rfloor.$$

This, of course, is not the case in general. As an example, we determine the $(2,0)$-spectrum of the wheel $W_n = C_n \vee K_1$ for each integer $n \geq 3$. In order to do this, we first present some preliminary results and determine the $(2,0)$-chromatic number of wheels.

Proposition 2.6 *If c is a monochromatic $(2,0)$-coloring of a connected $(2,0)$-colorable graph G, then c must assign the color 1 to an even number of even vertices of G. In particular, if G has exactly one even vertex x, then $c(x) = 0$.*

Proof. For a monochromatic $(2,0)$-coloring c of G, let

$$\sigma_c(G) = \sum_{v \in V(G)} \sigma(v).$$

Since $\sigma(v) = 0$ in $\mathbb{Z}_2$ for each $v \in V(G)$, it follows that $\sigma_c(G) = 0$ in $\mathbb{Z}_2$ and $\sigma_c(G)$ is even. Observe that a vertex colored 0 contributes 0 to $\sigma_c(G)$; while each vertex v colored 1 contributes $1 + \deg_G v$ to $\sigma_c(G)$ (namely, $c(v)$ contributes 1 to each color sum $\sigma(u)$ for every $u \in N[v]$). Let V_1 be the set of odd vertices of G and V_2 the set of even vertices of G such that each vertex in $V_1 \cup V_2$ is colored 1 by c. Then

$$\sigma_c(G) = \sum_{v \in V_1} (1 + \deg_G v) + \sum_{v \in V_2} (1 + \deg_G v).$$

Since $\sigma_c(G)$ and $\sum_{v \in V_1} (1 + \deg_G v)$ are both even, $\sum_{v \in V_2} (1 + \deg_G v)$ is even and so $|V_2|$ is even.

Observation 2.7 [1] *Let G be a connected $(2,0)$-colorable graph and let c be a monochromatic $(2,0)$-coloring of G. If S is any set of vertices of G that are colored 0 by c, then the restriction of c to $G-S$ is a monochromatic $(2,0)$-coloring of $G-S$ (where it is possible that the restriction of c assigns the color 0 to every vertex of some component of $G-S$).*

In Observation 2.7, if the restriction of a monochromatic $(2,0)$-coloring of G to the subgraph $G-S$ assigns the color 0 to every vertex of some component G' of $G-S$, then this restriction is called a *trivial* monochromatic $(2,0)$-coloring of G'; Otherwise, it is a *nontrivial* monochromatic $(2,0)$-coloring of G', in which case, c assigns the color 1 to at least one vertex of G'.

For a monochromatic k-coloring $c : V(G) \to \mathbb{Z}_k$, $k \geq 2$, the *complementary coloring* $\bar{c} : V(G) \to \mathbb{Z}_k$ of c is defined by $\bar{c}(v) = k-1-c(v)$. In particular, the complementary coloring $\bar{c}$ of a monochromatic $(2,0)$-coloring c is defined by $\bar{c}(v) = 1 - c(v)$ for each $v \in V(G)$. Thus $c(v) = 0$ if and only if $\bar{c}(v) = 1$ and $c(v) = 1$ if and only if $c(v) = 0$.

Proposition 2.8 [2] *If G is a connected odd-degree graph and c is a monochromatic $(2,0)$-coloring, then either $\bar{c}$ is a trivial coloring that assigns the color 0 to every vertex of G or c is a monochromatic $(2,0)$-coloring of G.*

Theorem 2.9 *For each $n \geq 3$, the wheel $W_n = C_n \vee K_1$ is $(2,0)$-colorable if and only if $n \not\equiv 2,4 \pmod 6$. Furthermore, if W_n is $(2,0)$-colorable, then*

$$\chi_{(2,0)}(W_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 6 \\ n+1 & \text{if } n \equiv 1,5 \pmod 6 \\ \frac{n}{3}+1 & \text{if } n \equiv 3 \pmod 6. \end{cases} \quad (5)$$

Proof. For an integer $n \geq 3$, let $W_n = C_n \vee K_1$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $V(K_1) = \{v\}$. First, suppose that W_n is $(2,0)$-colorable. Assume, to the contrary, that $n \equiv 2,4 \pmod 6$. Let c be a monochromatic $(2,0)$-coloring. Since n is even, v is the only even vertex of W_n and so $c(v) = 0$ by Proposition 2.6. However then, the restriction of c to the cycle $C_n = W_n - v$ is a (nontrivial) monochromatic $(2,0)$-coloring of C_n by Observation 2.7. This is a contradiction by Proposition 2.1.

For the converse, assume that $n \not\equiv 2,4 \pmod 6$. If n is odd, then W_n is an odd-degree graph and so W_n is $(2,0)$-colorable. Thus, we may assume that n is even and so $n \equiv 0 \pmod 6$. Since the coloring $c : V(W_n) \to \mathbb{Z}_2$ defined by

$$c(x) = \begin{cases} 1 & \text{if } x = v_i \text{ where } i \equiv 1,2 \pmod 3 \text{ and } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

is a monochromatic $(2,0)$-coloring of W_n, it follows that W_n is $(2,0)$-colorable. It remains to verify (5). First, suppose that $n \equiv 0 \pmod 6$. Since the monochromatic $(2,0)$-coloring described in (6) assigns the color 1 to exactly $2n/3$ vertices of W_n, it follows that $\chi_{(2,0)}(W_n) \le 2n/3$.

Let c be a minimum monochromatic $(2,0)$-coloring of W_n. Since v is the only even vertex in W_n, it follows that $c(v) = 0$. We may assume, without loss of generality, that $c(v_1) = 1$. Since $\sigma(v_1) = 0$, exactly one of $c(v_n)$ and $c(v_2)$ is 0, say $c(v_n) = 0$ and $c(v_2) = 1$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 0$ and then $c(v_4) = 1$. Continuing this procedure, we have $c(v_i) = 1$ if $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 0$ if $i \equiv 0 \pmod 3$. Thus c assigns the color 1 to at least $2n/3$ vertices of W_n and so $\chi_{(2,0)}(W_n) \ge 2n/3$. Hence $\chi_{(2,0)}(W_n) = 2n/3$ when $n \equiv 0 \pmod 6$.

Next, suppose that $n \equiv 1, 5 \pmod 6$ and we show that $\chi_{(2,0)}(W_n) = n + 1$. Assume, to the contrary, that $\chi_{(2,0)}(W_n) \le n$. Let c be a minimum monochromatic $(2,0)$-coloring of W_n. By Proposition 2.1, $c(v) = 1$ and so $c(v_i) = 0$ for some i with $1 \le i \le n$, say $c(v_1) = 0$. Since $\sigma(v_1) = 0$, exactly one of $c(v_n)$ and $c(v_2)$ is 0, say $c(v_n) = 1$ and $c(v_2) = 0$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 1$ and then $c(v_4) = 0$. Continuing this procedure, we have $c(v_i) = 0$ if $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 1$ if $i \equiv 0 \pmod 3$. Since $n \equiv 1, 5 \pmod 6$, it follows that $n \equiv 1, 2 \pmod 3$ and so $c(v_n) = 0$, which is a contradiction. Therefore, $\chi_{(2,0)}(W_n) = n + 1$ when $n \equiv 1, 5 \pmod 6$.

Finally, suppose that $n \equiv 3 \pmod 6$ and we show that $\chi_{(2,0)}(W_n) = 1 + n/3$. Since the coloring $c : V(W_n) \to \mathbb{Z}_2$ defined by

$$c(x) = \begin{cases} 1 & \text{if } x = v \text{ or } x = v_i \text{ where } i \equiv 1 \pmod 3 \text{ and } 1 \le i \le n \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

is a monochromatic $(2,0)$-coloring of W_n, it follows that $\chi_{(2,0)}(W_n) \le 1 + n/3$. To show that $\chi_{(2,0)}(W_n) \ge 1 + n/3$, let c be a minimum monochromatic $(2,0)$-coloring of W_n. First, suppose that $c(v) = 0$. Then the restriction of c to C_n is a monochromatic $(2,0)$-coloring of C_n. Since $\chi_{(2,0)}(C_n) = 2n/3$ when $n \equiv 0 \pmod 3$, a contradiction is produced. Next, suppose that $c(v) = 1$. Since $\chi_{(2,0)}(W_n) \le 1 + n/3$, it follows that $c(v_i) = 0$ for some i with $1 \le i \le n$, say $c(v_1) = 0$. Since $\sigma(v_1) = 0$, exactly one of $c(v_n)$ and $c(v_2)$ is 0, say $c(v_n) = 1$ and $c(v_2) = 0$. Since $\sigma(v_2) = 0$, it follows that $c(v_3) = 1$. Since $\sigma(v_3) = 0$, this implies that $c(v_4) = 0$ and so $c(v_5) = 0$. Continuing in this procedure, we have $c(v_i) = 0$ if $i \equiv 1, 2 \pmod 3$ and $c(v_i) = 1$ if $i \equiv 0 \pmod 3$. Hence c assigns the color 1 to at least $1 + n/3$ vertices of W_n and so $\chi_{(2,0)}(W_n) \ge 1 + n/3$. Therefore, $\chi_{(2,0)}(W_n) = 1 + n/3$ when $n \equiv 3 \pmod 6$.

We are now prepared to present the following result.

Theorem 2.10 *For each integer $n \geq 3$, if $W_n = C_n \vee K_1$ is $(2,0)$-colorable, then*

$$S_{(2,0)}(W_n) = \begin{cases} \{2n/3\} & \text{if } n \equiv 0 \pmod 6 \\ \{n+1\} & \text{if } n \equiv 1,5 \pmod 6 \\ \{(n/3)+1, 2n/3, n+1\} & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

Proof. Let $W_n = C_n \vee K_1$ where $C_n = (v_1, v_2, \ldots, v_n, v_1)$ for some integer $n \geq 3$ and $V(K_1) = \{v\}$. By Theorem 2.9, $n \not\equiv 2, 4 \pmod 6$. First, suppose that $n \equiv 0 \pmod 6$. Since $\chi_{(2,0)}(W_n) = 2n/3$, it follows that there is no monochromatic $(2,0)$-coloring of W_n that assigns the color 1 to ℓ vertices where $2 \leq \ell \leq 2n/3 - 2$. Now assume that there is a monochromatic $(2,0)$-coloring c of W_n that assigns the color 1 to k vertices where $2n/3 + 2 \leq k \leq n$. Since v is the only vertex that is even degree, $c(v) = 0$. Moreover, since $2n/3 + 2 \leq k \leq n$, there are three consecutive vertices of C_n that are assigned the color 1 by c; without loss of generality, say $c(v_1) = c(v_2) = c(v_3) = 1$. This, however, implies that $\sigma(v_2) = 1$, which is impossible. Thus such a monochromatic $(2,0)$-coloring c of W_n does not exist. Therefore, $S_{(2,0)}(W_n) = \{2n/3\}$.

Next, suppose that $n \equiv 1, 5 \pmod 6$. Since $\chi_{(2,0)}(W_n) = n + 1$, it follows that the only $(2,0)$-coloring of W_n is the coloring that assigns the color 1 to every vertex of W_n and so $S_{(2,0)}(W_n) = \{n+1\}$. Finally, we let $n \equiv 3 \pmod 6$. Note that since W_n is an odd-degree graph, it follows that the coloring that assigns the color 1 to every vertex of W_n is a monochromatic $(2,0)$-coloring of W_n. Since $\chi_{(2,0)}(W_n) = (n/3) + 1$, there is a monochromatic $(2,0)$-coloring of W_n that assigns the color 1 to exactly $(n/3) + 1$ vertices of W_n. By Proposition 2.8 then, W_n has a monochromatic $(2,0)$-coloring that assigns the color 1 to exactly $(n+1)-(n/3+1) = 2n/3$ vertices of W_n. Furthermore, there is no $(2,0)$-coloring of W_n that assigns the color 1 to exactly ℓ vertices for each ℓ with $2 \leq \ell \leq n/3 - 1$ or $2n/3 + 2 \leq \ell \leq n-1$. Now assume that there is a monochromatic $(2,0)$-coloring c of W_n that assigns the color 1 to t vertices where $n/3 + 3 \leq t \leq 2n/3 - 2$. If $c(v) = 0$, then the restriction of c to C_n is a monochromatic $(2,0)$-coloring of C_n that assigns the color 1 to t vertices of C_n, which is impossible since $\chi_{(2,0)}(C_n) = 2n/3$. If $c(v) = 1$, then since $\sigma(v) = 0$, there is a vertex v_i for some i $(1 \leq i \leq n)$ such that $c(v_i) = 1$; without loss of generality, assume that $c(v_1) = 1$. Since $\sigma(v_1) = 0$, it follows that c assigns the same color to v_2 and v_n. If $c(v_2) = c(v_n) = 1$, then it follows that every vertex of W_n is assigned the color 1, which is impossible. Now, if $c(v_2) = c(v_n) = 0$, then since $\sigma(v_2) = 0$, c assigns the color 0 to v_3. Since $\sigma(v_3) = 0$, $c(v_4) = 1$. Continuing this procedure, we have that

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 3 \\ 0 & \text{if } i \equiv 0, 2 \pmod 3). \end{cases}$$

So the number of vertices colored 1 by c is $(n/3)+1$, which is impossible. Hence $S_{(2,0)}(W_n) = \{(n/3)+1, 2n/3, n+1\}$. ∎

For each integer $n \geq 3$, let $C_n \,\square\, K_2$ be the Cartesian product of C_n and K_2 and let $\mathrm{cor}(C_n)$ be the corona of C_n. Since $C_n \,\square\, K_2$ and $\mathrm{cor}(C_n)$ are odd-degree graphs for $n \geq 3$, they are both $(2,0)$-colorable. The $(2,0)$-spectra of these two classes of graphs are presented in the next result.

Theorem 2.11 *For each integer $n \geq 3$, if $G \in \{C_n \,\square\, K_2, \mathrm{cor}(C_n)\}$, then*

$$S_{(2,0)}(G) = \begin{cases} \{n, 2n\} & \text{if } n \text{ is even} \\ \{2n\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $G \in \{C_n \,\square\, K_2, \mathrm{cor}(C_n)\}$. If n is odd, then G is a $(2,0)$-extremal graph of order $2n$ and so $S_{(2,0)}(G) = \{2n\}$. Thus, we may assume that n is even. Since G is an odd-degree graph, the coloring that assign the color 1 to each vertex of G is a monochromatic $(2,0)$-coloring of G and so $2n \in S_{(2,0)}(G)$. Since $\chi_{(2,0)}(G) = n$, each monochromatic $(2,0)$-coloring must assign the color 1 to at least n vertices of G. Suppose that c is a monochromatic $(2,0)$-coloring of G that assigns the color 1 to ℓ vertices of G where $n \leq \ell \leq 2n-2$. By Proposition 2.8, $\overline{c}$ is a monochromatic $(2,0)$-coloring of G that assigns the color 1 to $2n-\ell \leq n$ vertices of G. This implies that $\ell = n$. Consequently, every monochromatic $(2,0)$-coloring of G either assigns the color 1 to all vertices of G or to exactly n vertices of G. Therefore, $S_{(2,0)}(G) = \{n, 2n\}$. ∎

3 A Realization Result

We have seen that if G is a nontrivial connected $(2,0)$-colorable graph of order n, then $1 \leq |S_{(2,0)}(G)| \leq \lfloor n/2 \rfloor$. We now show that every pair k, n of integers with $1 \leq k \leq \lfloor n/2 \rfloor$ can be realized as the cardinality of the spectrum and the order of a connected $(2,0)$-colorable graph, respectively. The following observation will be useful to us.

Observation 3.1 *If uv is a pendant edge in a $(2,0)$-colorable graph G, then $c(u) = c(v)$ for every monochromatic $(2,0)$-coloring c of G.*

Theorem 3.2 *For every pair k, n of integers with $1 \leq k \leq \lfloor n/2 \rfloor$, there is a connected $(2,0)$-colorable graph G of order n such that $|S_{(2,0)}(G)| = k$.*

Proof. Let k and n be integers such that $1 \leq k \leq \lfloor n/2 \rfloor$. Since the result is true for $k=1$ or $k = \lfloor n/2 \rfloor$ by Propositions 2.4 and 2.2, we may assume that $2 \leq k \leq \lfloor n/2 \rfloor - 1$. Thus $n \geq 6$. We consider two cases, according to whether n is odd or n is even.

Case 1. *n is odd.* Let $H = kK_2 \vee K_1$ be the join of kK_2 and K_1, where $V(K_1) = \{v\}$ and $E(kK_2) = \{v_{i1}v_{i2} : 1 \le i \le k\}$, and let $F = K_{1,n-2k-2}$ be the star of order $n - 2k - 1$, where $V(F) = \{u, u_1, u_2, \ldots, u_{n-2k-2}\}$ and u is the central vertex of F. The graph G of order n is obtained from H and F by adding the edge uv. We show that $|S_{(2,0)}(G)| = k$. For each j with $1 \le j \le k$, the coloring c_j that assigns the color 1 to the vertices in $V_j = \{v_{i1}, v_{i2} : 1 \le i \le j\}$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G. Hence $\{2, 4, \ldots, 2k\} \subseteq S_{(2,0)}(G)$ and so $|S_{(2,0)}(G)| \ge k$.

Next, we show that $|S_{(2,0)}(G)| \le k$. If $|S_{(2,0)}(G)| > k$, then there is a monochromatic $(2,0)$-coloring c of G such that c assigns the color 1 to $\ell \ge 2(k+1)$ vertices of G. First, suppose that $c(v) = 0$. Since $|V(H)-\{v\}| = 2k$, there is a vertex in $V(F)$ that is colored 1 by c. It then follows by Observation 3.1 that every vertex in $V(F)$ must be colored 1 by c. Since $c(v) = 0$, it follows that $c(v_{i1}) = c(v_{i2})$ for $1 \le i \le k$. However then, $\sigma(v) = 1$ which is impossible. Next, suppose that $c(v) = 1$. Since $\sigma(u) = 0$ and $c(v) = 1$, it follows that every vertex in $V(F)$ must be colored 1 by c. Since n is odd, $n - 2k - 2$ is odd and so $\sigma(u) = 1$, which is impossible. Therefore, $|S_{(2,0)}(G)| \le k$ and so $|S_{(2,0)}(G)| = k$.

Case 2. *n is even.* Let H be defined as in Case 1. The graph G of order n is obtained from H by (i) adding $n - 2k - 1$ new vertices u, u_1, u_2, $\ldots$, u_{n-2k-2} and (ii) joining u to v_{11} and v_{12} and joining each of u_i $(1 \le i \le n-2k-2)$ to the vertex v. We show that $|S_{(2,0)}(G)| = k$. For each j with $1 \le j \le k$, the coloring c_j that assigns the color 1 to the vertices in $V_j = \{v_{i1}, v_{i2} : 1 \le i \le j\}$ and the color 0 to the remaining vertices of G is a monochromatic $(2,0)$-coloring of G. Hence $\{2, 4, \ldots, 2k\} \subseteq S_{(2,0)}(G)$ and so $|S_{(2,0)}(G)| \ge k$.

Next, we show that $|S_{(2,0)}(G)| \le k$. If $|S_{(2,0)}(G)| > k$, then there is a monochromatic $(2,0)$-coloring c of G such that c assigns the color 1 to $\ell \ge 2(k+1)$ vertices of G. First, suppose that $c(v) = 0$. Then $c(u_i) = 0$ for $1 \le i \le n - 2k - 2$ by Observation 3.1. Thus c assigns the color 1 to at most $2k + 1$ vertices in G, which is a contradiction. Next, suppose that $c(v) = 1$. Since $\sigma(v_{11}) = c(v) + c(v_{11}) + c(v_{12}) + c(u) = 0$, it follows that either exactly one vertex in $\{v_{11}, v_{12}, u\}$ is colored 1 or every vertex in $\{v_{11}, v_{12}, u\}$ are colored 1. In either case, however, we have that $\sigma(u) = 1$, which is impossible. Hence $|S_{(2,0)}(G)| \le k$ and so $|S_{(2,0)}(G)| = k$. ■

4 On Realizable Sets

In the proof of Theorem 3.2, we saw that for each pair k, n of positive integers with $2 \le k \le \lfloor n/2 \rfloor$, there is a connected graph G of order n such that $S_{(2,0)}(G) = \{2, 4, \ldots, 2k\}$. This gives rise to the following question: Let S be a set consisting of $k \geq 2$ positive even integers. Does there exist a connected graph G such that $S_{(2,0)}(G) = S$? If the answer is yes, then the set S is referred to as a *monochromatic* $(2,0)$*-realizable set* or simply a $(2,0)$-*realizable set*; otherwise, S is a non-$(2,0)$-realizable. Thus, $\{2, 4, \ldots, 2k\}$ is $(2,0)$-realizable for each integer $k \geq 2$. In fact, for each even integer $s \geq 2$, the singleton $\{s\}$ is $(2,0)$-realizable. For example, if $s = 2$, let $G = K_3$ and so $S_{(2,0)}(G) = \{2\}$; while if $s \geq 4$, let G be a $(2,0)$-colorable complete bipartite graph of order s and so $S_{(2,0)}(G) = \{s\}$ by Proposition 2.4. On the other hand, there are infinitely many sets of positive even integers that are not the $(2,0)$-spectrum of any $(2,0)$-colorable graph. To show this, we first present an additional definition and a useful observation.

For two monochromatic (s,t)-colorings c and c' of a connected graph G and two integers x and y, define *the linear combination* $xc+yc' : V(G) \to \mathbb{Z}_s$ of c and c' by

$$(xc + yc')(v) = xc(v) + yc'(v) \text{ for each } v \in V(G).$$

Lemma 4.1 *If c and c' are monochromatic (s,t)-colorings of a connected graph G, then each linear combination $xc + yc'$ of c and c' is a monochromatic $(s,(x+y)t)$-coloring of G for all integers x and y.*

Proof. For a vertex $v \in V(G)$, let $\sigma_c(v) = as + t$ and $\sigma_{c'}(v) = bs + t$, where $a, b \in \mathbb{Z}$. Let $c'' = xc + yc'$. Then $\sigma_{c''}(v) = x(as + t) + y(bs + t) = (xa + yb)s + (x + y)t$. Thus $\sigma_{c''}(v) = (x + y)t \in \mathbb{Z}_s$. ■

The following is an immediate consequence of Lemma 4.1.

Corollary 4.2 *If c and c' are monochromatic $(s,0)$-colorings of a connected graph G for some integer $s \geq 2$, then each linear combination is also a monochromatic $(s,0)$-coloring of G.*

We are now prepared to show that there are infinitely many sets of positive even integers that are not the $(2,0)$-spectrum of any $(2,0)$-colorable graph.

Proposition 4.3 *For each integer $k \geq 32$, the set $\{6, 14, k\}$ is not the monochromatic $(2,0)$-spectrum for any connected graph.*

Proof. Assume, to the contrary, that the set $S = \{6, 14, k\}$ is the monochromatic $(2,0)$-spectrum for some connected graph G. For each integer $i \in \{6, 14, k\}$, let c_i be a monochromatic $(2,0)$-coloring of G that assigns the color 1 to exactly i vertices of G and let V_i be the set of vertices colored 1 by c_i. By Corollary 4.2, for each pair i, j of distinct integers where $i, j \in S$ and $i \neq j$, the linear combination $c_i + c_j$ of c_i and c_j is also a monochromatic

(2, 0)-coloring of G. Let $V_{i,j}$ be the set of vertices colored 1 by $c_i + c_j$ where $i, j \in S$ and $i \neq j$. Then

$$|V_{i,j}| = |V_i| + |V_j| - 2|V_i \cap V_j|. \tag{8}$$

Also, the linear combination $c_6 + c_{14} + c_k$ is a monochromatic (2, 0)-coloring of G. Let $V_{6,14,k}$ be the set of vertices colored 1 by $c_6+c_{14}+c_k$. Since S is the monochromatic (2, 0)-spectrum G, it follows that $|V_{i,j}| \in S$ for each pair i, j of distinct integers where $i, j \in S$ and $|V_{6,14,k}| \in S$. Let $W_1 = V_6 - (V_{14} \cup V_k)$, $W_2 = (V_6 \cap V_{14}) - V_k$, $W_3 = V_{14} - (V_6 \cup V_k)$, $W_4 = (V_6 \cap V_k) - V_{14}$, $W_5 = V_6 \cap V_{14} \cap V_k$, $W_6 = (V_{14} \cap V_k) - V_6$ and $W_7 = V_k - (V_6 \cup V_{14})$. Let $w_i = |W_i|$ for $1 \leq i \leq 7$. Since $0 \leq |V_6 \cap V_{14}| \leq 6$ and $|V_{6,14}| \in S$, it follows by (8) that $|V_{6,14}| \leq 20$, which implies that $|V_6 \cap V_{14}| = 3$ (and so $|V_{6,14}| = 14$). Similarly, $|V_6 \cap V_k| = 3$ (and so $|V_{6,k}| = k$); while $|V_{14} \cap V_k| = 7$ (and so $|V_{14,k}| = k$). Since $w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = |V_6 \cup V_{14}| = 17$, it follows that $w_7 = |V_k - (V_6 \cup V_{14})| = k - 17 \geq 15$, which implies that $|V_{14,k}| \geq 15$ and $|V_{6,14,k}| \geq 15$. Since $S = \{6, 14, k\}$ is the monochromatic (2, 0)-spectrum of G, it follows that $|V_{6,14,k}| = k$. Therefore, we have the following:

$$\begin{aligned}
|V_6| &= 6 = w_1 + w_2 + w_4 + w_5 \\
|V_{14}| &= 14 = w_2 + w_3 + w_5 + w_6 \\
|V_k| &= k = w_4 + w_5 + w_6 + w_7 \\
|V_6 \cap V_{14}| &= 3 = w_2 + w_5 \\
|V_6 \cap V_k| &= 3 = w_4 + w_5 \\
|V_{14} \cap V_k| &= 7 = w_5 + w_6 \\
|V_{14,k}| &= k = w_2 + w_3 + w_4 + w_7 \\
|V_{6,14,k}| &= k = w_1 + w_3 + w_5 + w_7.
\end{aligned}$$

Since $w_2 + w_5 = w_4 + w_5 = 3$, it follows that $w_2 = w_4$. Now $w_2 + w_3 + w_4 + w_7 = w_1 + w_3 + w_5 + w_7 = k$ implies that $w_2 + w_4 = w_1 + w_5$. Because $w_1 + w_2 + w_4 + w_5 = 6$, it follows that $w_2 + w_4 = w_1 + w_5 = 3$. However then, $w_2 = w_4$ and $w_2 + w_4 = 3$, which is impossible.

With the aid of an argument similar to the one used in the proof of Proposition 4.3, we can show that for any given integer $t \geq 3$, there is a set of t positive even integers that is not (2, 0)-realizable; that is, the following is a consequence of Proposition 4.3.

Corollary 4.4 *For each integer $t \geq 3$, there is a set S of positive even integers such that $|S| = t$ and S is not the (2, 0)-spectrum of any (2, 0)-colorable graph.*

We have seen that there are infinitely many sets of positive even integers that are not the $(2,0)$-spectrum of any $(2,0)$-colorable graph. However, every set of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph. In order to establish this fact, we first present two additional definitions and some preliminary results. A vertex v is a *zero-vertex* of a $(2,0)$-colorable graph G if $c(v) = 0$ for every monochromatic $(2,0)$-coloring c of G. Similarly, a vertex v is a *one-vertex* of a $(2,0)$-colorable graph G if $c(v) = 1$ for every monochromatic $(2,0)$-coloring c of G. The following three results provide useful information on certain $(2,0)$-realizable sets related to the spectra of $(2,0)$-colorable graphs having zero-vertices or one-vertices. We illustrate the proofs of these results by verifying the first one.

Theorem 4.5 *For each integer $i = 1,2$, let G_i be a $(2,0)$-colorable graph having a zero-vertex and let $S_{(2,0)}(G_i) = S_i$. Then the following set is a $(2,0)$-realizable set:*

$$S_1 \cup S_2 \cup \{s_1 + s_2 : \ s_i \in S_i \text{ for } i = 1,2\}. \tag{9}$$

Proof. Let S be the set described in (9). For each integer $i = 1,2$, let v_i be a zero-vertex of G_i. Now, let G be the graph obtained from G_i $(i = 1,2)$ by adding two new vertices x and y and joining each of x and y to both v_1 and v_2. We show that G is $(2,0)$-colorable and $S_{(2,0)}(G) = S$.

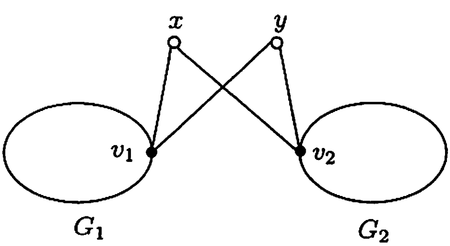

Figure 2: The graph G in the proof of Theorem 4.5

Let $s \in S$. Then either $s = s_i$ where $i \in \{1,2\}$ or $s = s_1 + s_2$, where $s_i \in S_i$ for $i = 1,2$. For $i = 1,2$, let c_i be a $(2,0)$-coloring of G_i such that c_i assigns the color 1 to exactly s_i vertices of G_i. First, suppose that $s = s_i$ for some $i \in \{1,2\}$, say $i = 1$. Since $c_1(v_1) = 0$, the coloring c_1 can be extended to a monochromatic $(2,0)$-coloring c of G by defining $c(v) = c_1(v)$ if $v \in V(G_1)$ and $c(v) = 0$ otherwise. Thus G is $(2,0)$-colorable and c assigns the color 1 to exactly s_1 vertices of G and so $s = s_1 \in S_{(2,0)}(G)$. Therefore, $S_1 \cup S_2 \subseteq S$. Next, suppose that $s = s_1 + s_2$. Define a monochromatic $(2,0)$-coloring c of G from both c_1 and c_2 by defining $c(v) = c_i(v)$ if $v \in V(G_i)$ for

$i = 1, 2$ and $c(x) = c(y) = 0$. Then c assigns the color 1 to exactly $s_1 + s_2$ vertices of G and so $s = s_1 + s_2 \in S_{(2,0)}(G)$. Therefore, $S \subseteq S_{(2,0)}(G)$.

To show that $S_{(2,0)}(G) \subseteq S$, let c be a monochromatic $(2,0)$-coloring of G that assigns the color 1 to exactly s vertices of G. For each $i = 1, 2$, let c_i be the restrictions of c to G_i. Since $N(x) = N(y) = \{v_1, v_2\}$, it follows by Observation 2.3 that $c(x) = c(y)$. We claim that $c(x) = c(y) = 0$; for otherwise, assume that $c(x) = c(y) = 1$. Since $\sigma_c(x) = 0$, exactly one of v_1 and v_2 is assigned the color 1 by c. We may assume, without loss of generality, that $c(v_1) = 1$. Since $\sigma_{c_1}(v) = \sigma_c(v) = 0$ in $\mathbb{Z}_2$ if $v \in V(G_1) - \{v_1\}$ and $\sigma_{c_1}(v_1) = \sigma_c(v_1) - 2 = 0 - 2 = 0$ in $\mathbb{Z}_2$, it follows that c_1 is a monochromatic $(2,0)$-coloring of G_1. Since $c_1(v_1) = 1$, this contradicts the defining property of v_1. Therefore, $c(x) = c(y) = 0$, as claimed. Observe that for each $i = 1, 2$, either c_i is a trivial coloring (that assigns the color 0 to each vertex of G_i) or c_i is a monochromatic $(2,0)$-coloring of G_i. Since $c(x) = c(y) = 0$, at least one of c_1 and c_2 is nontrivial; so the nontrivial coloring c_i $(i = 1, 2)$ is a monochromatic $(2,0)$-coloring of G_i. Suppose that c_i $(i = 1, 2)$ assigns the color 1 to exactly s_i vertices of G_i, where then either $s_i = 0$ or $s_i \in S_i$ and at least one of s_1 and s_2 is not zero. Thus c assigns the color 1 to exactly $s = s_1 + s_2$ vertices of G, which implies that either $s \in S_1$, or $s \in S_2$, or $s = s_1 + s_2$ where $s_1 \in S_1$ and $s_2 \in S_2$. Hence $S_{(2,0)}(G) \subseteq S$ and so $S_{(2,0)}(G) = S$. Therefore, S is a $(2,0)$-realizable set.

Theorem 4.6 *For each integer $i = 1, 2$, let G_i be a $(2,0)$-colorable graph having a one-vertex and let $S_{(2,0)}(G_i) = S_i$. Then the following set is a $(2,0)$-realizable set:*

$$\{s_1 + 2, s_2 + 2, s_1 + s_2 : \ s_i \in S_i \ for\ i = 1, 2\}. \tag{10}$$

Theorem 4.7 *For each integer $i = 1, 2$, let G_i be a $(2,0)$-colorable graph and let $S_{(2,0)}(G_i) = S_i$. If G_1 contain a zero-vertex and G_2 contains a one-vertex, then the following set is a $(2,0)$-realizable set:*

$$S_1 \cup \{s_2 + 2, s_1 + s_2 + 2 : s_i \in S_i \ for\ i = 1, 2\}. \tag{11}$$

We are now prepared to present the following result.

Theorem 4.8 *Every set of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph.*

Proof. Let S be a set of positive even integers. We proceed by induction on the cardinality of a set S to prove the following stronger result:

> Every set S of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph containing a zero-vertex.

First, suppose that $S = \{s\}$ for some positive integer s. For $s = 2$, let $G = K_{1,3} + e$ and so $S_{(2,0)}(G) = \{2\}$ and the end-vertex (and the vertex of degree 3) is a zero-vertex of G. For $s \geq 4$, let G be a path of order $3s/2 - 1$. Since $3s/2 - 1 \equiv 2 \pmod 3$, it follows by Proposition 2.1 that G is (2,0)-colorable. Moreover, $S_{(2,0)}(G) = \{s\}$ and G has zero-vertices by the proof in Proposition 2.4. Suppose for some integer $k \geq 2$ and every integer ℓ with $1 \leq \ell \leq k - 1$ that each ℓ-element set of positive even integers is a subset of the $(2,0)$-spectrum of some $(2,0)$-colorable graph containing a zero-vertex. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a set consisting of k positive even integers and consider $S_1 = S - \{s_k\}$ and $S_2 = \{s_k\}$. By induction hypothesis, there are $(2,0)$-colorable graphs G_1 and G_2 such that $S_i \subseteq S_{(2,0)}(G_i)$ for $i = 1, 2$ and each graph G_i contains a zero-vertex. Let v_i be a zero-vertex of G_i for $i = 1, 2$ and let G be the graph obtained from G_1 and G_2 by adding two new vertices x and y and joining each of x and y to both v_1 and v_2. It then follows by Theorem 4.5 that $S_{(2,0)}(G) = S_1 \cup S_2 \cup \{s_1 + s_k : s_1 \in S_1\}$ (so $S \subseteq S_{(2,0)}(G)$) and x is a zero-vertex of G. ■

By Theorem 4.8, every set of positive even integers is a subset of a $(2,0)$-realizable set. Consequently, we have the following result.

Corollary 4.9 *For each set S of positive even integers, there is an infinite sequence $S_1, S_2, \ldots, S_n, \ldots$ of $(2,0)$-realizable sets such that $S \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_n \subsetneq \cdots$.*

We conclude this work with the following problem.

Problem 4.10 *Under which conditions, is a set of positive even integers the $(2,0)$-spectrum of some $(2,0)$-colorable graph?*

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