On Twin Edge Colorings in Trees

Eric Andrews, Daniel Johnston and Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

Abstract

For a connected graph G of order at least 3 and an integer $k \geq 2$, a twin edge k-coloring of G is a proper edge coloring of G with the elements of $\mathbb{Z}_k$ so that the induced vertex coloring in which the color of a vertex v in G is the sum (in $\mathbb{Z}_k$) of the colors of the edges incident with v is a proper vertex coloring. The minimum k for which G has a twin edge k-coloring is called the twin chromatic index of G and is denoted by $\chi'_t(G)$. It was conjectured that $\Delta(T) \leq \chi'_t(T) \leq 2 + \Delta(T)$ for every tree of order at least 3, where $\Delta(T)$ is the maximum degree of T. This conjecture is verified for several classes of trees, namely brooms, double stars and regular trees.

Key Words: edge coloring, vertex coloring, induced coloring, trees.

AMS Subject Classification: 05C05, 05C15.

1 Introduction

Among the various colorings studied in graph theory, the best-known and most-studied are proper vertex colorings and proper edge colorings. In a *proper vertex coloring* of a graph G, each vertex of G is assigned a color from a given set of colors where adjacent vertices are colored differently. The minimum number of colors needed in a proper vertex coloring of a graph G is the chromatic number of G and denoted by $\chi(G)$. While $\chi(G) \leq \Delta(G)+1$ where $\Delta(G)$ is the maximum degree of G, one of the famous theorems in this area of research is due to Brooks [3], who proved that $\chi(G) \leq \Delta(G)$ for every connected graph G that is not an odd cycle or a complete graph. In a *proper edge coloring* of a graph G, each edge of G is assigned a color from a given set of colors where adjacent edges are colored differently. The minimum number of colors needed in a proper edge coloring of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. Thus $\chi'(G) \geq \Delta(G)$ for every nonempty graph G. The classic theorem in this connection is due to Vizing [7] who proved that $\chi'(G) \leq \Delta(G) + 1$ for every nonempty graph G.

A related and also well-studied graph coloring is the so-called *total coloring* of a graph G that assigns colors to both the vertices and edges of G so that not only the vertex coloring and edge coloring are proper but no vertex and an incident edge are assigned the same color. The minimum number of colors required for a total coloring of G is the *total chromatic number* of G, denoted by $\chi''(G)$. It then follows that $\chi''(G) \geq \Delta(G) + 1$. It was conjectured independently by Behzad and Vizing that $\chi''(G) \leq \Delta(G) + 2$ for every graph G (see [6, pp. 282]).

Recently, another related coloring was introduced by Chartrand and studied in [1, 2]. For a connected graph G of order at least 3, let $c : E(G) \to \mathbb{Z}_k$ be a proper edge coloring of G for some integer $k \geq 2$. A vertex coloring $c' : V(G) \to \mathbb{Z}_k$ is then defined by

$$c'(v) = \sum_{e \in E_v} c(e) \text{ in } \mathbb{Z}_k,$$

where E_v is the set of edges of G incident with a vertex v and the indicated sum is computed in $\mathbb{Z}_k$. If the induced vertex coloring c' is a proper vertex coloring of G, then c is referred to as a *twin edge k-coloring* or simply a *twin edge coloring* of G. The minimum k for which G has a twin edge k-coloring is called the *twin chromatic index* of G and is denoted by $\chi'_t(G)$. This concept was introduced by Gary Chartrand and studied in [1, 2]. Since a twin edge coloring is not only a proper edge coloring of G but induces a proper vertex coloring of G, it follows that

$$\chi'_t(G) \geq \max\{\chi(G), \chi'(G)\}. \tag{1}$$

Since $\max\{\chi(G), \chi'(G)\} = \chi'(G)$ except when G is a complete graph of even order, we have $\chi'_t(G) \geq \chi'(G)$ except when G is a complete graph of even order as we describe in Theorem 1.2. While $\chi'_t(G)$ does not exist if G is the connected graph of order 2, every connected graph of order at least 3 has a twin edge coloring. A twin edge 4-coloring of a graph G is shown in Figure 1. In fact, $\chi'_t(G) = 4$ for this graph G.

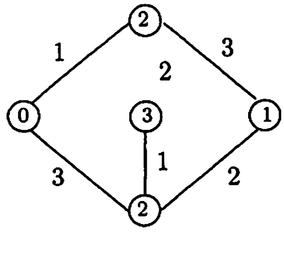

Figure 1: A twin edge 4-coloring of a graph

It was shown in [1] that $\chi'_t(C_5) = 5 = \Delta(C_5) + 3$. For each connected graph that is not the 5-cycle, the following conjecture analogous to the Total Coloring Conjecture.

Conjecture 1.1 [2] *If G is a connected graph of order at least 3 that is not a 5-cycle, then $\chi'_t(G) \leq \Delta(G) + 2$.*

Conjecture 1.1 was verified in [1] for several well-known classes of graphs, namely paths, cycles, complete graphs and complete bipartite graphs, which we state next.

Theorem 1.2 [1] *If n, a, b are integers with $n \geq 3$, $1 \leq a \leq b$ and $b \geq 2$, then*

$$\chi'_t(P_n) = 3$$

$$\chi'_t(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3 \\ 4 & \text{if } n \not\equiv 0 \pmod 3 \text{ and } n \neq 5 \\ 5 & \text{if } n = 5 \end{cases}$$

$$\chi'_t(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases}$$

$$\chi'_t(K_{a,b}) = \begin{cases} b & \text{if } b \geq a+2 \text{ and } a \geq 2 \\ b+1 & \text{if either } a = 1 \text{ and } b \not\equiv 1 \pmod 4 \\ & \text{or } b = a+1 \geq 3 \\ b+2 & \text{if either } a = 1 \text{ and } b \equiv 1 \pmod 4 \\ & \text{or } b = a \geq 2. \end{cases}$$

Conjecture 1.1 was verified in [2] for several classes of graphs having small maximum degree, namely all permutation graphs of the 5-cycle C_5 (see [4]), prisms, grids and all trees of maximum degree at most 6. We state the result on trees next.

Theorem 1.3 [2] *If T is a tree of order at least 3 such that $\Delta(T) \leq 6$, then*

$$\chi'_t(T) \leq \Delta(T) + 2.$$

In this work, we verify Conjecture 1.1 for several classes of trees, namely brooms, double stars and regular trees. The following two definitions will be useful to us. For integers a and b with $a < b$, let

$$[a..b] = \{a, a+1, \ldots, b\}$$

be the set of integers between a and b and let $\sigma(a,b)$ denote the sum of integers between a and b, that is,

$$\sigma(a,b) = \sum_{i=a}^{b} i = a + (a+1) + \cdots + b. \tag{2}$$

We refer to the books [5, 6] for graph theory notation and terminology not described in this paper. All graphs under consideration here are connected graphs of order at least 3.

2 Brooms and Double Stars

By Theorem 1.2, $\chi'_t(P_n) = 3$ for each integer $n \geq 3$ and for each integer $r \geq 2$,

$$\chi'_t(K_{1,r}) = \begin{cases} r+1 & \text{if } r \not\equiv 1 \pmod 4 \\ r+2 & \text{if } r \equiv 1 \pmod 4 \end{cases} \tag{3}$$

Paths and stars belong to a special class of trees, namely brooms. A *broom* is a tree obtained from a path by adding pendant edges at exactly one of the end-vertices of the path. We first verify Conjecture 1.1 for brooms. In fact, more can be said.

Theorem 2.1 *If T is a broom that is not a star, then T has a twin edge $(\Delta(T)+1)$-coloring and so $\chi'_t(T) \leq \Delta(T)+1$.*

Proof. By Theorem 1.2, we may assume that T is not a path and so $\Delta(T) = \Delta \geq 3$. Suppose that T is obtained from the path $P_\ell = (v = v_1, v_2, \ldots, v_\ell)$ by adding $\Delta - 1$ pendant edges u_iv $(1 \leq i \leq \Delta - 1)$ at the end-vertex v. We consider two cases, according to whether Δ is even or Δ is odd.

Case 1. *Δ is even.* Let $\Delta = 2t$ for some integer $t \geq 2$. Define a proper edge coloring $c : E(T) \to \mathbb{Z}_{2t+1}$ such that (i)

$$\{c(u_iv) : 1 \leq i \leq 2t-1\} = \{0, 2, 3, \ldots, 2t-1\} = [0..2t-1] - \{1\}$$

and (ii) $c(v_jv_{j+1}) = r$ if $j \equiv r \pmod 3$ where $r \in \{1,2,3\}$ and $1 \leq j \leq \ell - 1$. Then the induced vertex coloring c' satisfies that $c'(u_i) = c(u_iv) \neq 1$ for $1 \leq i \leq 2t-1$ and $c'(v) = \sigma(0, 2t-1) = 1$ in $\mathbb{Z}_{2t+1}$, where $\sigma(0,2t-1)$ is the sum of integers between 0 and $2t-1$, as described in (2). Let

$$s'_\ell = (c'(v_2), c'(v_3), \ldots, c'(v_\ell)). \tag{4}$$

If $\ell = 3$, then $s'_\ell = (3,2)$, if $\ell = 4$, then $s'_\ell = (3,5,3)$, if $\ell = 5$, then $s'_\ell = (3,5,4,1)$ and if $\ell \geq 6$, then

$$s'_\ell = \begin{cases} (3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{3}) & \text{if } \ell \equiv 1 \,(\text{mod } 3) \\ (3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{4},\mathbf{1}) & \text{if } \ell \equiv 2 \,(\text{mod } 3) \\ (3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{2}) & \text{if } \ell \equiv 0 \,(\text{mod } 3). \end{cases} \tag{5}$$

Note that when $t = 2$, each entry 5 of the sequences in (5) is 0 in $\mathbb{Z}_{2t+1}$. Since c' is a proper vertex coloring, c is a twin edge $(\Delta + 1)$-coloring.

Case 2. *Δ is odd.* Let $\Delta = 2t + 1$ for some integer $t \geq 1$. We consider two subcases, according to whether $t = 1$ or $t \geq 2$.

Subcase 2.1. $t = 1$. If $\ell = 3$, then define $c : E(T) \to \mathbb{Z}_4$ by $c(u_1v_1) = 0$, $c(u_2v_1) = 2$, $c(v_1v_2) = 1$ and $c(v_2v_3) = 0$. Hence $c'(u_1) = 0$, $c'(u_2) = 2$, $c'(v_1) = 3$, $c'(v_2) = 1$ and $c'(v_3) = 0$. Thus c is a twin edge 4-coloring of T. If $\ell \geq 4$, then define $c : E(T) \to \mathbb{Z}_4$ by $c(u_1v_1) = 1$, $c(u_2v_1) = 2$, $c(v_1v_2) = 0$ and $c(v_jv_{j+1}) = r$ if $j-1 \equiv r \pmod 3$ and $2 \leq j \leq \ell - 1$. That is, $(c(v_2v_3), c(v_3v_4), \ldots, c(v_{\ell-1}v_\ell)) = (1,2,3,1,2,3,\ldots)$, which ends at r if $\ell - 1 \equiv r \pmod 3$ for $r \in \{1,2,3\}$. Then $c'(u_1) = 1$, $c'(u_2) = 2$, $c'(v_1) = 3$ and $c'(v_2) = 1$. Furthermore, let s_ℓ be defined as in (4). If $\ell = 4$, then $s'_\ell = (1,3,2)$, if $\ell = 5$, then $s'_\ell = (1,3,5,3)$, if $\ell = 6$, then $s'_\ell = (1,3,5,4,1)$ and if $\ell \geq 7$, then

$$s'_\ell = \begin{cases} (1,3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{3}) & \text{if } \ell \equiv 2 \,(\text{mod } 3) \\ (1,3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{4},\mathbf{1}) & \text{if } \ell \equiv 0 \,(\text{mod } 3) \\ (1,3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{2}) & \text{if } \ell \equiv 1 \,(\text{mod } 3). \end{cases} \tag{6}$$

Note that each entry 5 of the sequences in (6) is 1 in $\mathbb{Z}_{2t+2} = \mathbb{Z}_4$. Thus c is a twin edge 4-coloring of T.

Subcase 2.2. $t \geq 2$. Define $c : E(T) \to \mathbb{Z}_{2t+2}$ such that (i)

$$\{c(u_iv) : 1 \leq i \leq 2t\} = \{1, \ldots, 2t+1\} - \{t+1\} = [1..2t+1] - \{t+1\}$$

(ii) $c(v_1v_2) = t + 1$ and (iii) $c(v_jv_{j+1}) = r$ if $j - 1 \equiv r \pmod 3$ where $r \in \{1,2,3\}$ and $2 \leq j \leq \ell - 1$. Then the induced vertex coloring c' satisfies that $c'(u_i) = c(u_iv) \neq t+1$ for $1 \leq i \leq 2t$, $c'(v) = \sigma(0, 2t+1) = t+1$ in $\mathbb{Z}_{2t+2}$ and $c'(v_2) = t + 2$. Note that $1, t+1, t+2$ are distinct in $\mathbb{Z}_{2t+2}$. Furthermore, if $\ell = 3$, then $s'_\ell = (t+2, 1)$, if $\ell = 4$, then $s'_\ell = (t+2, 3, 2)$, if $\ell = 5$, then $s'_\ell = (t+2,3,5,3)$, if $\ell = 6$, then $s'_\ell = (t+2,3,5,4,1)$ and if $\ell \geq 7$, then

$$s'_\ell = \begin{cases} (t+2,3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{3}) & \text{if } \ell \equiv 2 \,(\text{mod } 3) \\ (t+2,3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{4},\mathbf{1}) & \text{if } \ell \equiv 0 \,(\text{mod } 3) \\ (t+2,3,5,4,\ldots,3,5,4,\mathbf{3},\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{2}) & \text{if } \ell \equiv 1 \,(\text{mod } 3). \end{cases} \tag{7}$$

Since $t \geq 2$, it follows that $t + 2 \neq 3$ in (7) and so c' is a proper vertex coloring. Thus c is a twin edge $(\Delta + 1)$-coloring. ∎

A *double star* is a tree of diameter 3. Thus each double star has exactly two non-end-vertices called the *central vertices* of the double star. If the central vertices of a double star have degree a and b, respectively, then it is denoted by $S_{a,b}$ where the order of $S_{a,b}$ is $a+b$. If $a = b$, then $S_{a,b}$ is referred to as a *regular* double star; while if $a \neq b$, then $S_{a,b}$ is *irregular* (that is, every two non-end-vertices have different degrees). In particular, $S_{2,b}$ is a broom for each integer $b \geq 2$. In order to determine the twin chromatic indexes of all double stars, we first present some preliminary results on regular graphs or regular trees in general. The following observation appears in [1].

Observation 2.2 *If a connected graph G contains two adjacent vertices of degree $\Delta(G)$, then $\chi'_t(G) \geq 1 + \Delta(G)$. In particular, if G is a connected r-regular graph for some integer $r \geq 2$, then $\chi'_t(G) \geq 1 + r$.*

For an integer $r \geq 2$, a tree T is *r-regular* if each non-end-vertex of T has degree r. Thus, the degree set of an r-regular tree is $\{1, r\}$. In particular, a path P_n order $n \geq 3$ is 2-regular and a star $K_{1,r}$ is r-regular for $r \geq 2$. We first show that if T is an r-regular tree for some integer $r \geq 5$ such that $r \equiv 1 \pmod 4$, then $\chi'_t(T) \geq r + 2$. The following lemma is a useful observation.

Lemma 2.3 *Let $r \geq 5$ be an integer such that $r \equiv 1 \pmod 4$. Then*

$$\sigma(0, r) - j \not\equiv j \pmod{r + 1}$$

for each integer $j \in [0..r]$.

Theorem 2.4 *If T is a regular tree of order at least 6 such that $\Delta(T) \equiv 1 \pmod 4$, then*

$$\chi'_t(T) \geq \Delta(T) + 2.$$

Proof. Suppose that T is an r-regular tree for some integer $r \geq 5$ and $r \equiv 1 \pmod 4$. Then $\Delta(T) = r$. By Observation 2.2, it follows that $\chi'_t(T) \geq r + 1$. We first show that $\chi'_t(T) \neq r + 1$. Assume, to the contrary, that $\chi'_t(T) = r + 1$. Let $c : E(T) \to \mathbb{Z}_{r+1}$ be a twin edge $(r + 1)$-coloring of T. Let $v_1 \in V(T)$ such that $\deg v_1 = r$. Then there is exactly one color in $\mathbb{Z}_{r+1}$ that is not assigned to any edge incident with v_1 by c. Suppose that $\{c(v_1 w) : w \in N(v_1)\} = \mathbb{Z}_{r+1} - \{j_1\}$ for some integer j_1; that is, j_1 is the only color that is not assigned to any edge incident with v_1. Since $c'(v_1) = \sigma(0, r) - j_1$ and $r \equiv 1 \pmod 4$, it follows by Lemma 2.3 that $c'(v_1) \neq j_1$ and so $c'(v_1) \in \mathbb{Z}_{r+1} - \{j_1\}$. Hence $c'(v_1) = c(v_1 w)$ for some $w \in N(v_1)$.

Let $c'(v_1) = c(v_1v_2)$, where $v_2 \in N(v_1)$. If v_2 is an end-vertex of T, then $c'(v_2) = c(v_1v_2) = c'(v_1)$, which is impossible. Thus v_2 is not an end-vertex of T and so $\deg v_2 = r$. Suppose that $\{c(v_2w) : w \in N(v_2)\} = \mathbb{Z}_{r+1} - \{j_2\}$ for some integer j_2, where then $c(v_1v_2) \in \mathbb{Z}_{r+1} - \{j_2\}$. By Lemma 2.3 again, $c'(v_2) \neq j_2$ and so $c'(v_2) = c(v_2v_3)$ for some $v_3 \in N(v_2)$. Since c is a twin edge $(r+1)$-coloring of T, it follows that $c'(v_2) \neq c'(v_1) = c(v_1v_2)$; which implies that $v_3 \neq v_1$. A similar argument shows that v_3 is not an end-vertex and so $\deg v_3 = r$. Continuing in this manner, we arrive at a sequence $v_1, v_2, \ldots, v_k$ of distinct $k \geq 2$ vertices of degree r in T such that (1) $v_iv_{i+1} \in E(T)$ for $1 \leq i \leq k-1$ and (2) v_{k-1} is the only non-end-vertex to which v_k is adjacent and so v_k is adjacent to exactly $r-1$ end-vertices of T. Suppose that $\{c(v_kw) : w \in N(v_k)\} = \mathbb{Z}_{r+1} - \{j_k\}$ for some integer j_k, where then $c(v_{k-1}v_k) \in \mathbb{Z}_{r+1} - \{j_k\}$. It then follows by Lemma 2.3 that $c'(v_k) \neq j_k$. Hence $c'(v_k) = c(v_kv_{k+1})$ for some $v_{k+1} \in N(v_k) - \{v_{k-1}\}$. Since v_{k+1} is an end-vertex of T, it follows that $c'(v_{k+1}) = c(v_kv_{k+1}) = c'(v_k)$, which is impossible. Therefore, $\chi'_t(T) \neq r+1$ and so $\chi'_t(T) \geq r+2$.

We are now prepared to determine the twin chromatic indexes of all double stars.

Theorem 2.5 *If T is a regular double star, then*

$$\chi'_t(T) = \begin{cases} \Delta(T)+1 & \text{if } \Delta(T) \not\equiv 1 \pmod 4 \\ \Delta(T)+2 & \text{if } \Delta(T) \equiv 1 \pmod 4 \end{cases}$$

Proof. By Theorem 1.2, we may assume that T is not a path. Let $T = S_{r,r}$ for some integer r, where then $\Delta(T) = r \geq 3$. Suppose that the central vertices are u and v, where $\deg u = \deg v = r$. Let $u_1, u_2, \ldots, u_{r-1}$ be the end-vertices of T that are adjacent to u and let $v_1, v_2, \ldots, v_{r-1}$ be the end-vertices of T that are adjacent to v. First, suppose that $r \not\equiv 1 \pmod 4$. By Observation 2.2, it follows that $\chi'_t(T) \geq r+1$. It remains to show that T has a twin edge $(r+1)$-coloring $c : E(T) \to \mathbb{Z}_{r+1}$. We consider two cases, according to whether r is even or r is odd.

Case 1. *$r \geq 4$ is even.* Then $\sigma(1,r) = 0$ in $\mathbb{Z}_{r+1}$. Define $c(uv) = r$ and

$$\begin{aligned} \{c(uu_i) : 1 \leq i \leq r-1\} &= [1..r-1] \\ \{c(vv_i) : 1 \leq i \leq r-1\} &= [0..r-1] - \{1\}. \end{aligned}$$

Then c is a proper edge coloring. Observe that $c'(u) = \sigma(1,r) = 0$ in $\mathbb{Z}_{r+1}$ and $c'(v) = \sigma(1,r) - 1 = r$ in $\mathbb{Z}_{r+1}$. Thus $c'(u) \neq c'(v)$. Furthermore, $c'(u_i) \neq 0 = c'(u)$ and $c'(v_i) \neq r = c'(v)$ for $1 \leq i \leq r-1$. Hence, c' is a proper vertex coloring and so c is a twin edge $(r+1)$-coloring of T.

Case 2. *$r \geq 3$ is odd.* Since $r \not\equiv 1 \pmod 4$, it follows that $r \equiv 3 \pmod 4$ and so $r = 4t+3$ for some integer $t \geq 0$. First, suppose that

$t = 0$. Define $c : E(T) \to \mathbb{Z}_4$ by $c(uv) = 2$, $\{c(uu_1), c(uu_2)\} = \{0,1\}$ and $\{c(vv_1), c(vv_2)\} = \{1,3\}$. Then $c'(u) = 3 \neq c'(u_i)$ in $\mathbb{Z}_4$ and $c'(v) = 2 \neq c'(v_i)$ in $\mathbb{Z}_4$ for $i = 1,2$. Also, $c'(u) \neq c'(v)$. Therefore, c is a twin edge 4-coloring of T.

Next, suppose that $t \geq 1$. Observe that $\sigma(0, 4t+3) = 2t+2$ in $\mathbb{Z}_{r+1} = \mathbb{Z}_{4t+4}$ and so

$$\sigma(0, 4t+3) - (t+1) = t+1 \text{ in } \mathbb{Z}_{4t+4}. \tag{8}$$

Define $c(uv) = 2t+2$ and

$$\begin{aligned} \{c(uu_i) : 1 \leq i \leq 4t+2\} &= [0..4t+3] - \{t+1, 2t+2\} \\ \{c(vv_i) : 1 \leq i \leq 4t+2\} &= [1..4t+3] - \{2t+2\}. \end{aligned}$$

Then c is a proper edge coloring of T. By (8),

$$\begin{aligned} c'(u) &= \sigma(1, 4t+3) - (t+1) = t+1 \text{ in } \mathbb{Z}_{4t+4} \\ c'(v) &= \sigma(1, 4t+3) = 2t+2 \text{ in } \mathbb{Z}_{4t+4}. \end{aligned}$$

Thus $c'(u) \neq c'(v)$. Furthermore, $c'(u_i) \neq t+1 = c'(u)$ in $\mathbb{Z}_{4t+4}$ and $c'(v_i) \neq 2t+2 = c'(v)$ in $\mathbb{Z}_{4t+4}$ for $1 \leq i \leq 4t+2$. Hence, c' is a proper vertex coloring and so c is a twin edge $(r+1)$-coloring of T.

Next, suppose that $r \geq 5$ and $r \equiv 1 \pmod 4$. By Theorem 2.4, it follows that $\chi'_t(T) \geq r+2$. Thus, it remains to show that T has a twin edge $(r+2)$-coloring $c : E(T) \to \mathbb{Z}_{r+2}$. Let $r = 4t+1$ for some integer $t \geq 1$. Then $\sigma(0, 4t+2) = 0$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+3}$ and so

$$\begin{aligned} \sigma(0, 4t+2) - (2t+1) - 1 &= 2t+1 \text{ in } \mathbb{Z}_{4t+3} \\ \sigma(0, 4t+2) - (2t) - 3 &= 2t \text{ in } \mathbb{Z}_{4t+3} \end{aligned}$$

Define $c(uv) = 2t+2$ and

$$\begin{aligned} \{c(uu_i) : 1 \leq i \leq 4t\} &= [0..4t+2] - \{1, 2t+1, 2t+2\} \\ \{c(vv_i) : 1 \leq i \leq 4t\} &= [0..4t+2] - \{3, 2t, 2t+2\}. \end{aligned}$$

Then c is a proper edge coloring of T. Observe that

$$\begin{aligned} c'(u) &= \sigma(0, 4t+2) - (2t+1) - 1 = 2t+1 \text{ in } \mathbb{Z}_{4t+3} \\ c'(v) &= \sigma(0, 4t+2) - 2t - 3 = 2t \text{ in } \mathbb{Z}_{4t+3}, \end{aligned}$$

Thus $c'(u) \neq c'(v)$. Furthermore, $c'(u_i) \neq 2t+1 = c'(u)$ and $c'(v_i) \neq 2t = c'(v)$ for $1 \leq i \leq 4t$. The colorings c and c' are shown for $r = 5$ in Figure 2. Hence, c' is a proper vertex coloring and so c is a twin edge $(r+2)$-coloring of T. ■

We have seen in Theorem 2.1 that if T is a broom, then $\chi'_t(T) \leq \Delta(T) + 1$. This is also the case for irregular double stars, as we show next.

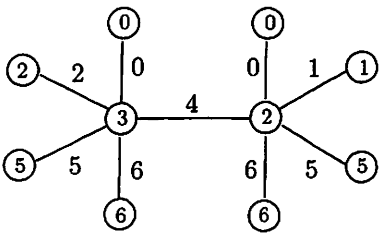

Figure 2: A twin edge 7-coloring of $S_{5,5}$

Theorem 2.6 *If T is an irregular double star, then T has a twin edge $(\Delta(T)+1)$-coloring and so $\chi'_t(T) \le \Delta(T)+1$.*

Proof. Let $T = S_{a,b}$ for some integers a and b with $2 \le a < b$. By Theorem 2.1, we may assume that $a \ge 3$ and so $b \ge 4$. Suppose that the central vertices are u and v, where $\deg u = a$ and $\deg v = b = \Delta(T)$. Let $u_1, u_2, \ldots, u_{a-1}$ be the end-vertices of T that are adjacent to u and let $v_1, v_2, \ldots, v_{b-1}$ be the end-vertices of T that are adjacent to v. To construct a twin edge $(b+1)$-coloring of T, we consider two cases, according to whether b is even or b is odd.

Case 1. *b is even.* Let $b = 2t$ for some integer $t \ge 2$. There are two subcases, according to (i) $a = 2t-1$ or $a = 2t-2 \ge 4$ and (ii) $3 \le a \le 2t-3$.

Subcase 1.1. *$a = 2t-1$ or $a = 2t-2$.* First, suppose that $a = 2t-1$. Define a proper edge coloring $c : E(T) \to \mathbb{Z}_{2t+1} = \{0, 1, \ldots, 2t\}$ by $c(uv) = 1$ such that

$$\begin{aligned} U_c &= \{c(uu_i) : \ 1 \le i \le a-1\} \\ &= \{0, 3, 4, \ldots, 2t-1\} = [0..2t-1] - \{1, 2\} \\ V_c &= \{c(vv_j) : \ 1 \le j \le b-1\} \\ &= \{0, 2, 3, \ldots, 2t-1\} = [0..2t-1] - \{1\}. \end{aligned}$$

Figure 3 shows an example of such a twin edge $(b+1)$-coloring of T.

Then the induced vertex coloring c' satisfies that

$$\begin{aligned} c'(u) &= 1+3+4+\cdots+(2t-1) = \sigma(0,2t) - 2 - (2t) = 2t \text{ in } \mathbb{Z}_{2t+1} \\ c'(v) &= 1+2+3+\cdots+(2t-1) = \sigma(0,2t) - (2t) = 1 \text{ in } \mathbb{Z}_{2t+1}. \end{aligned}$$

For each end-vertex u_i $(1 \le i \le 2t-2)$ or v_j $(1 \le j \le 2t-1)$ of T, it follows that

$$\begin{aligned} \{c'(u_i) : \ 1 \le i \le 2t-2\} &= [0..2t-1] - \{1, 2\} \\ \{c'(v_j) : \ 1 \le j \le 2t-1\} &= [0..2t-1] - \{1\}. \end{aligned}$$

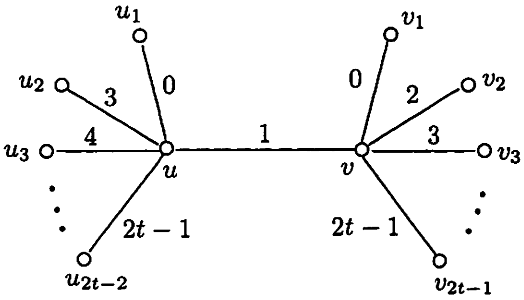

Figure 3: A twin edge $(b+1)$-coloring of T in Subcase 1.1

It follows that $c'(u_i) \neq c'(u) = 2t$ for $1 \leq i \leq 2t-2$ and $c'(v_j) \neq c'(v) = 1$ for $1 \leq j \leq 2t-1$ in $\mathbb{Z}_{2t+1}$. Hence c' is a proper vertex coloring and c is a twin edge $(2t+1)$-coloring of T.

For $a = 2t-2 \geq 4$, let $S_{2t-2,b}$ be obtained from $S_{2t-1,b}$ by deleting the edge uu_1 where $c(uu_1) = 0$ in the case $a = 2t-1$ (see Figure 3). Then the twin edge $(2t+1)$-coloring c for $S_{2t-1,b}$, as described above, gives rise to a twin edge $(2t+1)$-coloring of $S_{2t-2,b}$.

Subcase 1.2. $3 \leq a \leq 2t-3$. Define a proper edge coloring $c : E(T) \to \mathbb{Z}_{2t+1}$ by $c(uv) = 1$ such that $V_c = \{0, 2, 3, \ldots, 2t-1\} = [0..2t-1] - \{1\}$ (which is the same as in Subcase 1.1 and is shown in Figure 3). Hence $c'(v) = 1$ in $\mathbb{Z}_{2t+1}$ and $c'(v_j) \neq c'(v) = 1$ for $1 \leq j \leq 2t-1$. It remains to define the color $c(uu_i)$ for $1 \leq i \leq a-1$. We consider two situations when $a \geq 3$ is odd or $a \geq 4$ is even.

First, suppose that a is odd. Let $a - 1 = 2k$ for some positive integer k. If $k = 1$, then define $\{c(uu_1), c(uu_2)\} = \{0, 2t-1\}$; while if $k \geq 2$, then define

$$U_c = \{0, 2t-1, \underline{3, 2t-2}, \ldots, \underline{k+1, 2t-k}\}.$$

Figure 4 shows an example of a possible coloring of each edge uu_i for $1 \leq i \leq a-1$.

Since $c(uv) = 1$, it follows that $c'(u) = 1 + (2t-1) = 2t$ in $\mathbb{Z}_{2t+1}$. Since $a - 1 = 2k \leq 2t-4$, it follows that $k \leq t-2$ and so $k+1 \leq t-1$. Hence $c'(u_i) \neq 2t$ for $1 \leq i \leq a-1$. Therefore, c' is a proper vertex coloring and c is a twin edge $(2t+1)$-coloring. Next, suppose that $a \geq 4$ is even. Let c be a twin edge $(2t+1)$-coloring of $S_{a+1,b}$ as described above for the odd integer $a+1 \geq 5$. We may assume, without loss generality, that $c(uu_1) = 0$ (see Figure 4). Now let $S_{a,b}$ be obtained from $S_{a+1,b}$ by deleting the edge uu_1. Then the twin edge $(2t+1)$-coloring c for $S_{a+1,b}$, as described above, gives rise to a twin edge $(2t+1)$-coloring of $S_{a,b}$.

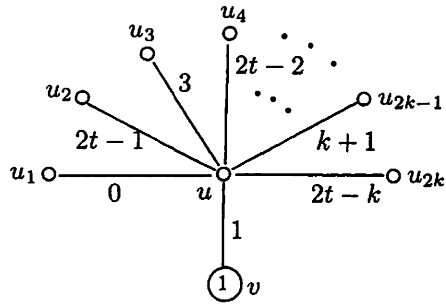

Figure 4: A possible coloring of uu_i for $1 \le i \le a-1$ in Subcase 1.2

Case 2. *b is odd.* Let $b = 2t+1$ for some integer $t \ge 2$. There are two subcases, according to $a = 2t$, $a = 2t-1 \ge 3$ or $3 \le a \le 2t-2$.

Subcase 2.1. $a = 2t$ *or* $a = 2t-1$. First, suppose that $a = 2t$. Define a proper edge coloring $c : E(T) \to \mathbb{Z}_{2t+2}$ by $c(uv) = t$ such that

$$\begin{aligned} U_c &= \{c(uu_i) : 1 \le i \le a-1\} \\ &= \{0, 1, \ldots, 2t+1\} - \{t-1, t, t+2\} = [0..2t+1] - \{t-1, t, t+2\} \\ V_c &= \{c(vv_j) : 1 \le j \le b-1\} \\ &= \{0, 2, 3, \ldots, t-1, t+1, \ldots, 2t+1\} = [0..2t+1] - \{1, t\}. \end{aligned}$$

Figure 5 shows an example of such a twin edge $(b+1)$-coloring of T.

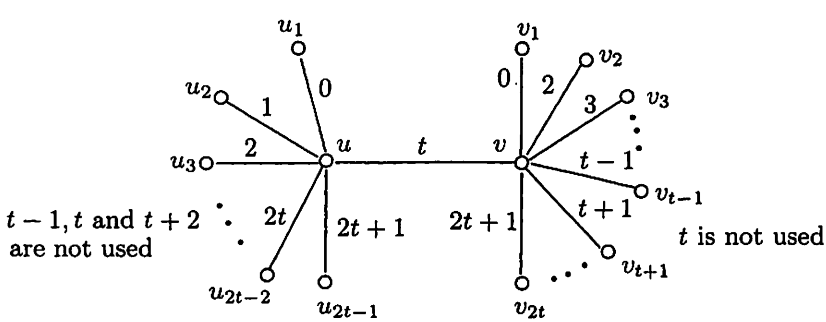

Figure 5: A twin edge $(b+1)$-coloring of T in Subcase 2.1

Then the induced vertex coloring c' satisfies that

$$\begin{aligned} c'(u) &= \sigma(0, 2t+1) - (t-1) - (t+2) = t(2t+1) = t+2 \text{ in } \mathbb{Z}_{2t+2} \\ c'(v) &= \sigma(0, 2t+1) - 1 = (2t+1)(t+1) - 1 = t \text{ in } \mathbb{Z}_{2t+2}. \end{aligned}$$

For each end-vertex u_i $(1 \le i \le 2t-1)$ or v_j $(1 \le j \le 2t)$ of T, it follows that

$$\begin{aligned}\{c'(u_i) : 1 \le i \le 2t-1\} &= [0..2t+1] - \{t-1, t, t+2\} \\ \{c'(v_j) : 1 \le j \le 2t\} &= [0..2t+1] - \{1, t\}.\end{aligned}$$

It follows that $c'(u_i) \neq c'(u) = t+2$ for $1 \le i \le 2t-1$ and $c'(v_j) \neq c'(v) = t$ for $1 \le j \le 2t$ in $\mathbb{Z}_{2t+2}$. Hence c' is a proper vertex coloring and c is a twin edge $(2t+2)$-coloring of T.

For $a = 2t-1 \ge 3$, let $S_{2t-1,b}$ be obtained from $S_{2t,b}$ by deleting the edge uu_1 where $c(uu_1) = 0$ in the case $a = 2t$ (see Figure 5). Then the twin edge $(2t+2)$-coloring c for $S_{2t,b}$ gives rise to a twin edge $(2t+2)$-coloring of $S_{2t-1,b}$.

Subcase 2.2. $3 \le a \le 2t-2$. Define a proper edge coloring $c : E(T) \to \mathbb{Z}_{2t+2}$ by $c(uv) = t$ such that $V_c = \{0, 2, 3, \ldots, t-1, t+1, 2t+1\} = [0..2t+1] - \{1, t\}$ (which is the same as in Subcase 2.1 and is shown in Figure 5). Hence $c'(v) = t$ in $\mathbb{Z}_{2t+2}$ and $c'(v_j) \neq c'(v) = t$ for $1 \le j \le 2t$. It remains to define the color $c(uu_i)$ for $1 \le i \le a-1$. We consider two situations when $a \ge 3$ is odd or $a \ge 4$ is even.

First, suppose that $a \ge 4$ is even. Let $a - 1 = 2k+1$ for some positive integer k. For $k = 1$, define $\{c(uu_1), c(uu_2), c(uu_3)\} = \{0, t+1, t+3\}$. Then $c'(u) = t + (t+1) + (t+3) = t+2$ in $\mathbb{Z}_{2t+2}$ and so $c'(u) \neq c'(u_i)$ for $i = 1, 2, 3$. For $k \ge 2$ and so $2k+1 \ge 5$, define

$$U_c = \{0, t+1, t+3, \underline{1, 2t+1}, \underline{2, 2t}, \ldots, \underline{k-1, 2t-k+3}\}.$$

Figure 6 shows an example of a possible coloring of each edge uu_i for $1 \le i \le a-1$.

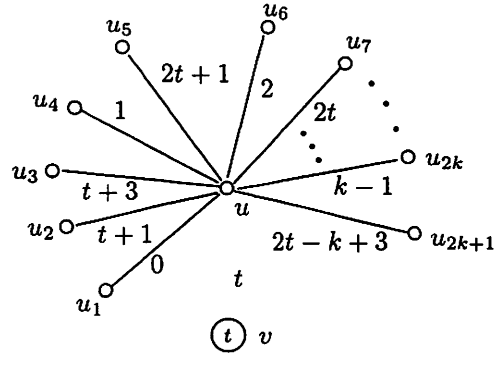

Figure 6: A possible coloring of uu_i for $1 \le i \le a-1$ in Subcase 2.2

Again, $c'(u) = t + (t+1) + (t+3) = t+2$ in $\mathbb{Z}_{2t+2}$. Since $k \le t-1$, it follows that $2t - k + 3 \ge t + 4$. Thus $c'(u) \ne c'(u_i)$ for $1 \le i \le a-1$. Therefore, c' is a proper vertex coloring and c is a twin edge $(2t+2)$-coloring. Next, suppose that $a \ge 3$ is odd. Let $a - 1 = 2k$ for some positive integer k. Let c be a twin edge $(2t+2)$-coloring of $S_{a+1,b}$ as described above for the even integer $a + 1 \ge 4$. We may assume, without loss of generality, that $c(uu_1) = 0$ (see Figure 6). Now let $S_{a,b}$ be obtained from $S_{a+1,b}$ by deleting the edge uu_1. Then the twin edge $(2t+2)$-coloring c for $S_{a+1,b}$, as described above, gives rise to a twin edge $(2t+2)$-coloring of $S_{a,b}$. ∎

The following is a consequence of Theorems 2.4, 2.5 and 2.6.

Corollary 2.7 *A double star T has $\chi'_t(T) = \Delta(T) + 2$ if and only if T is an r-regular tree for some integer $r \ge 5$ with $r \equiv 1 \pmod 4$.*

3 Regular Trees

Recall that a tree T is *r-regular* for an integer $r \ge 2$ if each non-end-vertex of T has degree r. In this section, we verify Conjecture 1.1 for regular trees of order at least 3. More precisely, we show that if T is a regular tree of order at least 3, then T has a twin edge $(\Delta(T)+2)$-coloring and so $\chi'_t(T) \le \Delta(T) + 2$. We begin with a lemma concerning stars only.

Lemma 3.1 *For each integer $r \ge 3$, the star $K_{1,r}$ has a twin edge $(r+2)$-coloring.*

Proof. Since $\chi'_t(K_{1,r}) = r + 2$ for $r \equiv 1 \pmod 4$, we may assume that $r \not\equiv 1 \pmod 4$. Let $T = K_{1,r}$ where $V(T) = \{v, v_1, v_2, \ldots, v_r\}$ and $\deg v = r$. We consider three cases.

Case 1. $r \equiv 0 \pmod 4$. Let $r = 4t$ for some integer $t \ge 1$. Observe that

$$\sigma(1, r+1) = \sigma(1, 4t+1) = 2t+1 \text{ in } \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+2}.$$

Define the edge coloring $c : E(T) \to \mathbb{Z}_{4t+2}$ such that

$$\{c(vv_i) : 1 \le i \le 4t\} = [1..4t+1] - \{2t+1\}.$$

Then $c'(v) = \sigma(1, 4t+1) - (2t+1) = 0 \ne c(v_i)$ for $1 \le i \le r = 4t$. Hence c is a twin edge $(r+2)$-coloring of T.

Case 2. $r \equiv 2 \pmod 4$. Let $r = 4t+2$ for some integer $t \ge 1$. Observe that

$$\sigma(1, r+1) = \sigma(1, 4t+3) = 2t+2 \text{ in } \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+4}.$$

Define the edge coloring $c : E(T) \to \mathbb{Z}_{4t+4}$ such that

$$\{c(vv_i) : 1 \le i \le 4t+2\} = [1..4t+3] - \{2t+2\}.$$

Then $c'(v) = \sigma(1, 4t+3) - (2t+2) = 0 \ne c(v_i)$ for $1 \le i \le r = 4t+2$. Hence c is a twin edge $(r+2)$-coloring of T.

Case 3. $r \equiv 3 \pmod 4$. Let $r = 4t+3$ for some integer $t \ge 0$. Observe that

$$\sigma(0, r) = \sigma(0, 4t+3) = 1 \text{ in } \mathbb{Z}_{r+2} = \mathbb{Z}_{4t+5}.$$

Define the edge coloring $c : E(T) \to \mathbb{Z}_{4t+5}$ such that

$$\{c(vv_i) : 1 \le i \le 4t+3\} = [0..4t+3] - \{2\}.$$

Then $c'(v) = \sigma(0, 4t+3) - 2 = -1 = 4t+4 \ne c(v_i)$ for $1 \le i \le r = 4t+3$. Hence c is a twin edge $(r+2)$-coloring of T.

Theorem 3.2 *If T is a regular tree of order at least* 3, *then*

$$\chi'_t(T) \le \Delta(T) + 2.$$

Proof. By Theorem 1.3, we may assume that $\Delta(T) \ge 7$. For a given integer $r \ge 7$, we proceed by induction on the number of vertices of degree r in an r-regular tree to show that every r-regular tree has a twin edge $(r+2)$-coloring. The star $K_{1,r}$ is the only r-regular tree that has exactly one vertex of degree r and $K_{1,r}$ has a twin edge $(r+2)$-coloring by Lemma 3.1. Assume that if T^* is an r-regular tree having exactly $k-1$ vertices of degree r for some integer $k \ge 2$, then T^* has a twin edge $(r+2)$-coloring.

Now, let T be an r-regular tree having exactly k vertices of degree r. Then T contains a vertex v of degree r such that v is adjacent to exactly $r-1$ end-vertices and exactly one non-end-vertex. Let $w \in V(T)$ for which $vw \in E(T)$ and $\deg w = r$. Next, let $T' = T - (N(v) - \{w\})$ be the tree that is obtained from T by removing the $r-1$ end-vertices of T that are adjacent to v. Then T' is an r-regular tree having exactly $k-1$ vertices of degree r. Furthermore, v is an end-vertex in T' and w is the only vertex that is adjacent to v in T'. By the induction hypothesis, T' has a twin edge $(r+2)$-coloring $c_0 : E(T') \to \mathbb{Z}_{r+2}$. Next, we extend the coloring c_0 to a twin edge coloring $c : E(T) \to \mathbb{Z}_{r+2}$ of T such that $c(e) = c_0(e)$ for each $e \in E(T')$ (and so $c'(x) = c'_0(x)$ for all $x \in V(T') - \{v\}$). First, we verify the following claim.

Claim. For each $r \ge 7$, there are six distinct elements

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}_{r+2}$$

such that

$$\sigma(0, r+1) - \alpha_1 - \alpha_2 = \alpha_1 \text{ in } \mathbb{Z}_{r+2} \tag{9}$$
$$\sigma(0, r+1) - \beta_1 - \beta_2 = \beta_1 \text{ in } \mathbb{Z}_{r+2} \tag{10}$$
$$\sigma(0, r+1) - \gamma_1 - \gamma_2 = \gamma_1 \text{ in } \mathbb{Z}_{r+2}. \tag{11}$$

To verify this claim, we consider four cases, according to whether r is congruent to $0, 1, 2$ or 3 modulo 4.

Case 0. $r \equiv 0 \pmod 4$. Let $r = 4t$ for some integer $t \geq 2$. Then $\sigma(0, 4t+1) = 2t + 1$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+2}$. If $t = 3$, then $4t + 2 = 14$ and

$$\begin{aligned}
&\sigma(0, 4t+1) - 3 - 1 = 3 \text{ in } \mathbb{Z}_{14}\\
&\sigma(0, 4t+1) - 0 - 7 = 0 \text{ in } \mathbb{Z}_{14}\\
&\sigma(0, 4t+1) - 8 - 5 = 8 \text{ in } \mathbb{Z}_{14}.
\end{aligned}$$

If $t = 7$, then $4t + 2 = 30$ and

$$\begin{aligned}
&\sigma(0, 4t+1) - 7 - 1 = 7 \text{ in } \mathbb{Z}_{30}\\
&\sigma(0, 4t+1) - 21 - 3 = 21 \text{ in } \mathbb{Z}_{30}\\
&\sigma(0, 4t+1) - 20 - 5 = 20 \text{ in } \mathbb{Z}_{30}.
\end{aligned}$$

If $t \geq 2$ and $t \neq 3, 7$, then

$$\begin{aligned}
&\sigma(0, 4t+1) - t - 1 = t \text{ in } \mathbb{Z}_{4t+2}\\
&\sigma(0, 4t+1) - 3t - 3 = 3t \text{ in } \mathbb{Z}_{4t+2}\\
&\sigma(0, 4t+1) - (3t-2) - 7 = 3t - 2 \text{ in } \mathbb{Z}_{4t+2}.
\end{aligned}$$

Case 1. $r \equiv 1 \pmod 4$. Let $r = 4t + 1$ for some integer $t \geq 2$. Since $\sigma(1, 4t+2) = 0$ in $\mathbb{Z}_{4t+3}$, it follows that

$$\begin{aligned}
&\sigma(0, 4t+2) - (2t+2) - (4t+2) = 2t + 2 \text{ in } \mathbb{Z}_{4t+3}\\
&\sigma(0, 4t+2) - (2t+1) - 1 = 2t + 1 \text{ in } \mathbb{Z}_{4t+3}\\
&\sigma(0, 4t+2) - (2t) - 3 = 2t \text{ in } \mathbb{Z}_{4t+3}.
\end{aligned}$$

Case 2. $r \equiv 2 \pmod 4$. Let $r = 4t + 2$ for some integer $t \geq 2$. Then $\sigma(0, 4t+3) = 2t + 2$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+4}$. If $t = 2$, then $4t + 4 = 12$ and

$$\begin{aligned}
&\sigma(0, 4t+3) - 3 - 0 = 3 \text{ in } \mathbb{Z}_{12}\\
&\sigma(0, 4t+3) - 4 - 10 = 4 \text{ in } \mathbb{Z}_{12}\\
&\sigma(0, 4t+3) - 5 - 8 = 5 \text{ in } \mathbb{Z}_{12}.
\end{aligned}$$

If $t = 6$, then $4t + 4 = 28$ and

$$\sigma(0, 4t+3) - 6 - 2 = 6 \text{ in } \mathbb{Z}_{28}$$
$$\sigma(0, 4t+3) - 5 - 4 = 5 \text{ in } \mathbb{Z}_{28}$$
$$\sigma(0, 4t+3) - 3 - 8 = 3 \text{ in } \mathbb{Z}_{28}.$$

If $t \geq 3$ and $t \neq 6$, then

$$\sigma(0, 4t+3) - t - 2 = t \text{ in } \mathbb{Z}_{4t+4}$$
$$\sigma(0, 4t+3) - 3t - 6 = 3t \text{ in } \mathbb{Z}_{4t+4}$$
$$\sigma(0, 4t+3) - (3t+4) - (4t+2) = 3t+4 \text{ in } \mathbb{Z}_{4t+4}.$$

Case 3. $r \equiv 3 \pmod 4$. Let $r = 4t + 3$ for some integer $t \geq 1$. Then $\sigma(0, 4t+4) = 0$ in $\mathbb{Z}_{r+2} = \mathbb{Z}_{4t+5}$. If $t = 5$, then $4t + 5 = 25$ and

$$\sigma(0, 4t+4) - 10 - 5 = 10 \text{ in } \mathbb{Z}_{25}$$
$$\sigma(0, 4t+4) - 11 - 3 = 11 \text{ in } \mathbb{Z}_{25}$$
$$\sigma(0, 4t+4) - 2 - 21 = 2 \text{ in } \mathbb{Z}_{25}.$$

If $t = 6$, then $4t + 5 = 29$ and

$$\sigma(0, 4t+4) - 10 - 9 = 10 \text{ in } \mathbb{Z}_{29}$$
$$\sigma(0, 4t+4) - 11 - 7 = 11 \text{ in } \mathbb{Z}_{29}$$
$$\sigma(0, 4t+4) - 12 - 5 = 12 \text{ in } \mathbb{Z}_{29}.$$

If $t \geq 1$ and $t \neq 5, 6$, then

$$\sigma(0, 4t+4) - 2t - 5 = 2t \text{ in } \mathbb{Z}_{4t+5}$$
$$\sigma(0, 4t+4) - 4t - 10 = 4t \text{ in } \mathbb{Z}_{4t+5}$$
$$\sigma(0, 4t+4) - 2(t-1) - 9 = 2(t-1) \text{ in } \mathbb{Z}_{4t+5}.$$

Therefore, the claim holds; that is, for each integer $r \geq 7$, there are six distinct elements $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}_{r+2}$ that satisfy in (9), (10) and (11), respectively.

We are now prepared to extend the coloring c_0 of T' to a twin edge coloring $c : E(T) \to \mathbb{Z}_{r+2}$ of T such that $c(e) = c_0(e)$ for each $e \in E(T')$. Note that $E(T) - E(T') = E_v - \{vw\}$, where E_v is the set of edges incident with v in T and $|E_v - \{vw\}| = r - 1$. Let $X = \{\alpha_1, \alpha_2\}$, $Y = \{\beta_1, \beta_2\}$ and $Z = \{\gamma_1, \gamma_2\}$ where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Z}_{r+2}$ are described in (9), (10) and (11), respectively. Since $|X| = |Y| = |Z| = 2$ and X, Y and Z are pairwise disjoint, there is at least one of X, Y and Z that is disjoint from the set $\{c_0'(w), c_0(vw)\}$. We may assume, without loss of generality, that $X \cap \{c_0'(w), c_0(vw)\} = \emptyset$. Define

$$\{c(e) : e \in E_v\} = [0..r+1] - X = [0..r+1] - \{\alpha_1, \alpha_2\}$$

where $c(wv) = c_0(wv)$. Then c is a proper edge coloring of G. By (9), it follows that $c'(v) = \alpha_1 \neq c_0'(w) = c'(w)$ and so c' is a proper vertex coloring of G. Therefore, c is a twin edge $(r+2)$-coloring of G.

By Theorems 2.4 and 3.2, if T is a regular tree of order at least 6 such that $\Delta(T) \equiv 1 \pmod 4$, then $\chi_t'(T) = \Delta(T) + 2$. Furthermore, we saw in Corollary 2.7 that if T is a double star, then $\chi_t'(T) = \Delta(T) + 2$ if and only if T is an r-regular tree for some integer $r \geq 5$ with $r \equiv 1 \pmod 4$ where then $r = \Delta(T)$. From the examples we are aware of, it suggests that Corollary 2.7 is true for all trees in general. In any case, this problem appears to be worthy of further study.

4 Acknowledgment

We are grateful to the anonymous referee whose valuable suggestions resulted in an improved paper.

References

[1] E. Andrews, L. Helenius, D. Johnston, J. VerWys and P. Zhang, On twin edge colorings of graphs. *Discuss. Math. Graph Theory*. To appear.

[2] E. Andrews, D. Johnston and P. Zhang, A twin edge coloring conjecture. *Bull. Inst. Combin. Appl.* **70** (2014) 28-44.

[3] R. L. Brooks, On coloring the nodes of a network. *Proc. Cambridge Philos. Soc.* **37** (1941) 194-197.

[4] G. Chartrand and F. Harary, Planar permutation graphs. *Ann. Inst. H. Poincaré (Sect. B)* **3** (1967) 433-438.

[5] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs: 5th Edition*, Chapman & Hall/CRC, Boca Raton, FL (2010).

[6] G. Chartrand and P. Zhang, *Chromatic Graph Theory*. Chapman & Hall/CRC, Boca Raton, FL (2008).

[7] V. G. Vizing, On an estimate of the chromatic class of a p-graph. (Russian) *Diskret. Analiz.* **3** (1964) 25-30.