Constructing γ-Sets of Grids

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Abstract

A lot of research has been spent determining the domination numbers, $\gamma_{m,n}$, of grid graphs. But relatively little effort has been given to constructing minimum dominating sets of grid graphs. In this paper, we introduce a method for constructing γ-sets of grid graphs $G_{m,n}$ for all $m \geq 16$ and $n \geq 16$. Further, for $G_{m,n}$, $m < 16$, $m \neq 12, 13$, we show how particular γ-sets can be used to construct γ-sets for other grid graphs.

1 Introduction

Let $G = (V, E) = (V(G), E(G))$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and *order* $n = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = \{u | uv \in E\}$ of vertices u that are adjacent to v; the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S, or equivalently if $N[S] = V$. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S of G; we say that such a set S is a γ*-set*. For more on domination theory, we refer the reader to [15, 16].

An $m \times n$ *grid graph* $G_{m,n}$ has vertex set $V = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$ with (i, j) adjacent to (k, l) if $i = k$ and $|j - l| = 1$ or $j = l$ and $|i - k| = 1$. We define the *rectilinear distance*, *dist*, between vertices (i, j) and (k, l) as

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$$dist((i,j),(k,l)) = |i-k| + |j-l|.$$

We note that $dist$ equals the number of edges in a shortest path in $G_{m,n}$ between (i,j) and (k,l). Further, the set of vertices of the form (i,j), $1 \leq j \leq n$, is called the ith row of $G_{m,n}$ and the set of vertices of the form (i,j), $1 \leq i \leq m$, is called the jth column of $G_{m,n}$. We will denote the domination number of $G_{m,n}$ by $\gamma_{m,n}$.

Numerous papers have been published on the problem of computing the domination number of grid graphs. Beginning in 1983, Jacobson and Kinch [17] determined $\gamma_{m,n}$ for $1 \leq m \leq 4$ and all n. In 1993, Chang and Clark [3] extended this to $m = 5, 6$ and all n. In 1989, Hare [4] settled specific cases for $m = 7, 8, 9, 10, 11$. In 1998, Fisher [8] developed a method for calculating $\gamma_{m,n}$ that is described in Spalding's 2001 Ph.D. thesis [21], who gave the values of $\gamma_{m,n}$ for $m \leq 19$ and all n. Several authors [5, 6, 7, 11, 13, 14, 19, 20] have developed techniques for either computing $\gamma_{m,n}$ exactly or establishing bounds. Alanko et al. [1] used dynamic programming to extend these results to $m \leq 29$ and all n. Back in 1992 Chang [2] devoted his Ph.D. thesis to studying the domination numbers of grid graphs, and he conjectured that

$$\gamma_{m,n} = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4,$$

for all values of $m, n \geq 16$. This conjecture has recently been proved by Goncalves et al. in 2011 [10]. The following theorem, given in [1] and [10], provides formulae for the value of $\gamma_{m,n}$ for all m, n.

Theorem 1.1 *Let $G_{m,n}$, $m, n \geq 1$, be a grid graph. Then*

$$\gamma_{1,n} = \left\lfloor \frac{n+2}{3} \right\rfloor$$

$$\gamma_{2,n} = \left\lfloor \frac{n+2}{2} \right\rfloor$$

$$\gamma_{3,n} = \left\lfloor \frac{3n+4}{4} \right\rfloor$$

$$\gamma_{4,n} = \begin{cases} n+1 & \text{if } n = 5, 6, 9 \\ n & \text{otherwise} \end{cases}$$

$$\gamma_{5,n} = \begin{cases} \left\lfloor \frac{6n+6}{5} \right\rfloor & \text{if } n = 7 \\ \left\lfloor \frac{6n+8}{5} \right\rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{6,n} = \begin{cases} \lfloor \frac{10n+10}{7} \rfloor & \text{if } n \equiv 1 \pmod 7 \\ \lfloor \frac{10n+12}{7} \rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{7,n} = \left\lfloor \frac{5n+3}{3} \right\rfloor$$

$$\gamma_{8,n} = \left\lfloor \frac{15n+14}{8} \right\rfloor$$

$$\gamma_{9,n} = \left\lfloor \frac{23n+20}{11} \right\rfloor$$

$$\gamma_{10,n} = \begin{cases} \lfloor \frac{30n+37}{13} \rfloor & \text{if } n \neq 13, 16,\ n \equiv 0, 3 \pmod{13} \\ \lfloor \frac{30n+24}{13} \rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{11,n} = \begin{cases} \lfloor \frac{38n+21}{15} \rfloor & \text{if } n = 11, 18, 20, 22, 33 \\ \lfloor \frac{38n+36}{15} \rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{12,n} = \left\lfloor \frac{80n+66}{29} \right\rfloor$$

$$\gamma_{13,n} = \begin{cases} \lfloor \frac{98n+111}{33} \rfloor & \text{if } n \not\equiv 14, 15, 17, 20 \pmod{33} \\ \lfloor \frac{98n+78}{33} \rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{14,n} = \begin{cases} \lfloor \frac{35n+40}{11} \rfloor & \text{if } n \equiv 18 \pmod{22} \\ \lfloor \frac{35n+29}{11} \rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{15,n} = \begin{cases} \lfloor \frac{44n+27}{13} \rfloor & \text{if } n \equiv 5 \pmod{26} \\ \lfloor \frac{44n+40}{13} \rfloor & \text{otherwise} \end{cases}$$

$$\gamma_{m,n} = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4, m \geq 16, n \geq 16.$$

Most of the literature on this problem has focused on developing techniques for computing the value of $\gamma_{m,n}$, but little focus has been given to finding a method for constructing γ-sets of grid graphs. Hare [14] displayed many γ-sets for $m \leq 11$, and Cockayne et al. [7] gave a method for constructing γ-sets for square grids $G_{n,n}$. In this paper, we extend the work of Cockayne et al. to construct γ-sets for many of the values of $\gamma_{m,n}$ cited in Thorem 1.1.

In Section 2 we extend the work in [7] to give a method for constructing γ-sets of grid graphs for all $m, n \geq 16$. In Section 3 we show how this method can be used to construct γ-sets for some values of $m, n < 16$, $m, n \neq 12, 13$, and how some of these γ-sets for a fixed m can be used to construct γ-sets for other values of n. We also show how these γ-sets define the rate of growth given in Theorem 1.1 for values of $m < 16$.

2 Constructing γ-Sets when $m, n \geq 16$

In this section we show how to construct a γ-set for $G_{m,n}$ of size $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$ for $m, n \geq 16$. These same techniques can be applied in some cases to construct γ-sets for $G_{m,n}$ when $m, n < 16$. Assume that $m, n \geq 4$. Further, let G^* be an $(m+2) \times (n+2)$ grid graph where $(i, j) \in V(G^*)$ if $0 \leq i \leq m+1$ and $0 \leq j \leq n+1$, and $G_{m,n}$ is a subgraph of G^*, such that the (i, j) entry of $G_{m,n}$ is the $(i+1, j+1)$ entry of G^*. We call any vertex in $V(G^*) - V(G_{m,n})$ a *boundary vertex*. Further, we define a *(2,1)-slant grid* $S(2,1)$ as an infinite graph with vertex set $V(S(2,1)) = \{(i,j) : i \in \mathcal{Z}, j \in \mathcal{Z}$ with the property that if $(i, j) \in V(S(2,1))$ then so are $(i+2, j+1)$, $(i+1, j-2)$, $(i-2, j-1)$, and $(i-1, j+2)\}$. The edges that connect vertices $(i+2, j+1)$, $(i+1, j-2)$, $(i-2, j-1)$, and $(i-1, j+2)$ to (i, j) are in the edge set $E(S(2,1))$. We note that there are 5 isomorphic graphs of $S(2,1)$ depending on the smallest value of k, $0 \leq k \leq 4$ for which the vertex $(0, k)$ of G^* is a vertex of $S(2,1)$. We denote these five slant grids as $S^k(2,1)$, $0 \leq k \leq 4$. Figure 1 shows a partial slant grid with $k = 4$.

Given G^*, we can overlay $S^k(2,1)$ on top of G^* in 5 ways depending on the 'starting' vertex of $(0, k)$, $0 \leq k \leq 4$. Let $V^k = V(S^k(2,1)) \cap G^*$. In Figure 1, we show this overlay for $m = 10$, $n = 10$, $k = 4$. To highlight $G_{m,n}$ as a subgraph of G^*, we have removed the edges connecting boundary vertices of G^* to vertices in $G_{m,n}$.

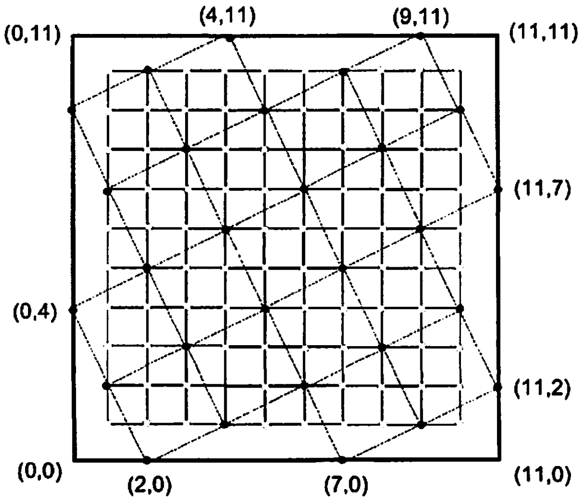


Figure 1: G^* and $S^4(2,1)$ when $m = 10$, $n = 10$, $k = 4$

Lemma 2.1 *For all* $0 \le k \le 4$ *and all* $m, n \ge 1$, $V^k = V(S^k(2,1)) \cap G^*$ *is a dominating set of* $V(G_{m,n})$.

Proof. Let $(x, y) \in V(G_{m,n})$, and let $k \in \{0,1,2,3,4\}$. Let $(i, j) \in V^k$ so that (i, j) minimizes the rectilinear distance *dist* to (x, y). If *dist* $= 0$, then $(x, y) = (i, j) \in V^k$. Otherwise, we wish to show that *dist* $= 1$. Assume that *dist* ≥ 2. By definition of $S^k(2,1)$, if $(i, j) \in V(S^k(2,1))$ then so are $(i+2, j+1)$, $(i+1, j-2)$, $(i-2, j-1)$, and $(i-1, j+2)$. Note that each of these 4 vertices are rectilinear distance 3 from (i, j). Further, any vertex of rectilinear distance 2 to (i, j) is adjacent to one of these four vertices. Hence, if *dist* ≥ 2, (x, y) is closer to one of $(i+2, j+1)$, $(i+1, j-2)$, $(i-2, j-1)$, and $(i-1, j+2)$ than it is to (i, j), a contradiction. Hence, *dist* ≤ 1 and (x, y) is dominated by $(i, j) \in V^k$. □

We now show that we can move the boundary vertices in V^k to an adjacent vertex in $G_{m,n}$ to construct a dominating set for $G_{m,n}$ that is composed entirely of vertices in $G_{m,n}$. Let $A = (0, k_1)$, $B = (k_2, m+1)$, $C = (n+1, m+1-k_3)$, and $D = (n+1-k_4, 0)$, be the closest boundary vertices in V^k, moving counterclockwise, to the four corners ($(0,0)$, $(0, m+1)$, $(n+1, m+1)$, and $(n+1, 0)$) of G^*. We note that $0 \le k_i \le 4$ for $i \in \{1,2,3,4\}$. In Figure 1, each of k_1, k_2, k_3, and k_4 is 4 and $A = (0,4)$, $B = (4,11)$, $C = (11,7)$, and $D = (7,0)$.

Depending on the value of k, we make one of five adjustments to V^k so that these four boundary vertices in V^k are moved to vertices in $G_{m,n}$ to form a dominating set of $G_{m,n}$. Some of these adjustments are given in [7] and are referred to as *redomination.* Since these redomination techniques are not exhaustive, we extend them here. To start, all boundary vertices in V^k that are not involved in one of these five adjustments are replaced by the vertices in $G_{m,n}$ that are closest in rectilinear distance. Let S^k equal the set V^k where the boundary vertices that are not involved in an adjustment have been replaced by their nearest rectilinear distance neighbor in $G_{m,n}$. We describe the adjustments to the four boundary vertices closest to the four corners of G^* in terms of the boundary vertex $B = (k_2, m+1)$. However, these adjustments are equivalent for all four corners.

We first describe the adjustments if k_2 is even. When $k_2 = 0$ and $B = (0, m+1)$, we simply delete the boundary vertex B from S^k, and let $S^k = S^k - B$. When $k_2 = 2$ and $B = (2, m+1)$, we let $S^k = S^k - \{B, (0, m)\} \cup \{(1, m)\}$. When $k = 4$ and $B = (4, m+1)$, let $S^k = S^k - \{B, (2, m), (0, m-1)\} \cup \{(3, m), (1, m-1)\}$. Note that in each of these cases we remove boundary vertices from the dominating set and add vertices in $G_{m,n}$ so that all of the vertices of $G_{m,n}$ remain dominated by S. Figure 2 shows these adjustments when k is even.

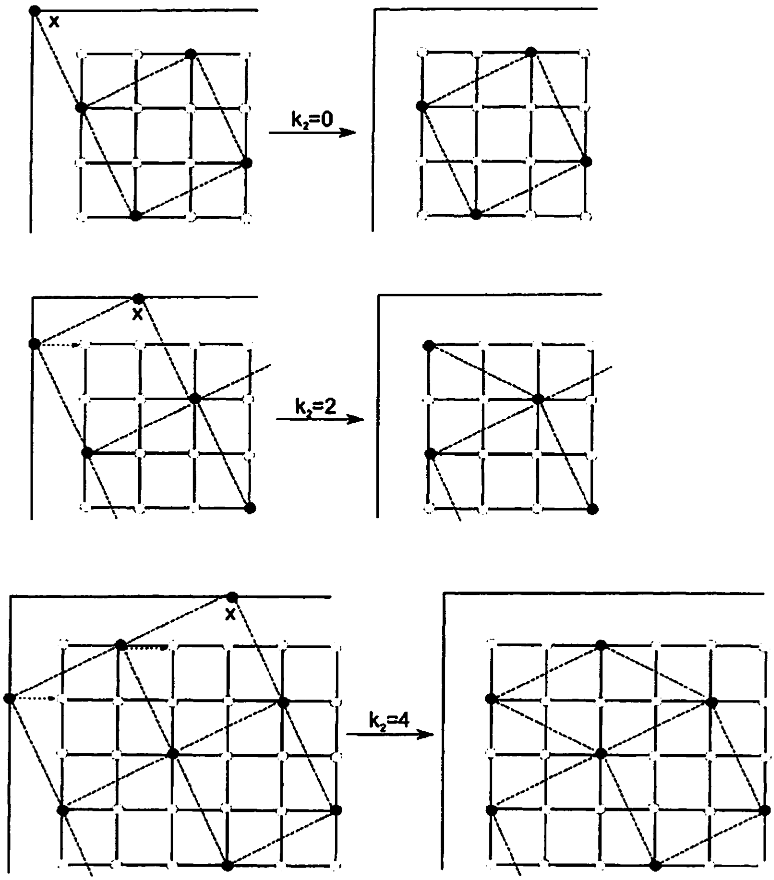


Figure 2: Adjustments for $k_2 = 0, 2, 4$

When k_2 is odd, there are two types of adjustments: a full adjustment and a partial adjustment. When $k_2 = 1$, we first describe the partial adjustment. Here we let $S^k = S^k - \{B, (0, m-2)\} \cup \{(1, m-1)\}$. For the full adjustment, we let $S^k = S^k - \{B, (0, m-2), (2, m-1), (4, m), (6, m+1)\} \cup \{(1, m), (1, m-2), (3, m-1), (5, m)\}$. These adjustments are shown in Figure 3. We note in Figure 3, the arrows and the X that have subscripts with a p denote movement that occurs only in the partial adjustment. The arrows without the p subscript denote movement that occurs only in the full adjustment.

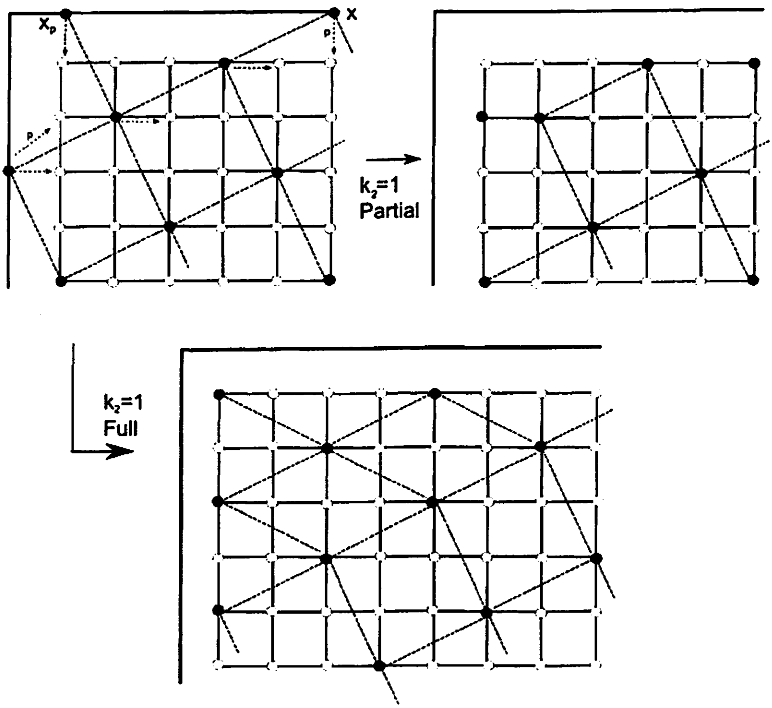


Figure 3: Full and Partial Adjustments for $k_2 = 1$

When $k_2 = 3$ and $B = (3, m+1)$, we first describe the partial adjustment. Here we let $S^k = S^k - \{B, (1, m), (0, m-3)\} \cup \{(2, m), (1, m-2)\}$. For the full adjustment, we let $S^k = S^k - \{B, (1, m), (0, m-3), (2, m-2), (4, m-1), (6, m), (8, m+1)\} \cup \{(1, m-1), (3, m), (1, m-3), (3, m-2), (5, m-1), (7, m)\}$. Figures 4 and 5 show the full and partial adjustments when $k_2 = 3$.

Figure 6 shows the full adjustments at each of the four corners of $G_{10,10}$ in Figure 1.

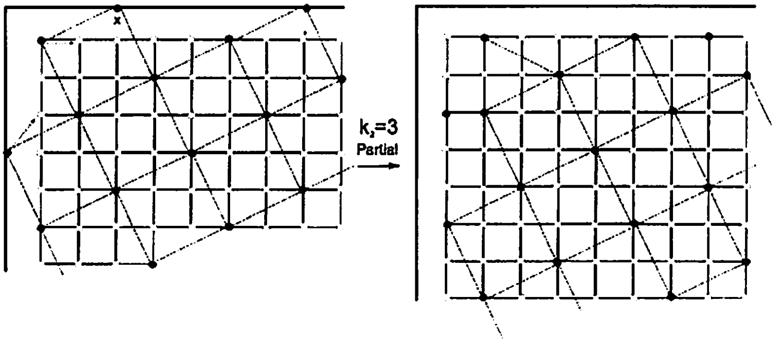


Figure 4: Partial Adjustment for $k_2 = 3$

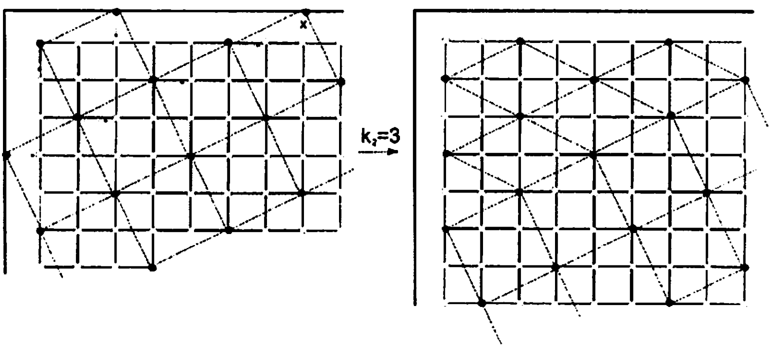


Figure 5: Full Adjustment for $k_2 = 3$

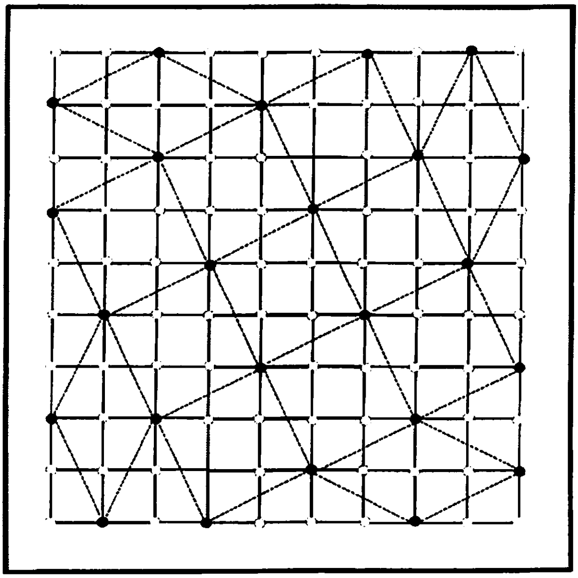


Figure 6: A γ-set for $G_{10,10}$

We note that not all adjustments at a given corner are possible in conjunction with adjustments at other corners for some small values of m and n. For instance, consider the example in Figure 7, where $m = 5$, $n = 7$, and $k = 4$.

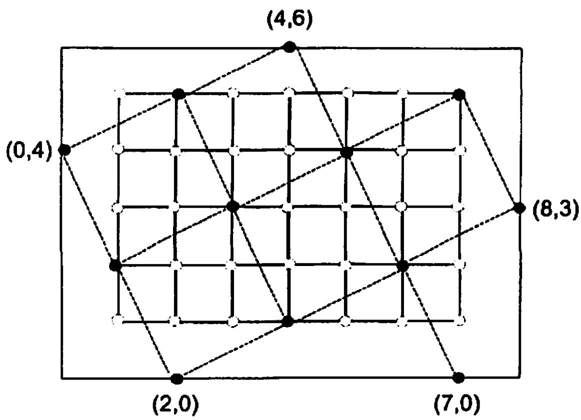


Figure 7: $S^4(2,1)$ overlaid on $G_{5,7}$

Note that $k_1 = 4$ and $k_2 = 4$, but the adjustment for k_1 cannot be completed in conjunction with the adjustment for k_2 since the k_1 adjustment requires the deletion of vertex $(0, 4)$ from S^k without replacement whereas the k_2 adjustment requires that $(0, 4)$ be replaced by $(1, 4)$. If these two adjustments are made together, we do not end up making full adjustments around each corner. For grids $G_{m,n}$ where m and n take on values that do not allow for full adjustments, we simply make as many adjustments as possible to form a dominating set for $G_{m,n}$. Table 1 shows the lower bounds that m and n need to be if $k_i = a$ and $k_{i+1} = b$ in order to ensure full adjustments. Partial adjustments can be made in the cases where $k = 1$ and $k = 3$. Each of these partial adjustments require $m, n \geq 7$.

		b				
		0	1	2	3	4
a	0	4	10	6	12	8
	1	7	13	9	15	11
	2	5	11	7	13	9
	3	8	14	10	16	12
	4	6	12	8	14	10

Table 1: Lower Bounds on m and n Ensuring Full Adjustments

Knowing k, it is easy to compute the four values of k_1, k_2, k_3, and k_4 for each value of m and n modulo 5. Given $k = k_1$, we note that the vertex $(0, (-3k \mod 5))$ is in $S^k(2, 1)$ but not G^*. However, each vertex of the form $(0, (5q - 3k) \mod 5) \in G^*$ for all $1 \leq q \leq \lfloor \frac{n+1-(3k \mod 5)}{5} \rfloor$. Hence, $k_4 = (n + 1 - 3k) \mod 5$. We can apply the same reasoning to see that $k_3 = (m + 1 - k - 3(m + 1)) \mod 5$ and $k_2 = (3k + 2(m + 1)) \mod 5$. Table 5, in the Appendix, shows the values of k_1, k_2, k_3, and k_4 for each value of m and n modulo 5. In the table, k_1 is listed in the lower left-hand corner, k_2 the upper left-hand corner, k_3 the upper right-hand corner, and k_4 the lower right-hand corner for each m and n.

So under the conditions on m and n as defined by Tables 1 and 5 that allow for full adjustments or partial adjustments in the case of $k = 1$ or $k = 3$, adjusting V^k to form set S^k, we see that S^k contains one fewer vertex per corner adjustment than V^k, so that the final set S^k has four fewer vertices than V^k. Since in each case we replace a boundary vertex with one from $G_{m,n}$, we have shown the following lemma.

Lemma 2.2 *The set $S^k \subset V(G_{m,n})$ is a dominating set for $G_{m,n}$ of size $|S^k| = |V^k| - 4$, provided m, n are of sufficient size, as defined in Table 1, that allow for 4 full or partial corner adjustments to be performed.*

Since we know that S^k is a dominating set for $G_{m,n}$ for certain values of m and n, we turn our attention to the size of V^k. We will show that either $|V^k| = \lfloor\frac{(m+2)(n+2)}{5}\rfloor$ or $|V^k| = \lfloor\frac{(m+2)(n+2)}{5}\rfloor + 1$, by examining how many vertices in V^k appear in each column of G^*. For each $1 \le i \le n+2$, let $\bar{j}_i$ be the smallest value of j such that $(i,j) \in V^k$. Note that $\bar{j}_0 = k$ and for each $1 \le i \le n+2$, $\bar{j}_i = (\bar{j}_{i-1}+2) \mod 5$. Further, $(i, \bar{j}_i+5q), 1 \le q \le \lfloor\frac{m+1-\bar{j}_i}{5}\rfloor$, will also be in V^k. Hence, given $\bar{j}_i$, there will be $\lfloor\frac{m+1-\bar{j}_i}{5}\rfloor + 1 = \lfloor\frac{m+6-\bar{j}_i}{5}\rfloor$ vertices in column i. We note that j ranges from 0 to 4 and depends on the value of k. Note that any span of 5 columns of G^* contains

$$\sum_{q=0}^{4}\left\lfloor\frac{m+6-q}{5}\right\rfloor = \sum_{q=0}^{4}\left\lfloor\frac{m+2+q}{5}\right\rfloor = m+2 \text{ vertices of } V^k. \quad (1)$$

Thus, if $n+2 = 5s+r$,

$$\begin{aligned}|V^k| &= (m+2)\left\lfloor\frac{n+2}{5}\right\rfloor + \sum_{q=0}^{r-1}\left\lfloor\frac{m+2+(4-\bar{j}_q)}{5}\right\rfloor, \\ &= \frac{(m+2)(n+2-r)}{5} + \sum_{q=0}^{r-1}\left\lfloor\frac{m+2+(4-\bar{j}_q)}{5}\right\rfloor.\end{aligned}$$

If $m+2+(4-\bar{j}_q) = 5t_q + d_q$, for each $0 \le q \le r-1$, we have

$$\begin{aligned}|V^k| &= \frac{(m+2)(n+2)}{5} - \frac{r(m+2)}{5} + \sum_{q=0}^{r-1}\frac{m+2+(4-\bar{j}_q)-d_q}{5}, \\ &= \frac{(m+2)(n+2)}{5} - \frac{r(m+2)}{5} + \frac{r(m+2)}{5} + \sum_{q=0}^{r-1}\frac{(4-\bar{j}_q-d_q)}{5}, \\ &= \frac{(m+2)(n+2)}{5} + \sum_{q=0}^{r-1}\frac{q^*}{5},\end{aligned}$$

where $q^* = 4 - \bar{j}_q - d_q$. Let's examine the term q^* more closely. This term represents the amount we round up or round down from $m+2$ depending on the value of $\bar{j}_q$. So, if $4 - \bar{j}_q \ge 5 - (m+2)$, then $q^* = 5 - (m+2)$ otherwise $q^* = -(m+2)$. Given (1) above, we know that

$$\sum_{q=0}^{4} \frac{(4 - \bar{j}_q - d)}{5} = 0.$$

However, for values of $r < 5$, this sum could vary away from 0. Since $\bar{j}_i = (\bar{j}_{i-1} + 2) (\text{mod } 5)$,

$$-\frac{4}{5} \leq \sum_{q=0}^{4} \frac{(4 - \bar{j}_q - d)}{5} \leq \frac{4}{5}.$$

Hence,

$$\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor \leq |V^k| \leq \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor + 1.$$

Table 2 shows the cases for each value of m, n and k for which V^k either is exactly $\lfloor \frac{(m+2)(n+2)}{5} \rfloor$ or is one more than this value.

n	$m = 0$ (mod 5)					$m = 1$ (mod 5)				
(mod 5)	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
0	1	1	1	1	0	0	0	1	0	0
1	1	0	0	0	0	1	1	1	0	1
2	1	1	1	0	0	0	1	1	0	0
3	0	0	0	0	0	0	0	0	0	0
4	1	1	0	0	0	1	1	1	0	0
n	$m = 2$ (mod 5)					$m = 3$ (mod 5)				
(mod 5)	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
0	1	0	1	1	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	1	1	1	1	0	0	0	0	0	0
n	$m = 4$ (mod 5)									
(mod 5)	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$					
0	1	0	1	0	0					
1	1	0	1	0	1					
2	1	1	0	0	1					
3	0	0	0	0	0					
4	1	0	0	0	0					

Table 2: $|V^k| - (\lfloor \frac{(m+2)(n+2)}{5} \rfloor)$

It is easy to see from Table 2 that for each value of m and n, there is at least one value of k in which we can construct a set V^k of cardinality $\lfloor \frac{(m+2)(n+2)}{5} \rfloor$ and hence a dominating set S^k of cardinality $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$. From [10], we know that for each $m, n \geq 16$, $\gamma_{m,n} = \lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$. Hence, we can use the above construction methods to create a γ-set for $G_{m,n}$, for $m, n \geq 16$. Thus, we have proven the following.

Theorem 2.3 *For all $m, n \geq 16$, there exists a k, $0 \leq k \leq 4$, such that S^k is a γ-set of $G_{m,n}$.*

3 Constructing γ-sets when $m, n \leq 15$, $m \neq 12, 13$

As outlined in [10] and [1], when $m, n \leq 15$, $\gamma_{m,n}$ may or may not equal $\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4$. In this section we will show that in some of the cases where $m, n \leq 15$, $\gamma_{m,n} = \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4$, and our methods can be used to construct a γ-set for $G_{m,n}$. In other cases, we will show how a repeatable γ-pattern can be used to construct γ-sets.

We begin with an example. Consider $G_{5,6}$. As in the previous section, we form G^* and overlay a slant grid. From Table 5 we see that when $m = 0 \pmod 5$ and $n = 1 \pmod 5$ there is a value of k, namely $k = 0$, that produces values $k_1 = 0$, $k_2 = 2$, $k_3 = 0$ and $k_4 = 2$ that allow for full adjustments to be made at each corner. Given m, n, we say that any value of k that allows for four full or partial adjustments to be made is a *compatible* value of k. Also, note for this value of k, $k_1 = k_3$ and $k_2 = k_4$. In these cases, we say that this value of k produces *symmetric* adjustments. In the previous example given in Figures 1 and 6, the compatible value of $k = 4$ produced symmetric adjustments. In Figure 8, we show how performing these adjustments produces a dominating set of size 8 for $G_{5,6}$ which, from Theorem 1.1, is optimum.

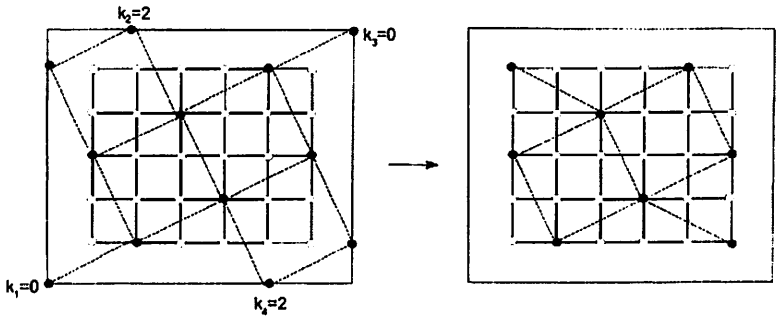


Figure 8: Constructing a γ-set of $G_{5,6}$

Further, note that when symmetric adjustments are made, in some cases of m and n, the resulting γ-set can be combined with itself to produce a

γ-sets of larger grids. For example, the γ-set we constructed for $G_{5,6}$ can be flipped over a line of symmetry drawn on column 6 (or column 1) to produce a dominating set of $G_{5,11}$ of size $2 * \gamma_{5,6} - \gamma(n)$, where $\gamma(n)$ equals the number of γ-vertices that lie in column n of $G_{m,n}$, in this case $n = 6$. We note that flipping $\gamma_{5,6}$ over a line of symmetry drawn on row 1 or row m also produces a γ-set for $\gamma_{9,6}$. Figure 9 shows how this flipping can create a γ-set for $G_{5,16}$.

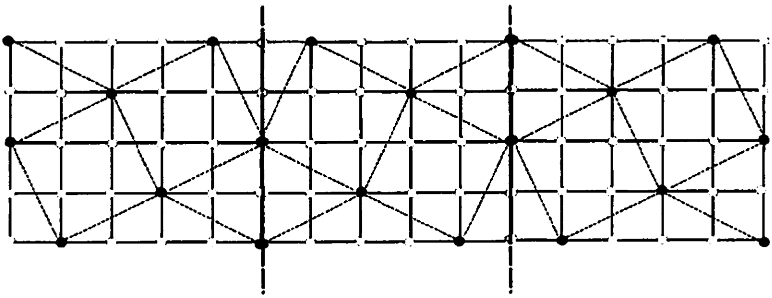


Figure 9: Constructing a γ-set of $G_{5,16}$

In general, for $m, n \leq 15$, a compatible value for k that allows symmetric adjustments to produce a γ-set for $G_{m,n}$ is called *repeatable*, and this γ-set can be flipped over a line of symmetry drawn at column n (or column 1) of $G_{m,n}$. We say that this repeatable γ-set has a *rate of growth*, R, where

$$R = \frac{\gamma_{m,n} - \gamma(n)}{n-1}.$$

For our example highlighted in Figure 8, note that the rate of growth is $R = \frac{8-2}{5} = \frac{6}{5}$ which matches the rate of growth highlighted in Theorem 1.1 for $\gamma_{5,n}$. If $R < \frac{(m+2)(n+2)}{5}$, then a repeatable γ-set has a rate of growth slower than that of our construction method to produce set S, and thus can be used to produce a dominating set of size smaller than $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$. In Figures 10 - 14, we show the repeatable γ-sets matching the growth rates indicated in Theorem 1.1 for $2 \leq m \leq 11$ and $14 \leq m \leq 15$.

The key to achieving a rate of growth slower than $\frac{(m+2)(n+2)}{5}$ lies in maximizing $\gamma(n)$ or $\gamma(1)$. As an example, note that when $m = 8$, $k = k_1 = 2$ implies $k_2 = 4$, and there are 3 γ-vertices on the first column border. This value of k maximizes the number of γ-vertices that exist on the first column border for $m = 8$. From Table 2, we note that $k = 2$ allows for the creation of a γ-set of size $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$ for all values of $n \geq 8$. In Figures 15 and

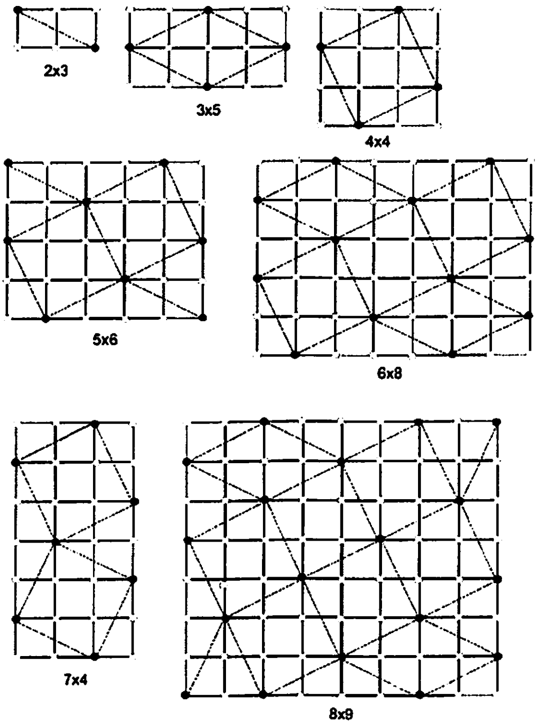
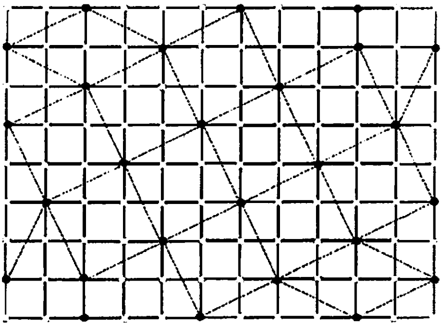


Figure 10: Repeatable Patterns for $n = 2$ through $n = 8$



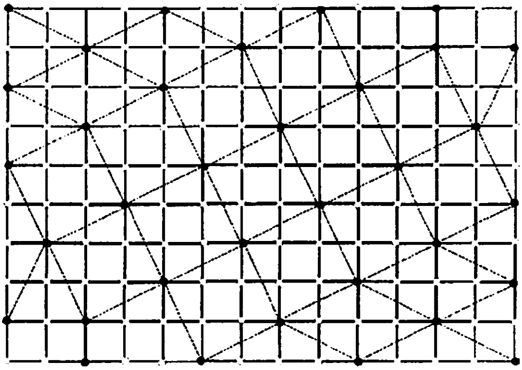


Figure 11: Repeatable Patterns for $n = 9$ and $n = 10$

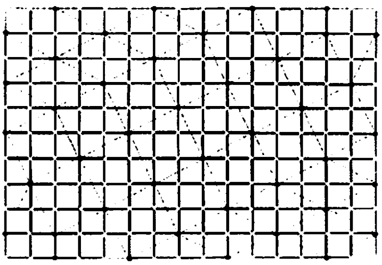


Figure 12: Repeatable Pattern for $n = 11$

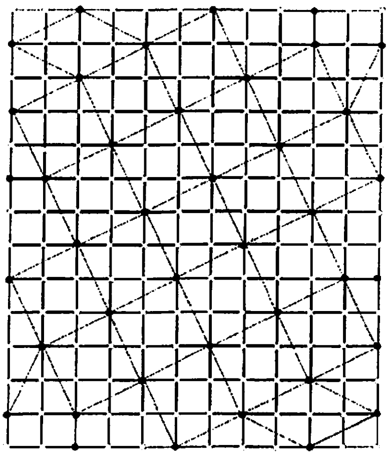


Figure 13: Repeatable Pattern for $n = 14$

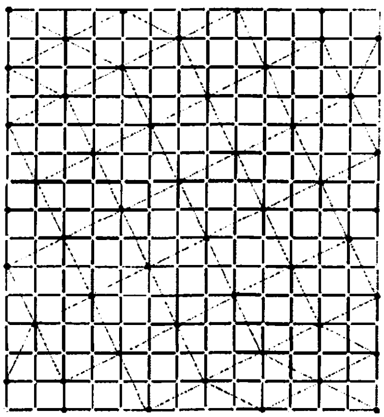


Figure 14: Repeatable Pattern for $n = 15$

16 we show the optimal γ-sets for $G_{8,n}$ for $8 \leq n \leq 14$.

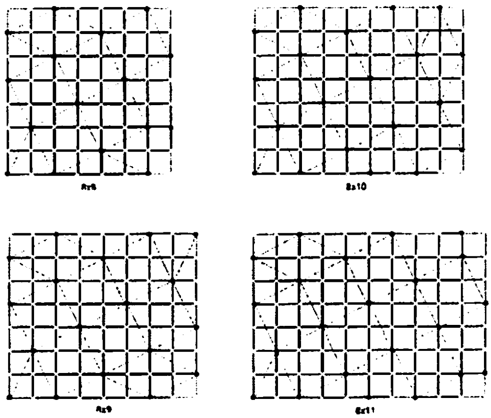


Figure 15: $k = 2$ γ-sets for $n = 8$

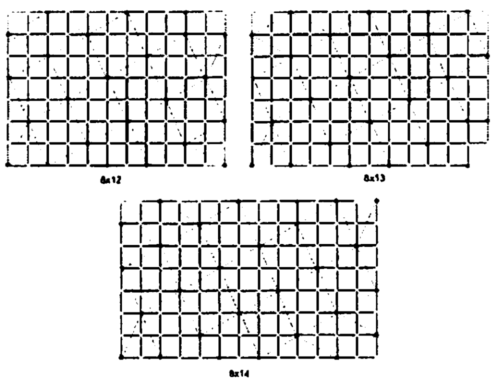


Figure 16: $k = 2$ γ-sets for $n = 8$

Although the γ-set for $G_{8,9}$ is used to define the rate of growth of γ-sets for $G_{8,n}$ for all $n \geq 8$, the γ-set in Figure 15 for $G_{8,8}$ can be repeated one time (since it is not symmetric) to create a γ-set for $G_{8,15}$ as seen in the Figure 17. It can then be combined with the symmetric γ-set for $G_{8,9}$ to create a γ-set for $G_{8,23}$. Continuing this process for other values of n shows that we can create a γ-set for $G_{8,n}$ for any value of $n \geq 8$.

Tables 3 and 4 shows how to construct a γ-set for $G_{m,n}$ by indicating the value of k to use or by showing how to use a smaller, repeatable γ-

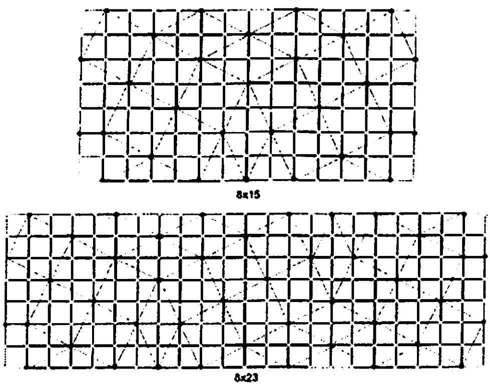


Figure 17: γ-sets for $G_{8,15}$ and $G_{8,23}$

set to construct the γ-set. The notation $a|a$ indicates that the γ-set for $G_{m,2a-1}$ is constructed by flipping the repeatable γ-set for $G_{m,a}$ once. The notation $a|a|a|a$ indicates that the γ-set for $G_{m,4a-3}$ is formed by flipping the symmetric, repeatable γ-set for $G_{m,a}$ four times. The notation $a-a$ means that the γ-set for $G_{m,2a}$ is formed from two copies of $G_{m,a}$ that share no vertices. This occurs for some special cases when $m \leq 4$ and $m = 7$. When $m = 10$, the γ-set for $G_{10,4}$ is the same as the γ-set indicated in the chart of $G_{4,10}$. It is denoted as 4*. This is a repeatable γ-set and can be used to generate γ-sets of $G_{10,3j+1}$ for $1 \leq j \leq 16$.

<table>
<tr><th>m : n</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>13</th><th>14</th><th>15</th></tr>
<tr><td>2</td><td>k = 0</td><td>k = 2</td><td>k = 2</td><td>3|3</td><td>3-3</td><td>3|3|3</td><td>3|3-3</td><td>3|3|3|3</td><td>3|3|3-3</td><td>3|3|3|3|3</td><td>...</td><td>...</td><td>...</td><td>...</td></tr>
<tr><td>3</td><td></td><td>k = 0</td><td>k = 3</td><td>k = 2*</td><td>k = 2</td><td>k = 2</td><td>k = 2</td><td>5|5</td><td>5 − 5</td><td>k = 2</td><td>k = 2</td><td>5|5|5</td><td>5|5 − 5</td><td>5 − 5 − 5</td></tr>
<tr><td>4</td><td></td><td></td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>4|4</td><td>4 − 4</td><td>k = 0</td><td>4|4|4</td><td>4|4 − 4</td><td>4 − 4 − 4</td><td>4|4|4|4</td><td>4|4|4 − 4</td><td>4|4 − 4 − 4</td></tr>
<tr><td>5</td><td></td><td></td><td></td><td>k = 4</td><td>k = 0</td><td>k = 1</td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>6|6</td><td>6|k = 0</td><td>6|8</td><td>6|9</td><td>6|10</td></tr>
<tr><td>6</td><td></td><td></td><td></td><td></td><td>k = 3</td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>k = 0</td><td>8|7</td><td>8|8</td></tr>
<tr><td>7</td><td></td><td></td><td>4</td><td>k = 0</td><td>k = 3</td><td>4|4</td><td>4 − 4</td><td>k = 2</td><td>4|4|4</td><td>4|4 − 4</td><td>4 − 4 − 4</td><td>4|4|4|4</td><td>4|4|4 − 4</td><td>4|4 − 4 − 4</td></tr>
<tr><td>8</td><td></td><td></td><td></td><td></td><td></td><td></td><td>k = 2</td><td>k = 2</td><td>k = 2</td><td>k = 2</td><td>k = 2</td><td>k = 2</td><td>k = 2</td><td>8|8</td></tr>
<tr><td>9</td><td></td><td></td><td></td><td></td><td></td><td></td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>8|8</td></tr>
<tr><td>10</td><td></td><td></td><td>4*</td><td></td><td></td><td>4|4</td><td>k = 2</td><td>k = 3</td><td>4|4|4</td><td>k = 3</td><td>k = 3</td><td>4|4|4|4</td><td>k = 3</td><td>k = 4</td></tr>
<tr><td>11</td><td></td><td></td><td></td><td></td><td></td><td></td><td>k = 4</td><td>k = 4</td><td>k = 4</td><td>k = 3</td><td>k = 4</td><td>k = 4</td><td>k = 3</td><td>8|8</td></tr>
<tr><td>14</td><td></td><td></td><td></td><td></td><td></td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>8|8</td></tr>
<tr><td>15</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>k = 3</td><td>k = 4</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 3</td><td>k = 4</td></tr>
</table>

Table 3: Generation of γ-sets of $G_{m,n}$ for $m \leq 15$ and $n \leq 15$, $m \neq 12, 13$

There are a couple of special cases in the table above. The γ-sets for $G_{11,18}$, $G_{11,20}$, and $G_{11,22}$ do not follow the construction methods from Section 1 or the repeatable pattern of this section. Figure 18 shows the γ-set for $G_{11,18}$. The γ-sets for $G_{11,20}$ and $G_{11,22}$ have a similar construction to $\gamma_{11,18}$. The construction method for these γ-sets are more complicated and resemble the construction of γ-sets when $m = 12, 13$. These constructions do not use

<table>
<tr><th>m : n</th><th>16</th><th>17</th><th>18</th><th>19</th><th>20</th><th>21</th><th>22</th><th>23</th><th>24</th><th>25</th><th>26</th><th>27</th></tr>
<tr><td>8</td><td>8|9</td><td>9|9</td><td>9|10</td><td>9|11</td><td>9|12</td><td>9|13</td><td>9|14</td><td>8|8|9</td><td>8|9|9</td><td>9|9|9</td><td>9|9|10</td><td>...</td></tr>
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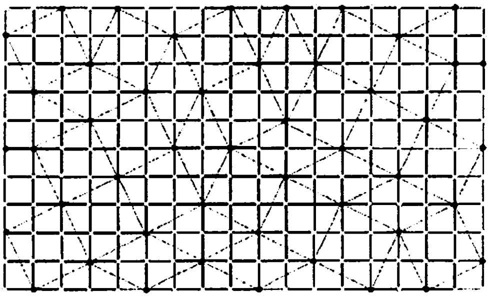
<tr><td>10</td><td>4|4|4|4|4</td><td>4|4|4|4 − 4</td><td>15|4</td><td>4|4|4|4|4|4</td><td>$k = 4$</td><td>15|4|4</td><td>4|4|4|4|4|4|4</td><td>20|4|4</td><td>15|4|4|4</td><td>4|4|4|4|4|4|4|4</td><td>20|4|4</td><td>14|14</td></tr>
<tr><td>11</td><td>$k = 3$</td><td>8|10</td><td>**</td><td>8|12</td><td>**</td><td>12|10</td><td>**</td><td>12|12</td><td>9|16</td><td>12|14</td><td>11|16</td><td>14|14</td></tr>
<tr><td>14</td><td>8|9</td><td>8|10</td><td>8|11</td><td>8|12</td><td>9|12</td><td>10|12</td><td>11|12</td><td>12|12</td><td>12|13</td><td>12|14</td><td>8|12|8</td><td>8|12|9</td></tr>
<tr><td>15</td><td>$k = 3$</td><td>$k = 3$</td><td>$k = 3$</td><td>$k = 3$</td><td>$k = 4$</td><td>11|11</td><td>11|12</td><td>12|12</td><td>11|14</td><td>12|14</td><td>13|14</td><td>14|14</td></tr>
</table>

Table 4: Generation of γ-sets of $G_{m,n}$, $m \leq 15$, $m \neq 12, 13$, $16 \leq n \leq 27$

a repeatable pattern to create larger γ-sets and are left for future work.



11x18

Figure 18: γ-set for $G_{11,18}$

4 Conclusion and Future Work

In this paper, we present a method for constructing γ-sets for $G_{m,n}$, for all $m \neq 12, 13$ and all n, that exhibits the rate of growth noted in Theorem 1.1. However, there is still work to be done in this area. For $m < 16$, our method constructs one γ-set which can be used to create other γ-sets through a repeatable transformation. In some cases, when $m = 12, 13$ for instance, known γ-sets of $G_{m,n}$ do not follow these repeatable transformations but are much more complicated. From Theorem 1.1, the growth rate R when $m = 12$ is $\frac{80}{29}$ and when $m = 13$ is $\frac{98}{33}$. If our construction approach works in this case, $G_{12,30}$ should have a γ-set of size 85 with 5 γ-vertices on columns 1 and 30. Further, $G_{13,34}$ should have a γ-set of size 103 with 5 γ-vertices on columns 1 and 34. However, the authors have not been able to construct such γ-sets using various construction techniques. In fact, known γ-sets for these cases indicate more complicated transformations that do not conform with the ones presented here.

Although we have given a method for constructing a γ-set of a grid graph

$G_{m,n}$, there are other γ-sets of $G_{n,m}$ that are not constructible using our method, and that are not just flips and rotations of the γ-sets that we construct. We do not know of techniques or transformations to create these other γ-sets.

It seems reasonable that the techniques discussed here could be used to construct γ-sets of related graphs. For instance, the methods presented here could be extended to construct γ-sets of $P_m \Box C_n$ and $C_m \Box C_n$. Further, these techniques could construct sets that are optimal for other types of domination. The γ-sets constructed here for $m, n \geq 16$ are independent sets, proving that $\gamma_{m,n} = i_{m,n}$ for $m, n \geq 16$, where $i_{m,n}$ denotes the independent domination number of $G_{n,m}$. However, if $m, n \leq 15$, many of the γ-sets created have adjacent vertices (c.f. Figures 11-14). For these cases, we do not know if it is possible to transform the vertices so that there is an independent γ-set of the same size. Identifying the values of m and n where $\gamma_{m,n} < i_{m,n}$ for $m, n < 16$ is an open question for future work.

Finally, the computation of the total domination numbers of grid graphs has not been studied as much as the domination numbers of grid graphs. Gravier [9] has determined the total domination numbers of $G_{m,n}$ for $m \leq 4$ and all n, while Klobucar [18] has determined these numbers for $m = 5, 6$ and all n. However, it appears that an analysis similar to that given here applies for the construction of γ_t-sets of grids. Figure 1 in [9] suggests that a transformation of the slant grid overlay given here could be applied to produce γ_t-sets of grid graphs.

5 Appendix

References

[1] S. Alanko, S. Crevals, A. Insopoussu, P. Östergärd, V. Petterson, Computing the domination number of grid graphs, *The Electronic J. Comb.* **18** (2011), #P141.

[2] T. Y. Chang. Domination numbers of grid graphs, *Ph.D. Thesis* , Dept. of Mathematics, University of South Florida, 1992.

[3] T. Y. Chang, W. E. Clark, The domination numbers of the $5 \times n$ and $6 \times n$ grid graphs, *J. Graph Theory*, **17** (1993), 81-108.

[4] T. Y. Chang, W. E. Clark, E. O. Hare, Dominations of complete grid graphs I, *Ars Combin.* **38** (1994), 97-112.

	n=0		n=1		n=2		n=3		n=4	
$m = 0$	2	3	4	4	1	0	3	1	0	2
	0	1	0	1	0	1	0	1	0	1
	0	2	2	3	4	4	1	0	3	1
	1	3	1	3	1	3	1	3	1	3
	3	1	0	2	2	3	4	4	1	0
	2	0	2	0	2	0	2	0	2	0
	1	0	3	1	0	2	2	3	4	4
	3	2	3	2	3	2	3	2	3	2
	4	4	1	0	3	1	0	2	2	3
	4	4	4	4	4	4	4	4	4	4
	$n = 0$		$n = 1$		$n = 2$		$n = 3$		$n = 4$	
$m = 1$	2	0	4	1	1	2	3	3	0	4
	0	2	0	2	0	2	0	2	0	2
	0	4	2	0	4	1	1	2	3	3
	1	4	1	4	1	4	1	4	1	4
	3	3	0	4	2	0	4	1	1	2
	2	1	2	1	2	1	2	1	2	1
	1	2	3	3	0	4	2	0	4	1
	3	3	3	3	3	3	3	3	3	3
	4	1	1	2	3	3	0	4	2	0
	4	0	4	0	4	0	4	0	4	0
	$n = 0$		$n = 1$		$n = 2$		$n = 3$		$n = 4$	
$m = 2$	2	2	4	3	1	4	3	0	0	1
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	1	4	3	0	0	1	2	2	4	3
	3	4	3	4	3	4	3	4	3	4
	4	3	1	4	3	0	0	1	2	2
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	$n = 0$		$n = 1$		$n = 2$		$n = 3$		$n = 4$	
$m = 3$	2	4	4	0	1	1	3	2	0	3
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	0	3	2	4	4	0	1	1	3	2
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	$n = 0$		$n = 1$		$n = 2$		$n = 3$		$n = 4$	
$m = 4$	2	1	4	2	1	3	3	4	0	0
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	0	0	2	1	4	2	1	3	3	4
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	3	1	3	1	3	1	3	1	3	1
	4	2	1	3	3	4	0	0	2	1
	4	3	4	3	4	3	4	3	4	3

Table 5: Values of k_1, k_2, k_3, k_4 for each value of m, n for $m, n \geq 16$

[5] R. Cherifi, S. Gravier, X. Lagraula, C. Payan, I. Zighem, Domination number of the cross product of paths, *Discrete Appl. Math.*, **94** (1999), 101-139.

[6] R. Cherifi, S. Gravier, I. Zighem, Bounds on domination number of complete grid graphs, *Ars Combin.*, **60** (2001), 307-311.

[7] E. J.Cockayne, E. O. Hare, S. T. Hedetniemi, T. V. Wimer, Bounds for the domination number of grid graphs, *Cong. Numer.*, **47** (1985), 217-228.

[8] D. C. Fisher, The domination number of complete grid graphs, *manuscript*, circa 1998.

[9] S. Gravier, Total domination number of grid graphs, *Discrete Appl. Math.*, **121** (2002), no.1-3, 119-128.

[10] D. Goncalves, A. Pinlou, M. Rao, S. Thomasse, The domination number of grids, *SIAM J. Disc. Math.* **25** (3) (2011), 1443-1453.

[11] D. R. Guichard, A lower bound for the domination number of complete grid graphs, *J. Combin. Math. Combin. Comput.* **49** (2004), 215-220.

[12] E. O. Hare, Algorithms for grid and grid-like graphs, Ph.D. Thesis, Dept. of Computer Science, Clemson University, 1989.

[13] E. O. Hare, D. C. Fisher, An application of beatable dominating sets to algorithms for complete grid graphs, *Proceedings of the Seventh Quadrenial International Conference on the Theory and Applications of Graphs*, **1**, Y. Alavi, A. Schwenk, eds., (1995), 497-506.

[14] E. O. Hare, S. T. Hedetniemi, W. R. Hare, Algorithms for computing the domination number of $k \times n$ complete grid graphs, *Cong. Numer.* **55** (1986), 81-92.

[15] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[16] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, editors, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[17] M. S. Jacobson, L. F. Kinch, On the domination number of products of a graph, I, *Ars Combin.* **10** (1983), 33-44.

[18] A. Klobucar, Total domination numbers of Cartesian products, *Math. Commun.* **9** (2004), no. 1, 35-44.

[19] M. Livingston, Q. F. Stout, Constant time computation of minimum dominating sets, *Cong. Numer.*, **105** (1994), 116-128.

[20] H. G. Singh, R. P. Pargas, A parallel implementation for the domination number of grid graphs, *Cong. Numer.* **59** (1987), 297-312.

[21] A. Spalding. A min-plus algebra scheme for computing domination numbers, *Ph.D. Thesis*, Dept. of Mathematics, University of Colorado, 2001.