On a K_4-UH self-dual 1-configuration $(102_4)_1$

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Abstract

Self-dual 1-configurations $(n_d)_1$ have the most K_d-separated Menger graph $\mathcal{Y}$ for connected self-dual configurations (n_d). Such $\mathcal{Y}$ is most symmetric if it is K_d-ultrahomogeneous. In this work, such a graph $\mathcal{Y}$ is presented for $(n,d) = (102,4)$ and shown to relate n copies of the cuboctahedral graph $L(Q_3)$ to the n copies of K_4. These are shown to share each copy of K_3 with two copies of $L(Q_3)$. Vertices and copies of $L(Q_3)$ in $\mathcal{Y}$ are the points and lines of a self-dual $(104_{12})_1$.

1 Introduction

Let $1 < d < n \in \mathbf{Z}$ and $1 < c < m \in \mathbf{Z}$. A *configuration* $R = (m_c, n_d)$ is an incidence structure of m points and n lines such that there are c lines through each point and d points on each line [8]. Thus, $cm = dn$. Let $L = L(R) = L(m_c, n_d)$ be the *Levi graph* of R, namely the bipartite graph with: **(a)** m "black" vertices representing the points of R; **(b)** n "white" vertices representing the lines of R; and **(c)** an edge between each two vertices representing a point and a line incident in R. To each configuration $R = (m_c, n_d)$ corresponds the *dual* configuration $\overline{R} = (n_d, m_c)$ by reversing the roles of points and lines in R. If $(m,n) = (c,d)$, then R is *balanced* [17]. If R is isomorphic to its dual $\overline{R}$, then $R = (n_d)$ is *self-dual*. A corresponding isomorphism is called a *duality*. Both R and $\overline{R}$ share the same Levi graph, but the black-white coloring of their vertices is reversed. To any such configuration (n_d) we can associate its *Menger graph*, in which the points of (n_d) are represented by vertices, each two joined by an edge whenever the two corresponding points are in a common line in (n_d). Let $1 \leq \lambda < d$. If any two different points of R are in at most λ lines, then R is a λ-*configuration* $(n_d)_\lambda$ [15]. The 4-cube Q_4 is the Levi graph of the Möbius

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$(8_4)_2$ with "white" (resp. "black") vertices being those of even (resp. odd) weight, (and so on for the remaining Cox 2-configurations, in relation to the respective d-cube Q_d) [8]. Let H be a connected regular graph. A graph G is *$\mathcal{C}$-ultrahomogeneous* [20], or *$\mathcal{C}$-UH*, if every isomorphism between two induced copies of $H \in \mathcal{C}$ in G extends to an automorphism of G. If $\mathcal{C} = \{H\}$ then G is said to be *H-UH*.

The motivation of this paper is the study of connected Menger graphs [8] of self-dual 1-configurations $(n_d)_1$ [7, 15] expressible as K_d-ultrahomogeneous graphs [20]. The question of for which values of n such graphs exist is interesting because it would yield the most symmetric, connected, edge-disjoint unions of n copies of K_d on n vertices in which the roles of vertices and copies of K_d are interchangeable. For $d = 4$, known values of n are: $n = 13, 21$ (see [17, 18, 21]) and $n = 42$ [9]. It is of interest to determine the spectrum and multiplicities of the involved values of n. To this aim, Theorem 4.1 below contributes the value of $n = 102$. This is obtained via the Biggs-Smith association scheme [6]. This is shown in Theorem 6.1 to control attachment of 102 (cuboctahedral) copies of $L(Q_3)$ to the 102 (tetrahedral) copies of K_4. These copies share each (triangular) copy of K_3 with two copies of $L(Q_3)$. So, Theorem 7.1 guarantees the distance 3-graph of the Biggs-Smith graph $\mathcal{S}$ [3, 5] as the Menger graph $\mathcal{Y}$ of a self-dual 1-configuration $(102_4)_1$. On the other hand, the Möbius 2-configuration $(8_4)_2$ for example, and more generally the Cox 2-configurations $((2^{d-1})_d)_2$ [8], have their Menger graphs with copies of K_4 and K_d respectively not edge-disjoint, even though these are K_4- and K_d-ultrahomogeneous graphs. Some questions arising at this level are: Are variations of the latter graphs as in [21] (5.3.7) K_d-ultrahomogeneous? Does there exist a relation between K_d-ultrahomogeneous Menger graphs and geometric configurations [4]? Do there exist two different configurations with common K_d-ultrahomogeneous Menger graph? Must K_d-ultrahomogeneous duality be involutory [19, 21]?

A connected graph G is an *$\{H\}^d_n$-graph* if it is an edge-disjoint union of n induced copies of H with no other copies of H as subgraphs and each vertex incident to exactly d copies of H, no two such copies sharing more than one vertex. If $H = K_r$ is the complete graph of order r ($0 < r \in \mathbb{Z}$) then the vertices and copies of H in G can be seen as the points and lines of a 1-configuration R_G with its points representing the vertices of G and its lines representing the copies of H in G. If R_G is a self-dual 1-configuration, then it can be denoted $(n_d)_1$ and G can be recovered as the Menger graph of $R_G = (n_d)_1$. Let us illustrate these concepts with some examples. Clearly, a connected graph G is m-regular if and only if it is a $\{K_2\}^m_{|E(G)|}$-graph. In this case, G is arc-transitive if and only if G is $\{K_2\}$-UH. On the other hand:

(A) for $1 < r \in \mathbf{Z}$, the complete graph K_r and its Cartesian powers $K_r^2 = K_r \square K_r, K_r^3 = K_r^2 \square K_r, \ldots, K_r^s = K_r^{s-1} \square K_r, \ldots$ etc. are K_r-UH $\{K_r\}_n^m$-graphs; their orders form a sequence $r, r^2, r^3, \ldots, r^s, \ldots$ of integers corresponding to the respective K_r-UH $\{K_r\}_1^1$-, $\{K_r\}_{2r}^2$-, $\{K_r\}_{3r^2}^3$-, …, $\{K_r\}_{sr^{s-1}}^r$-, …-graphs;

(B) for $3 \leq r \in \mathbf{Z}$ the line graph $L(Q_r)$ of the r-cube Q_r is a $\{K_r, K_{2,2}\}$-UH $\{K_r\}_{2^r}^2 \{K_{2,2}\}_{r(r-1)2^{r-3}}^{r-1}$-graph. A similar argument yields a K_r-UH $\{K_r\}_n^m$-graph out of any other regular-polytopal graph via its line graph.

There is only one case in **(A)-(B)** that is Menger graph of a self-dual configuration, namely K_2^2 (duality sending for example the points $00, 10, 11, 01$ resp. onto the lines $x0, 0x, x1, 1x$, where $0 \leq x \leq 1$), even though all graphs K_r^r have equal numbers of vertices and of copies of K_r so they are Menger graphs of *balanced* configurations (but not self-dual). If $r = 4$, then the orders of the K_d-UH $\{K_d\}_n^m$-graphs in **(A)-(B)** are divisible by 4. Beside ours ($n = 132$), a case of even order indivisible by 4 is the one mentioned above on $n = 42$ vertices [9]. Its construction was based on the ordered pencils of the Fano plane. Extensions of that construction of [9], based on ordered pencils of binary projective spaces, are introduced in [13] which provides K_4-UH $\{K_4\}_n^m$-graphs whose even orders are indivisible by 4, the smallest of which being 210. However, the latter graphs are not Menger graphs of self-dual configurations. A configuration $(n_d)_1$ is said to be K_d-*UH* if its Menger graph is. Are there any UH-K_4 self-dual configurations $(n_4)_1$ with even $n < 42$? Or $42 < n < 102$?

In Section 4, the claimed Menger graph $\mathcal{Y}$ is constructed by means of the distance-3 graphs of the 9-cycles of the Biggs-Smith graph $\mathcal{S}$. Theorem 4.1 proves our claim about $\mathcal{Y}$ as an application of a transformation of distance-transitive graphs into $\mathcal{C}$-UH graphs that took in [10] from the Coxeter graph of order 28 onto the Klein graph of order 56. A similar application allowed in [11] to confront, as digraphs, the Pappus graph of order 18 to the Desargues graph of order 20. These applications as well as [12] use the following definitions. Given a family $\mathcal{C}$ of digraphs, a digraph G is said to be $\mathcal{C}$-*UH* if every isomorphism between two induced members of $\mathcal{C}$ in G extends to an automorphism of G. If $\mathcal{C} = \{H\}$ then G is said to be H-*UH*. By removing the suffix "di" here, the definition of $\mathcal{C}$-UH graph is recovered. A presentation of $\mathcal{S}$ is given in Section 2 by means of Biggs-Hoare sextets mod 17 [2] which provide a convenient notation to present $\mathcal{Y}$ in Section 3 in preparation for Section 4.

We set one more definition to be used from Section 2 on. If M is a subgraph of H and if G is both M-UH, and H-UH, then G is an $\{H\}_M$-*UH graph* if, for each induced copy H_0 of H in G containing an induced copy M_0 of M, there exists exactly one induced copy $H_1 \neq H_0$ of H in G such that

$V(H_0) \cap V(H_1) = V(M_0)$ and $E(H_0) \cap E(H_1) = E(M_0)$.

2 The Biggs-Smith graph

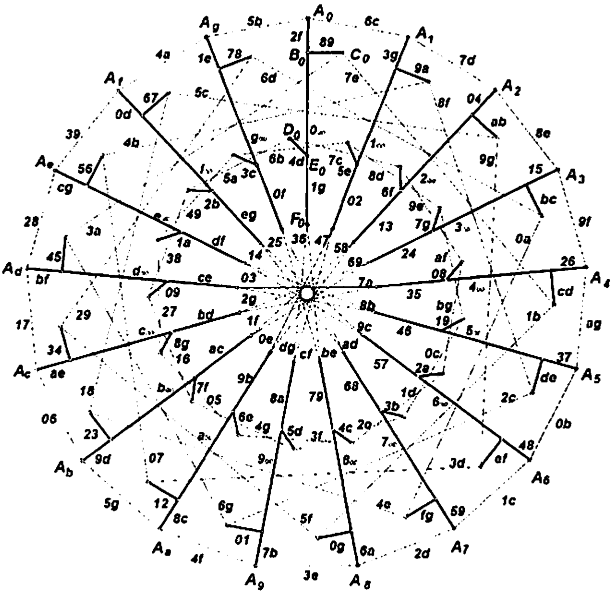

Figure 1: Representation of $\mathcal{S}$ via sextets and thick subtrees T_i^∞

The Biggs-Smith graph $\mathcal{S}$ has order $n = 102$, diameter $d = 7$, girth $g = 9$ and automorphism group $\mathcal{A} = PSL(2,17)$ [6]. By letting k be the largest integer s such that $\mathcal{S}$ is s-arc transitive, it is seen that $k = 4$. In addition, the number η of 9-cycles of $\mathcal{S}$ is $\eta = 136$. Taking into account the definition in the last paragraph of Section 1 and by denoting a 3-path by P_4 and a 9-cycle by γ_9, the following particular case of Theorem 3 of [12] holds (which cannot be refined to a result of $\{\vec{\gamma_9}\}_{\vec{P_4}}$-UH digraphs; see (4) below):

$$\mathcal{S} \text{ is } \{\gamma_9\}_{P_4}\text{-UH}. \qquad (1)$$

Properties of $\mathcal{S}$ we need are presented via *sextets* [2], where heptadecimal notation is used to denote elements of $GF(17)$ (for example $g = 16 = -1$ and $d = 13 = -4$). In fact, we view $\mathcal{S}$ as a connected graph whose vertex set $V(\mathcal{S})$ comprises 102 sextets mod 17, namely 102 unordered triples

$$\{a_0b_0, a_1b_1, a_2b_2\}$$

composed by unordered pairs a_ib_i of points a_i, b_i of the projective line $PG(1,17) = GF(17) \cup \{\infty\}$ satisfying

$$(a_i - a_j)(b_i - b_j)(a_i - b_j)^{-1}(b_i - a_j)^{-1} = -1,$$

if $a_i \neq \infty$ and satisfying

$$(b_i - b_j)(b_i - a_j)^{-1} = -1,$$

if $a_i = \infty$, whenever $i \neq j$ in $\{0, 1, 2\}$, including the vertices

$$\begin{array}{lll} A_0 = \{2f, 5b, 6c\}, & B_0 = \{0\infty, 2f, 89\}, & C_0 = \{3a, 7e, 89\}, \\ D_0 = \{5a, 7c, 4d\}, & E_0 = \{0\infty, 1g, 4d\}, & F_0 = \{1g, 36, be\}. \end{array} \tag{2}$$

Any two of the resulting 102 vertices are adjacent in S whenever they share one such pair a_ib_i, in which case the resulting edge is labeled a_ib_i. It is shown in [2] that this S is unique and that the *edge labels* a_ib_i are pairwise distinct, so they determine an edge labeling of S represented in Figure 1 with the following notation. The six vertices in (2) are those of a subtree T_0^∞ (of S) which is the edge-disjoint union of the paths

$$(A_0, 2f, B_0, 89, C_0), (D_0, 4d, E_0, 1g, F_0) \text{ and } (B_0, 0\infty, E_0)$$

of lengths 3, 3 and 2, respectively. By adding to all elements of $GF(17)$ in T_0^∞ a constant $i \in GF(17)$, a similar tree T_i^∞ is obtained. The trees $T_0^\infty, \dots, T_g^\infty$, represented in Figure 1 via dark traces, are pairwise disjoint and cover $V(S)$. The complement of their union in S is formed by 4 17-cycles

$$\begin{array}{ll} A = (A_0, 6c, A_1, \dots, A_g, 5b), & D = (D_0, 7c, D_2, \dots, D_f, 5a), \\ C = (C_0, 7e, C_4, \dots, C_d, 3a), & F = (F_0, be, F_8, \dots, F_9, 36). \end{array}$$

Each of these cycles $y = A, D, C, F$ has vertices y_r with $r \in GF(17)$ advancing in 1, 2, 4, 8 units mod 17 stepwise from left to right, respectively.

Employed in [12] in proving (1) above, there is a set C_9 of 136 directed 9-cycles of S, of which a generating subset

$$\{\Pi^0 = (\Pi_0^0\Pi_1^0 \dots \Pi_8^0); \Pi = S, T, \dots, Z\}$$

(written without commas and accompanied to the right by auxiliary permutations, as explained below) is as follows:

$$\begin{array}{ll} S^0 = (B_2A_2A_1A_0A_gA_fB_fC_fC_2) & s^0 = (07cb4d65a)(\infty 8g2e3f19) \\ T^0 = (E_dD_dD_fD_0D_2D_4E_4F_4F_d) & t^0 = (03ac9857e)(\infty 12d6b4fg) \\ U^0 = (B_9C_9C_dC_0C_4C_8B_8A_8A_9) & u^0 = (06371gacb)(\infty 249c58df) \\ V^0 = (E_gF_gF_8F_0F_9F_1E_1D_1D_g) & v^0 = (05b3f2e6c)(\infty d9ga7184) \\ W^0 = (B_9E_9F_9F_0F_8E_8B_8A_8A_9) & w^0 = (\infty a3b986c7)(0df15cg24) \\ X^0 = (E_gB_gA_gA_0A_1B_1E_1D_1D_g) & x^0 = (\infty ebcg1563)(084f7a2d9) \\ Y^0 = (B_2E_2D_2D_0D_fE_fB_fC_fC_2) & y^0 = (\infty 6ca2f75b)(01943ed8g) \\ Z^0 = (E_dB_dC_dC_0C_4B_4E_4F_4F_d) & z^0 = (\infty 5aed437c)(0fg9b6812) \end{array} \tag{3}$$

where the permutation $\pi^0 = (\pi_0^0\pi_1^0\dots\pi_8^0)(\xi_0^0\xi_1^0\dots\xi_8^0)$ of $PG(1,17)$ to the right of each Π^0 is such that: **(i)** the pair $\pi_i^0\pi_{i+4}^0$ labels the edge $\Pi_i^0\Pi_{i+1}^0$; **(ii)** the pair $\xi_i^0\xi_{i+3}^0$ labels the only edge incident to Π_i^0 outside Π^0, where $i = 0,\dots,8$ and index addition is taken modulo 9. C_9 also contains the directed cycles Π^r with accompanying permutations π^r obtained from Π^0 and π^0 by uniformly adding $r \in \mathbf{Z}_{17}$ mod 17 to all subscripts and superscripts. Observe that: **(iii)** passing from s^0 to t^0 to u^0 to v^0 and again to s^0, (resp. from w^0 to x^0 to y^0 to z^0 and again to w^0) amounts to multiplying uniformly and successively the participating entries of the permutations π^0 by either 2 or -2 mod 17; and **(iv)** $S^0,\dots,Z^0$ are invariant with respect to their change-of-sign involutions mod 17, with corresponding involutions on $s^0,\dots,z^0$ around the initial entries of their two composing cycles, which are either 0 and ∞, or ∞ and 0.

3 Distance-3 digraphs of oriented 9-cycles

A *k-arc* in a (di)graph is a sequence of vertices $v_0v_1\dots v_k$ (written without parentheses or commas), where consecutive vertices are adjacent and $v_{i-1} \neq v_{i+1}$, for $0 < i < k$ [14]. A k-arc can be interpreted as a directed walk of length k in which consecutive edges are distinct [16]. Thus, an arc in a (di)graph Γ is a 1-arc of Γ. The form in which the directed 9-cycles Π^r in Section 2 share 3-arcs, either oppositely oriented or not, to be used in Figure 3 below, can be encoded as in the following table that for each Π^0 presents details (explained below) of the 9-cycles $\Xi_r \neq \Pi^0$ in C_9 that intersect Π^0 either in the succeeding 3-arcs $\Pi_i^0\Pi_{i+1}^0\Pi_{i+2}^0\Pi_{i+3}^0$ or in their respective reversed arcs, for $i = 0,\dots,8$, with sums involving i taken mod 9:

$$\begin{array}{l}
S^0:(-X_2^1,\ S_2^1,\ S_1^g,-X_1^g,-U_5^7,\ U_8^6,\ Y_6^0,\ U_4^b,-U_7^a);\\
T^0:(-Y_2^f,\ T_2^f,\ T_1^2,-Y_1^2,-V_5^3,\ V_8^5,\ Z_6^0,\ V_4^c,-V_7^c);\\
U^0:(\ Z_1^d,\ U_2^d,\ U_1^4,\ Z_2^4,\ S_7^6,-S_4^a,\ W_6^0,-S_8^7,\ S_5^b);\\
V^0:(-W_2^8,\ V_2^8,\ V_1^9,-W_1^9,\ T_7^5,-T_4^c,\ X_6^0,-T_8^3,\ T_5^c);\\
W^0:(-Z_7^d,-V_3^8,-V_0^9,-Z_5^4,-W_8^g,\ X_0^9,\ U_6^0,\ X_3^8,-W_4^1);\\
X^0:(\ W_5^8,-S_3^1,-S_0^g,\ W_7^9,-X_8^2,\ Y_0^g,\ V_6^0,\ Y_3^1,-X_4^f);\\
Y^0:(\ X_5^1,-T_3^f,-T_0^2,\ X_7^g,-Y_8^d,\ Z_0^2,\ S_6^0,\ Z_3^f,-Y_4^4);\\
Z^0:(\ Y_5^f,\ U_0^4,\ U_3^d,\ Y_7^2,-Z_8^8,-W_3^d,\ T_6^0,-W_0^4,-Z_4^9).
\end{array} \qquad (4)$$

Each such Ξ^r has: either **(I)** a preceding minus sign, if the corresponding 3-arcs in Π^0 and Ξ^r are oppositely oriented, or **(II)** no preceding sign, otherwise. Each shown $-\Xi_j^r$ (resp. Ξ_j^r) has a subscript j indicating the equality of initial vertices $\Xi_j^r = \Pi_{i+3}^0$ (resp. $\Xi_j^r = \Pi_i^0$) of those 3-arcs, for $i = 0,\dots,8$.

Given a (di)graph Γ and a positive integer $k \leq$ diameter(Γ), the *distance-k (di)graph* Γ_k of Γ, with vertex set $V(\Gamma_k) = V(\Gamma)$, is such that from every

$u \in V(\Gamma_k)$ an arc of Γ_k departs to a vertex $v \neq u$ whenever there is a shortest k-arc of length k in Γ from u to v. Let $(\mathcal{C}_9)_3$ be the family of distance-3 digraphs of directed 9-cycles in $\mathcal{C}_9$. On a representation of an arc $e = w_0w_1$ of a member $(\zeta_9)_3$ of $(\mathcal{C}_9)_3$, we label its *tail*, or initial vertex, w_0, its *initial flag* $\{w_0, e\}$, its *terminal flag* $\{e, w_1\}$ and its *head*, or terminal vertex, w_1, respectively by the names of the vertices v_0, v_1, v_2, v_3 of the 3-arc $v_0v_1v_2v_3$ in ζ_9 for which w_0w_1 stands in $(\zeta_9)_3$. For example, if $\zeta_9 = U^9 = (B_1C_1C_5C_9C_dC_0B_0A_0A_1)$, so that $(\zeta_9)_3 = (U^9)_3 = (B_1C_9B_0)(C_1C_d A_0)(C_5C_0A_1)$, then the initial flag of the arc B_1C_9 in $(\zeta_9)_3 = (U^9)_3$ is labeled by C_1, the terminal flag by C_5, while B_1 and C_9 are labeled exactly by B_1 and C_9, respectively. We get the labels over $(\zeta_9)_3 = (U^0)_3$ shown in Figure 2.

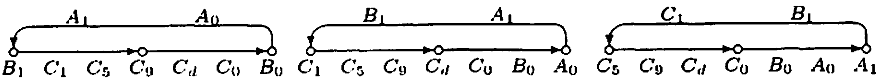

Figure 2: Labels of vertices and flags of $(\zeta_9)_3 = (U^9)_3$

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We are to fasten pairs of arcs of the digraphs $(\zeta_9)_3$ defined in Section 3 in such a way that a graph $\mathcal{Y}$ with the properties claimed in Section 1 is produced. A sequence of operations $\mathcal{S} \to \mathcal{C}_9 \to (\mathcal{C}_9)_3 \to \mathcal{Y}$ (compare with [10]) is performed in order to transform $\mathcal{S}$ into the claimed $\mathcal{Y}$. Each distance-3 digraph $(\zeta_9)_3$ of a 9-cycle ζ_9 in the collection $\mathcal{C}_9$ generated via (3) is formed by 3 disjoint directed triangles. It yields a total of 3×136 directed triangles so $\mathcal{C}_9$ determines a family of 408 directed triangles in the claimed $\mathcal{Y}$ with each edge shared by exactly two such directed triangles in arcs that are either oppositely or identically oriented. It amounts to 102 copies of K_4; these can be subdivided into 6 subfamilies $\{\Sigma^i\}$ of 17 copies each, say with $\Sigma \in \{A, B, C, D, E, F\}$ and $i \in \{0, 1, \ldots, 16 = g\} = \mathbf{Z}_{17}$. The vertex sets $V(\Sigma^i)$, each followed by the set $\Lambda(\Sigma_i)$ of copies of K_4 containing the corresponding vertex Σ_i can be taken as follows, showing $\mathbf{Z}_2$-symmetry produced by change of sign mod 17:

$$\begin{array}{ll}
V(A^i)=\{C_i, \; D_i, \; E_{i+4}, E_{i-4}\}; & \Lambda(A_i)=\{C^i, \; D^i, \; E^{i+7}, E^{i-7}\}; \\
V(B^i)=\{D_{i+3}, D_{i-3}, F_{i+5}, F_{i-5}\}; & \Lambda(B_i)=\{D^{i+2}, D^{i-2}, F^{i+8}, F^{i-8}\}; \\
V(C^i)=\{A_i, \; F_i, \; E_{i+1}, E_{i-1}\}; & \Lambda(C_i)=\{A^i, \; F^i, \; E^{i+6}, E^{i-6}\}; \\
V(D^i)=\{A_i, \; D_i, \; B_{i+2}, B_{i-2}\}; & \Lambda(D_i)=\{A^i, \; D^i, \; B^{i+3}, B^{i-3}\}; \\
V(E^i)=\{C_{i+6}, C_{i-6}, A_{i+7}, A_{i-7}\}; & \Lambda(E_i)=\{C^{i+1}, C^{i-1}, A^{i+4}, A^{i-4}\}; \\
V(F^i)=\{C_i, \; F_i, \; B_{i+8}, B_{i-8}\}; & \Lambda(F_i)=\{C^i, \; F^i, \; B^{i+5}, B^{i-5}\};
\end{array} \quad (5)$$

where i varies in $\mathbf{Z}_{17}$. This reveals a duality ϕ from the 102 vertices of $\mathcal{S}$ onto the 102 copies of K_4 in $\mathcal{S}$. In fact, these copies of K_4 are the vertices of a graph $\phi(\mathcal{S}) = \mathcal{S}^* \equiv \mathcal{S}$ determined by

$$\begin{array}{lll} \phi(A_i)=A^{3i}=A_i^*, & \phi(B_i)=B^{-7i}=B_i^*, & \phi(C_i)=C^{3i}=C_i^*, \\ \phi(D_i)=D^{5i}=D_i^*, & \phi(E_i)=E^{6i}=E_i^*, & \phi(F_i)=F^{5i}=F_i^*, \end{array} \quad (6)$$

($i \in \mathbf{Z}_{17}$), with a structure similar to that of the vertices $A_i, \ldots, F_i$ of $\mathcal{S}$, the copies of K_4 in $\mathcal{S}^*$ precisely being $\Sigma_i = A_i, \ldots, F_i$ and corresponding vertex sets $\Lambda(\Sigma_i)$ as specified above. Moreover, $\phi : \mathcal{S} \to \mathcal{S}^*$ is a graph isomorphism, with the adjacency of $\mathcal{S}^*$ equivalent to that of $\mathcal{S}$.

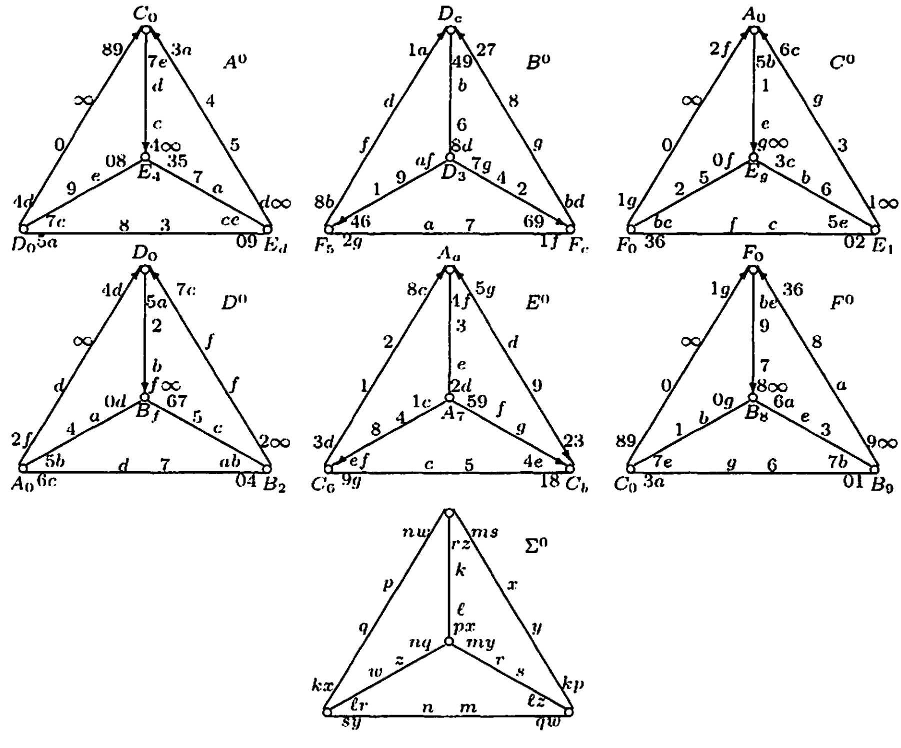

Figure 3: Symmetry of edge labels in copies of K_4 in $\mathcal{Y}$, for $i = 0$

Figure 3 illustrates the left side of (5) for $i = 0$ in terms of edge labels, where edges of $\mathcal{Y}$ arising from pairs of 3-arcs of $\mathcal{S}$ identically (resp. oppositely) fastened according to (1) are shown oriented (resp. unoriented) accordingly. Observe the edges oriented in

$$\begin{array}{lll} A^0 : D_0C_0,\ C_0E_4,\ C_0E_d; & B^0 : D_3F_5,\ F_5D_c,\ D_3F_c,\ F_cD_c; & C^0 : F_0A_0,\ A_0E_1,\ E_gA_0; \\ D^0 : A_0D_0,\ B_2D_0,\ D_0B_f; & E^0 : A_7C_6,\ A_7C_b,\ C_bA_a,\ C_6A_a; & F^0 : F_0B_8,\ B_9F_0,\ C_0F_0. \end{array}$$

By uniformly adding successively $1 \in \mathbf{Z}_{17}$, each of these 6 cases yields 16 additional ones. This yields the 102 edge-labeled copies of K_4 in $\mathcal{Y}$. If the two points of $PG(1,17)$ labeling near its center each edge ϵ in the figure are disposed as shown, labeling the respective flags of ϵ, then the 6 cases may be indicated uniquely as $(kl,mn)(pq,rs)(xy,zw)$, where the position of the labels $k,\ell,m,n,p,q,r,s,w,x,y,z$ is as in the referential depiction Σ^0 of a copy of K_4 in the lower part of the figure. Then, the flag-label triples at the upper, middle, lower-right and lower-left vertices of this depiction are respectively kpx, ℓrz, msy and nqw. Moreover, the 6 points of $PG(1,17)$ in each of these copies of K_4 not participating of its edge labeling conform a unique sextet χ which is not a vertex of $\mathcal{S}$ as characterized in Section 2. However, χ is a sextet of an alternative labeling of $\mathcal{S}$ happening via the remaining 102 sextets (of the total of 204). These 102 alternative sextets are the images of the 102 vertices of $\mathcal{S}$ via multiplication of indices in $PG(1,17)$ times $3 \in GF(17)$, operation that coincides with the duality ϕ expressed in (6) above. This proves the assertion in Theorem 4.1 below that the vertices and copies of K_4 of $\mathcal{S}$ are the points and lines of a self-dual 1-configuration $(102_4)_1$, which in turn has $\mathcal{Y}$ as its Menger graph. Correspondingly, the vertex labels in Σ^i are the sextets (rz,ms,nw), $(px,nq,\ my)$ $(kp,\ell z,qw)$ and $(kx,\ell r,sy)$.

A procedure that allows to determine which point of $PG(1,17)$ labels which flag in a copy of K_4 as in Figure 3 is given as follows:

(i) A triangle Δ in a copy ∇ of K_4 in $\mathcal{Y}$, say $\Delta = (C_0E_4D_0)$ in $\nabla = A^0$, arises from a 9-cycle $\Pi^j = (\Pi^j_0 \dots \Pi^j_8)$ in $\mathcal{S}$ with associated permutation $\pi^j = (\pi^j_0 \dots \pi^j_8)(\xi^j_0 \dots \xi^j_8)$ as displayed in Section 2, in this case $\Pi^j = Y^2$ with $\pi^j = x^2$; and

(ii) by labeling each edge $\Pi^j_i\Pi^j_{i+1}$ of Π^j just by π^j_i, it holds that the flag label of edge $\epsilon = \Pi^j_i\Pi^j_{i+3}$ at Π^j_i is π^j_{i+1}, while the flag label of ϵ at Π^j_{i+3} is π^j_{i+5}, where $i = 0,3,6$.

The distance-3 digraphs of the directed 9-cycles Π^0 of $\mathcal{S}$ are composed by the following triples of disjoint directed triangles of $\mathcal{Y}$:

$$\begin{aligned}
S^0 &\to \{D^0\backslash D_0=(B_2A_0B_f),\ E^9\backslash C_3=(A_2A_gC_f),\ E^8\backslash C_8=(A_1A_fC_2)\};\\
T^0 &\to \{A^0\backslash C_0=(E_dD_0E_4),\ B^g\backslash F_b=(D_dD_2F_4),\ B^1\backslash F_1=(D_fD_4F_d)\};\\
U^0 &\to \{F^0\backslash F_0=(B_9C_0B_8),\ E^f\backslash A_5=(C_9C_4A_8),\ E^2\backslash A_2=(C_dC_8A_9)\};\\
V^0 &\to \{C^0\backslash A_0=(E_gF_0E_1),\ B^4\backslash D_7=(F_gF_9D_1),\ B^d\backslash D_d=(F_8F_1D_g)\};\\
W^0 &\to \{F^0\backslash C_0=(B_9F_0B_8),\ C^8\backslash E_7=(E_9F_8A_8),\ C^9\backslash C_9=(F_9E_8A_9)\};\\
X^0 &\to \{C^0\backslash F_0=(E_gA_0E_1),\ D^1\backslash B_3=(B_gA_1D_1),\ D^g\backslash D_g=(A_gB_1D_g)\};\\
Y^0 &\to \{D^0\backslash A_0=(B_2D_0B_f),\ A^f\backslash E_b=(E_2D_fC_f),\ A^2\backslash A_2=(D_2E_fC_2)\};\\
Z^0 &\to \{A^0\backslash D_0=(E_dC_0E_4),\ F^4\backslash D_c=(B_dC_4F_4),\ F^d\backslash F_d=(C_dB_4F_d)\}.
\end{aligned}$$

This way, it can be seen that $\mathcal{Y}$ is a K_4-UH graph. However, in view of Beineke's characterization of line graphs [1] and observing that $\mathcal{Y}$ contains induced copies of $K_{1,3}$, which are forbidden for line graphs of simple graphs, we conclude that $\mathcal{Y}$ is non-line-graphical.

Theorem 4.1 *$\mathcal{Y}$ is both the Menger graph of a K_4-UH self-dual 1-configuration $(102_4)_1$ and a non-line-graphical $\{K_4\}^4_{102}$-graph. Moreover, $\mathcal{Y}$ is arc-transitive with regular degree 12, diameter 3, distance distribution (1, 12, 78, 11) and automorphism group $PSL(2,17)$ of order 2448. Its associated Levi graph is a 2-arc-transitive graph with regular degree 4, diameter 6, distance distribution $(1,4,12,36,78,62,11)$ and automorphism group $SL(2,17)$ of order 4896.*

Proof. It remains to prove that $\mathcal{Y}$ is K_4-UH, which uses (1) and more specifically (4) above. In fact, consider an isomorphism $\Psi : \Theta_1 \to \Theta_2$ between copies Θ_1, Θ_2 of K_4 in $\mathcal{Y}$. Each Θ_i, $(i = 1,2)$, arises from 4 9-cycles $\gamma_9 = \theta_i^j$ in $\mathcal{S}$, $(j = 1,2,3,4)$, whose union is a subgraph $\overline{\Theta}_i$ of $\mathcal{S}$ with 4 vertices v_i^j of degree 3 and 12 vertices of degree 2 that are the internal vertices of 6 3-paths P_4 whose ends are the vertices v_i^j. For example, the vertices $v_1^1 = B_0, v_1^2 = B_1, v_1^3 = F_9, v_1^4 = C_9, v_2^1 = B_1, v_2^2 = B_2, v_2^3 = F_a, v_2^4 = C_a$ in $\mathcal{S}$ determine such subgraphs Θ_1, Θ_2 in $\mathcal{Y}$ and $\overline{\Theta}_1, \overline{\Theta}_2$ in $\mathcal{S}$. Clearly, Ψ induces an isomorphism $\overline{\Psi} : \overline{\Theta}_1 \to \overline{\Theta}_2$ that sends say each v_1^j onto its corresponding v_2^j, $(j = 1,2,3,4)$. As an automorphism $\overline{\overline{\Psi}}$ of $\mathcal{S}$ exists that extends $\overline{\Psi}$, then $\overline{\overline{\Psi}}$ determines an automorphism of $\mathcal{Y}$ that restricts to Ψ, showing that $\mathcal{Y}$ is a K_4-UH graph. □

5 Definitions to deal with the copies of $L(Q_3)$

If H is a graph with an edge partition $\Omega = \Omega(H)$ into 2-paths, then a graph G is *Ω-preserving H*-UH if every Ω-preserving isomorphism between two induced copies of H in G extends to an automorphism of G. If M is a subgraph of H and if G is both M-UH, and Ω-preserving H-UH, then G is an *Ω-preserving $\{H\}_M$-UH graph* if, for each induced copy H_0 of H in G containing an induced copy M_0 of M, there is just one induced copy $H_1 \neq H_0$ of H in G such that:

(a) $V(H_0) \cap V(H_1) = V(M_0)$;

(b) $E(H_0) \cap E(H_1) = E(M_0)$; and

(c) the edges of M_0 are in distinct 2-paths both in $\Omega(H_0)$ and $\Omega(H_1)$.

A graph G is *rK_s-frequent* if every edge e of G is intersection of exactly r induced copies of K_s, these copies having only e and its ends in common. For example, K_4 is $2K_3$-frequent and $L(Q_3)$ is $1K_3$-frequent. A graph G is *$\{H_2, H_1\}_{K_3}$-UH*, where H_i is iK_3-frequent $(i = 1,2)$ if:

(d) G is H_2-UH and edge-disjoint union of induced copies of H_2;

(e) there is a partition Ω of H_1 into 2-paths and G is Ω-preserving $\{H_1\}_{K_3}$-UH; and

(f) each induced copy of H_2 in G has each induced copy of K_3 in common with exactly two induced copies of H_1 in G.

Theorem 6.1 shows that $\mathcal{Y}$ is $\{K_4, L(Q_3)\}_{K_3}$-UH. This allows to gather information on $\mathcal{S}_2$ and $\mathcal{S}_4$, leading to $\mathcal{Y} = \mathcal{S}_3$ in Theorem 7.1.

6 The K_4-UH graph $\mathcal{Y}$ is $\{K_4, L(Q_3)\}_{K_3}$-UH

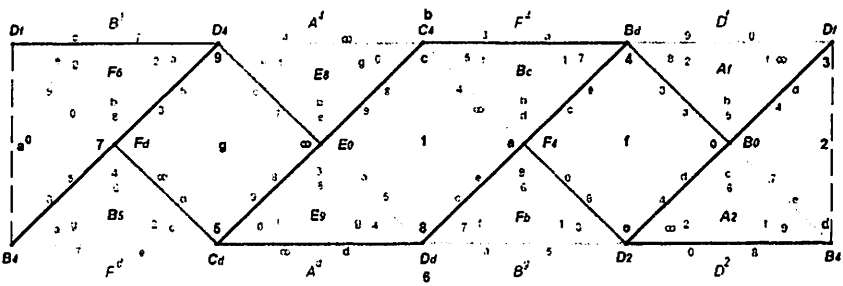

Figure 4: Toroidal cutout representation of a^0

Recall from Section 4 that each copy of K_4 in $\mathcal{Y}$ arises from the distance-3 digraphs of 4 directed 9-cycles of $\mathcal{S}$. The subgraph of $\mathcal{S}$ spanned by these 4 9-cycles contains 4 degree-3 vertices (which are tails and heads of corresponding 3-arcs) and 12 degree-2 vertices (internal vertices of those 3-arcs). These 12 vertices induce a copy $\mathcal{L}$ of $L(Q_3)$ in $\mathcal{Y}$. For the copy A^0 of K_4 in $\mathcal{Y}$, the corresponding copy $\mathcal{L} = a^0$ of $L(Q_3)$ in $\mathcal{Y}$ can be represented as in the big rectangle $\mathcal{R}$ in Figure 4, where:

(a) the leftmost and rightmost dashed lines of $\mathcal{R}$ are to be identified by parallel translation;

(b) each of the 8 shown triangles Δ forms part of a corresponding copy ∇ of K_4 cited on the exterior of $\mathcal{R}$ about the horizontal edge of Δ, while its 4th vertex is cited at the center of Δ; and

(c) the edges are colored via a partition Ω into 2-paths P_3, the edges of each P_3 with a common color from a set of 3 colors: (i) black; (ii) light-gray; (iii) dark-gray; the 3 colors are present together in every triangle, and opposite edges in every induced 4-cycle, or 4-*hole*, have a common color, a total of two colors per 4-hole.

For $\sigma = a, b, c, d, e, f$, the copies σ^0 of $L(Q_3)$ are expressed by means of the data contained in Figure 4 as follows:

$a^0:(D_f D_4 C_4 B_d)(B_4 C_d D_d D_2) F_d E_0 F_4 B_0 (B^1 F_6 A^4 E_8 F^4 B_e D^f A_f)(F^d B_5 A^d E_9 B^g F_b D^2 A_2)$
$b^0:(D_5 D_g E_e F_d)(D_e F_4 E_5 D_1) F_e E_3 E_e F_3 (B^2 F_7 A^g C_g C^d A_d B^8 D_b)(B^9 D_6 C^4 A_4 A^1 C_1 B^a D_f)$
$c^0:(F_8 F_1 A_1 B_g)(B_1 A_g F_g F_9) D_g E_0 D_1 B_0 (B^d D_a C^1 E_2 D^1 B_3 F^8 C_8)(D^g B_e C^g E_f B^4 D_7 F^9 C_9)$
$d^0:(A_1 A_f D_f E_2)(E_f D_2 A_2 A_g) C_2 B_0 C_f E_0 (E^8 C_c D^f B_d A^f E_b C^1 F_1)(A^2 E_6 D^2 B_4 E^9 C_3 C^g F_g)$
$e^0:(A_6 A_9 B_b C_2)(A_b C_f B_6 A_8) C_a B_7 B_a C_7 (E^g C_5 D^0 D_9 F^2 F_2 E^d A_3)(E^4 A_e F^f F_f D^a E_8 E^1 C_e)$
$f^0:(C_4 C_9 F_9 E_8)(E_9 F_8 C_8 C_d) A_8 B_0 A_9 E_0 (E^f A_5 F^0 B_1 C^9 E_a A^4 D_4)(C^8 E_7 F^8 B_g E^2 A_c A^d D_d)$

and their translations mod 17 are denoted σ^i, for $0 \neq i \in \mathbf{Z}_{17}$ (uniformly translating all involved subscripts and superscripts). Each copy σ^i of $L(Q_3)$ admits an edge partition $\Omega = \Omega(\sigma^i)$ into j-colored 2-paths ($j \in \{1,2,3\}$) so that each (monochromatic) 2-path in an $\Omega(\sigma^i)$ is shared only by one other copy of $L(Q_3)$ in $\mathcal{Y}$ (as in Theorem 6.1(**3**), below). We may write

$$\sigma^i = \sigma_1^i \cup \sigma_2^i \cup \sigma_3^i, \tag{7}$$

to stress the color partition of σ^i into its black, light-gray and dark-gray subgraphs, which are copies of the disconnected graph $4P_3$ (formed by 4 disjoint copies of P_3) as in Figure 4 for $\sigma^i = a^0$. The edge labels of σ^0 in Figure 4 (shown in gray type) and of all the other σ^is are taken as the flag labels for $i = 0, \ldots, g$ in Figure 3. The relation and location of these flag labels justifies a labeling of the 12 vertices and 6 4-holes as shown with symbols $0, \ldots, g, \infty$ (in black type) in Figure 4, the sole edge-label notation to be used ahead.

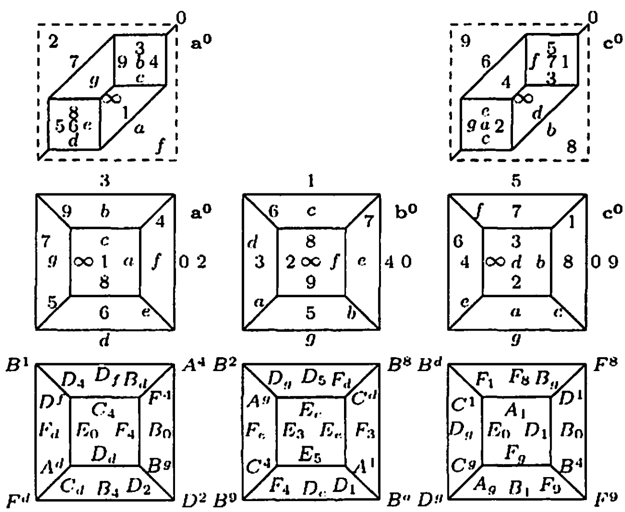

Figure 5: Label and vertex-tetrahedron representations of a^0, b^0, c^0 in Q_3

The labels of the 12 vertices and 6 4-holes of each of $\sigma^0 = a^0, \ldots, f^0$ are depicted again on the middle thirds of Figures 5 and 6, this time on a copy

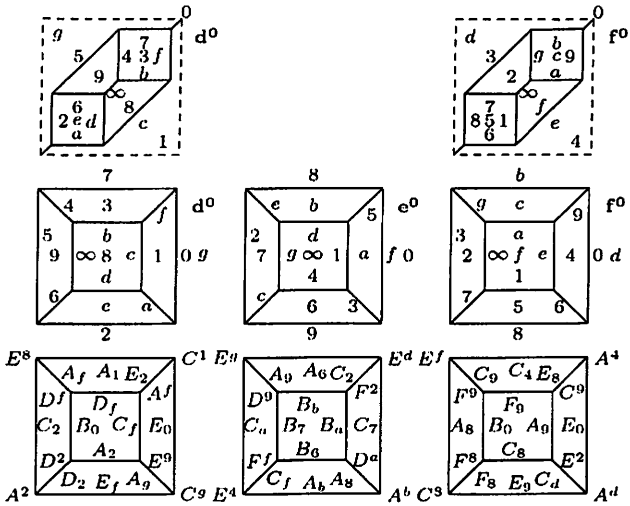

Figure 6: Label and vertex-tetrahedron representations of d^0, e^0, f^0 in $\mathcal{Q}_3$

$\mathcal{Q}_3$ of the 3-cube Q_3 from which a corresponding copy of $L(Q_3)$ in $\mathcal{Y}$ is obtained with its vertices taken as the middle points of the edges of $\mathcal{Q}_3$, tracing an edge between two such vertices whenever the edges they represent have a vertex in common in $\mathcal{Q}_3$, with the convention that labels of vertices and 4-holes of σ^0 label now respectively the corresponding edges and faces of $\mathcal{Q}_3$. (On the bottom thirds those edges are labeled by the corresponding vertices of $\mathcal{S}$ and their vertices by the corresponding containing copies of K_4; on the upper thirds, 4 different cutouts of $\mathcal{Q}_3$ are depicted to show involution symmetry around edges labeled ∞, where $\mathcal{Q}_3$ is regained by identifying the upper and left sides and the lower and right sides via 90° rotations at the upper-left and lower-right corners). Opposite faces in such σ^j determine pairs of points of $PG(1,17)$, a total of 3 such pairs leading to a unique sextet which is not a vertex of $\mathcal{S}$ but uniformly 3 times a vertex of $\mathcal{S}$. For example, these 3 pairs for $\sigma^0 = a^0$ form the sextet $\{12, 6b, fg\} = 3 \times \{6c, 2f, 5b\} = A_0$, mod 17. By denoting $a^0 = \{12, 6b, fg\}$ and so on for the 101 remaining copies of $L(Q_3)$ in $PG(1,17)$, we obtain a self-dual configuration that uses again the duality ϕ of Section 4, this time with points and lines taken as the vertices and copies of $L(Q_3)$ in $\mathcal{S}$, This is a self-dual 1-configuration $(102_4)_1$, as claimed in Theorem 6.1(8) below, depending on the facts that $L(Q_3)$ has 12 vertices and that each vertex of $\mathcal{Y}$ belongs to 12 copies of $L(Q_3)$.

Figure 7 shows the complements of vertex A_0 in 4 of the 12 copies of $L(Q_3)$ containing A_0, namely e^b, d^2, c^1, f^9, which share the long vertical edges,

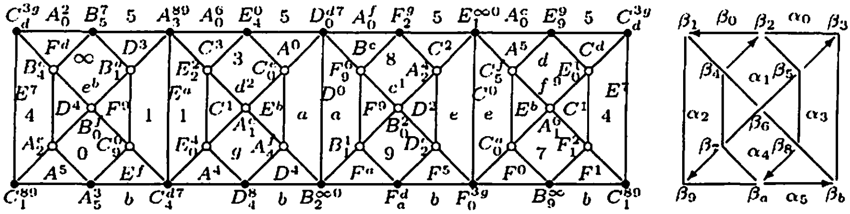

Figure 7: Covering graph Υ_0 of $e^b \cup d^2 \cup c^1 \cup f^9 - A_0$ and α-β denotations

successively present in the copies E^a, D^0, C^0, E^7 of K_4, the last long vertical edge both as the leftmost and rightmost edges in the shown covering graph, say Υ_0, of $e^b \cup d^2 \cup c^1 \cup f^9 - A_0$, where:

(a) black vertices participate of the 8 4-holes containing A_0, namely those labeled 5 on top and b at the bottom; other labels of 4-holes internal to them, respectively;

(b) the labels j of vertices Σ_i appear as superindices, as in Σ_i^j, (with j also in the citations A_0^j of A_0 on top), or $\Sigma_i^{jj'}$, in case labels j and j' happen in contiguous copies of $L(Q_3)$;

(c) each triangle contains the name Σ^ℓ of the copy of K_4 containing it;

(d) for each $\sigma^i = e^b, d^2, c^1, f^9$, the partition $\Omega(\sigma^i)$ restricts as in the rightmost diagram, in which darts indicate the first edges of monochromatic 2-paths whose final vertex is A_0; as a result, the 4 mentioned long vertical edges belong each to two different monochromatic 2-paths of contiguous copies of $L(Q_3)$ in $\mathcal{Y}$;

(e) alternate internal anti-diagonal monochromatic 2-paths (i.e. from top-right to bottom-left) coincide with directions reversed; (the middle vertices of these 4 2-paths are just two neighbors of A_0 in S, and their degree-1 vertices are at distance 2 from A_0 in S); and

(f) the rightmost diagram contains denotations β_i, ($i \in [0, b]$), and α_j, ($j \in [0.5]$), respectively for the vertex and 4-hole labels in their positions in the 4 copies of $L(Q_3)$.

Apart from the union $e^b \cup d^2 \cup c^1 \cup f^9$ of copies of $L(Q_3)$ sharing A_0 in Figure 7, there are two other unions of 4 copies of $L(Q_3)$ in $\mathcal{Y}$ sharing A_0. The following display of the data in Figure 7 contains at its left the α-β denotations of **(f)**. Moreover, the data corresponding to the 3 unions of 4 copies of $L(Q_3)$ sharing A_0 in $\mathcal{Y}$ are set (or encoded) in the arrays to the right and below the α-β denotations (these solely for e^b, d^2, c^1, f^9, respectively), where the leftmost array summarizes Υ_0, the two doubly repeated middle vertices in Υ_0 (as in **(e)**) parenthesized to the right of A_0

and the remaining data displayed in similar order, with the two rightmost arrays preceded by the first one of their 4 corresponding α-β denotations, which condenses all needed information of $\mathcal{Y}$ around A_0:

$\alpha_0\beta_0\alpha_5$=52b,56b,5fb,5cb	$A_0(B_0A_1)$	f62	$A_0(A_1A_g)$
$\beta_1\beta_2\beta_3$=g78,90d,7g∞,093	$(E^7e^bE^ad^2D^0c^1C^0f^9)$	41g	$(E^7d^gC^0d^1E^ae^9D^0e^8)$
$\beta_4\alpha_1\beta_5$=c∞a,23e,684,fd1	$(C_dB_5A_3E_4D_0F_2E_1E_9)$	5d9	$(C_1E_eE_gD_3A_3C_7B_fC_6)$
$\alpha_2\beta_6\alpha_3$=4f1,1ca,a2e,e64	$(B_4B_1E_2C_0F_9A_2C_5E_0)$	8c0	$(D_1A_fC_3B_1C_2B_gB_eA_2)$
$\beta_7\alpha_4\beta_8$=e06,4gf,19c,a72	$(A_2C_9E_0A_4B_1D_2C_0F_1)$	∞7b	$(B_gC_eA_2D_gA_fB_3B_1C_f)$
$\beta_9\beta_a\beta_b$=93d,78∞,0d3,g∞8	$(C_1A_5C_4D_4B_2F_aF_0B_9)$	3ae	$(A_cD_cE_1E_3C_gC_bB_2C_a)$
		cb6	$A_0(A_gB_0)$
		804	$(E^7d^fD^0c^gC^0f^8E^ae^6)$
		fe3	$(A_cE_dD_0F_fE_gE_8C_4B_c)$
		g57	$(E_fC_0F_8A_fC_eE_0B_dB_g)$
		d12	$(E_0A_dB_gD_fC_0F_gA_fC_8)$
		a9∞	$(C_dD_dB_fF_7F_0B_8C_gA_c)$

Some edges are shared by two of these 3 unions. In fact, each of the edges bordering the central 2-paths ω in anti-diagonal 4-paths in Υ_0 is present also in one of the two covering graphs, say Υ_1 and Υ_2, corresponding to the two rightmost arrangements above, one encoded on top and the other at the bottom of the display, respectively. For example, the edge B_1A_3 of e^b on Υ_0 appears in Υ_1. Also, the labels $\{\alpha_0\alpha_4, \alpha_1\alpha_5, \alpha_2\alpha_3\}$ of opposite copies of $L(Q_3)$, just sharing vertex A_0, are images of vertices at distance 3 in S via the duality ϕ (but copies of $L(Q_3)$ sharing a triangle containing A_0 are images of vertices at distance 7). The following permutations on the set $\{\alpha_0, \ldots, \alpha_5, \beta_0, \ldots, \beta_{11}\}$ relate the labels of the 12 copies of $L(Q_3)$ sharing A_0:

$$e^b \to d^2 \to c^1 \to f^9 \to e^b:$$
$$(\alpha_0)(\alpha_5)(\beta_0\beta_4\beta_6\beta_8)(\beta_1\alpha_4\beta_2\beta_9)(\beta_3\beta_a\alpha_1\beta_b)(\beta_5\alpha_3\alpha_2\beta_7);$$

$$e^b d^2 c^1 f^9 \to d^g d^1 e^9 e^8 \to d^f c^g f^8 e^6 \to e^b d^2 c^1 f_9:$$
$$(\alpha_0\beta_4\beta_6)(\beta_0\alpha_5\beta_8)(\beta_1\beta_3\alpha_2)(\beta_2\alpha_4\alpha_3)(\alpha_1\beta_7\beta_b)(\beta_5\beta_a\beta_9).$$

The following permutations allow to relate the labels of the 12 cuboctahedral subgraphs sharing A_0 to those sharing B_0, C_0, D_0, E_0, F_0:

$$A_0 \to B_0 : (\alpha_0\alpha_3\beta_a\alpha_1\beta_5\alpha_4\beta_7\beta_b\beta_2\beta_1\beta_3\beta_0\alpha_5\beta_4\beta_8\alpha_2\beta_9\beta_6);$$
$$A_0 \to C_0 : (\alpha_0\beta_1\beta_2\beta_0\alpha_4\beta_3)(a1\beta_9\beta_6)(\alpha_2\beta_a\beta_4)(a3\beta_7\alpha_5)(\beta_5\beta_8\beta_b);$$
$$A_0 \to D_0 : (\alpha_0\beta_8\alpha_2\beta_0\beta_a\beta_b\beta_6\beta_5\beta_4)(\alpha_1\beta_9\beta_7a3\alpha_4\beta_3\beta_2\alpha_5\beta_1):$$
$$A_0 \to E_0 : (\alpha_0\beta_b\beta_0\beta_a\beta_8\alpha_2\beta_6\beta_3\beta_2\alpha_4\beta_5\beta_1)(\alpha_1\beta_7\alpha_3\alpha_5\beta_4)(\beta_9):$$
$$A_0 \to F_0 : (\alpha_0\beta_b\alpha_4\beta_3\beta_5\alpha_2\alpha_1\beta_9\beta_a)(\alpha_3\beta_0\beta_2\beta_1\beta_6\beta_7\beta_8\alpha_5\beta_4).$$

Additions mod 17 yield the remaining information for copies of K_4 and $L(Q_3)$ neighboring each vertex of $\mathcal{Y}$. In sum, we have the following theorem.

Theorem 6.1 *In addition to* Theorem 4.1, *the following properties of* $\mathcal{Y}$ *hold:*

(1) $\mathcal{Y}$ *is a connected union of* 102 *copies* σ *of* $L(Q_3)$, *each with an edge partition* $\Omega(\sigma)$ *into 2-paths;*

(2) *each edge in* $\mathcal{Y}$ *is shared exactly by* 4 *copies of* $L(Q_3)$ *in* $\mathcal{Y}$;

(3) *each copy* Δ *of* K_3 *(resp. each* 2*-path* $\omega \in \Omega(\sigma)$*) in a copy* σ *of* $L(Q_3)$ *in* $\mathcal{Y}$ *is shared exactly by two copies* σ, σ' *of* $L(Q_3)$ *in* $\mathcal{Y}$;

(4) *Each two copies of* $L(Q_3)$ *sharing a copy* Δ *of* K_3 *in* $\mathcal{Y}$ *share* Δ *with exactly one copy of* K_4 *in* $\mathcal{Y}$;

(5) *each* 4*-hole in* $\mathcal{Y}$ *happens in just one copy of* $L(Q_3)$ *in* $\mathcal{Y}$;

(6) $\mathcal{Y}$ *is an* Ω*-preserving* $\{L(Q_3)\}_{K_3}$-UH *graph;*

(7) $\mathcal{Y}$ *is* $\{K_4, L(Q_3)\}_{K_3}$-UH;

(8) *the vertices and copies of* $L(Q_3)$ *in* $\mathcal{Y}$ *are the points and lines of a self-dual* 1*-configuration* $(102_{12})_1$.

In Theorem 6.1(3), for each triangle Δ in σ, the copies σ, σ' of $L(Q_3)$ intersect exactly in Δ, while for each 2-path $\omega \in \Omega(\sigma)$ in σ, not only ω is shared by σ, σ', but these also share a vertex at distance 2 from the ends of ω. This common distance, 2, is realized by 2-paths in the other two colors distinct from the color of ω, in each of σ and σ', as in Figure 4, where for example the dark-gray-colored 2-path $F_4D_2B_4$ (present both in a^0 and c^3) is at distance 2 from vertex D_4 (also present in a^0 and c^3) via the black-colored path $B_4F_dD_4$ and the light-gray-colored path $F_4C_4D_4$.

Proof. It only remains to prove item (6). We explain how a monochromatic 2-path-preserving isomorphism $\Psi' : \sigma'_1 \to \sigma'_2$ between two copies of $L(Q_3)$ σ'_1, σ'_2 in $\mathcal{Y}$ extends to an automorphism of $\mathcal{S}$. Both σ'_1 and σ'_2 are colored as in Figure 4 with Ψ' respecting the color structure, thus inducing a 1-1 correspondence between the color classes of σ'_1 and σ'_2. In each copy of $L(Q_3)$ in $\mathcal{Y}$ there are exactly 12 monochromatic 2-paths, 4 in each of the 3 colors, and exactly 12 dichromatic 2-paths not contained in any triangle, a total of 24 2-paths not contained in any triangle. A $\Psi' : \sigma'_1 \to \sigma'_2$ as mentioned can be extended to an automorphism of $\mathcal{Y}$ because the information gathered in σ'_i comes via sextets from corresponding information in a subgraph $\overline{\sigma'}_i$ of $\mathcal{S}$, $(i = 1, 2)$, so that Ψ' arises from an isomorphism $\overline{\Psi'} : \overline{\sigma'}_1 \to \overline{\sigma'}_2$. However, $\overline{\sigma'}_i = \overline{\sigma}_i$, $(i = 1, 2)$, for a corresponding copy σ_i of $L(Q_3)$ in $\mathcal{Y}$, but while the vertices of σ'_i are denoted like the degree-2 vertices of $\overline{\sigma'}_i = \overline{\sigma}_i$, the vertices of σ_i are denoted like the degree-3 vertices of $\overline{\sigma}_i = \overline{\sigma'}_i$. Here the pairs (σ_i, σ'_i) are of the form (Σ^j, σ^j), where $(\Sigma, \sigma) \in \{(A, a), (B, b), (C, c), (D, d), (E, e), (F, f)\}$ and $j \in \mathbf{Z}_{17}$. Then $\overline{\Psi'} = \overline{\Psi} : \sigma_1 \to \sigma_2$ is a corresponding map as in the proof of Theorem 4.1. But now $\overline{\Psi'} = \overline{\Psi}$ extends to an automorphism of $\mathcal{S}$. This takes us to an automorphism of $\mathcal{Y}$ that extends Ψ', as claimed above.

For example, the black 2-path $B_4F_dD_4$ in the copy a^0 of $L(Q_3)$ in $\mathcal{Y}$ represented in Figure 4 arise from the 3-paths $B_4E_4F_4F_d$ and $F_dF_4E_4D_4$ in $\mathcal{S}$, which share the 2-path $F_dF_4E_4$ and differ otherwise, so their union $(B_4E_4F_4F_d) \cup (F_dF_4E_4D_4)$ is realized by a tree T_1 with just one vertex of degree 3, namely E_4, from which two 1-paths and one 2-path depart. A similar tree T_2 is obtained from the black 2-path $D_dF_4B_d$ in Figure 4. However $T_1 \bigcap T_2 = F_dF_4$, a terminal 1-path of T_i on its 2-path departing from t_i, for both $i = 1, 2$, where $t_1 = E_4$ and $t_2 = E_d$, the vertex of degree 3 in T_2. The other two black 2-paths in Figure 4 behave similarly, leading to trees T_3 and T_4 intersecting at the 1-path B_0E_0. Similar behavior holds for the dark gray and the light gray quadruples of 2-paths in Figure 4, leading to pairs of trees that intersect respectively at the 1-paths D_4D_2, B_dC_d and the 1-paths B_4C_4, D_fD_d. Thus, if σ'_1 is this copy of $L(Q_3)$ in $\mathcal{Y}$, then $\overline{\sigma'}_1$ coincides with $\overline{\sigma}_1$, where $\sigma_1 = A^0$. □

7 Using the Biggs-Smith association scheme

The 2-paths ω of Theorem 6.1(3) rearrange into an edge partition $\mathcal{I}$ of $\mathcal{Y}$ into 102 4-holes. In fact, each 4-hole in $\mathcal{I}$ is the union of 4 successive 2-paths $\omega_0, \omega_1, \omega_2, \omega_3$ from 4 respective partitions $\Omega(\sigma^0), \Omega(\sigma^1), \Omega(\sigma^2), \Omega(\sigma^3)$ of $L(Q_3)$ into 2-paths, with each two successive 2-paths ω_i, ω_{i+1} here overlapping in just one edge, (subindex addition taken mod 4).

$\mathcal{I}$ can be reconstructed by adding $r \in \mathbf{Z}_{17}$ uniformly mod 17 to all indexes in the following generating-set table of its member 4-holes, from those 4-holes shown in the left column of the table. In each line of the table, the 4 pairs of copies σ^i_j of the disconnected graph $4P_3$ shown to the right (as in (7) above) overlap at succeeding pairs of 2-paths of the 4-hole shown on their left. This is continued to its right by the citation of two vertices that alternatively are at distance 2 from the ends of those composing 2-paths:

$(A_2B_0B_1A_g)\ A_0A_1$	$(c_3^1\ e_2^b)$	$(e_2^7\ c_2^0)$	$(d_3^1\ e_3^8)$	$(e_3^a\ d_2^0)$
$(C_0A_gE_0A_1)\ A_0B_0$	$(d_2^f\ f_1^8)$	$(c_1^0\ d_1^0)$	$(d_3^2\ f_1^9)$	$(e_1^7\ e_1^a)$
$(C_4E_0C_dA_0)\ B_0C_0$	$(a_1^0\ f_1^0)$	$(f_2^9\ d_1^f)$	$(e_2^6\ e_2^b)$	$(d_1^2\ f_3^8)$
$(D_0A_0F_0C_0)\ B_0E_0$	$(c_2^g\ c_3^1)$	$(f_2^8\ f_3^9)$	$(a_2^d\ a_3^4)$	$(d_3^f\ d_2^2)$
$(C_8B_0B_4C_d)\ C_0C_4$	$(a_3^4\ e_1^a)$	$(e_1^b\ a_2^0)$	$(f_3^4\ e_3^f)$	$(e_3^6\ f_2^0)$
$(D_4D_fE_2E_0)\ D_0D_2$	$(a_2^2\ b_3^g)$	$(b_2^e\ d_3^0)$	$(d_2^2\ b_2^5)$	$(b_3^3\ a_3^0)$
$(F_0D_2B_0D_f)\ D_0E_0$	$(c_1^1\ a_2^4)$	$(a_3^0\ d_1^0)$	$(a_3^a\ c_1^g)$	$(b_1^3\ b_1^e)$
$(F_8B_0F_9D_0)\ E_0F_0$	$(c_1^0\ f_1^0)$	$(c_2^1\ a_1^d)$	$(b_2^5\ b_2^c)$	$(a_1^4\ c_3^g)$
$(E_8E_0F_gF_9)\ F_0F_8$	$(b_1^3\ f_2^8)$	$(b_3^c\ c_2^0)$	$(c_3^8\ b_3^d)$	$(f_3^0\ b_1^5)$

The vertices of each such 4-hole coincide in notation with the degree-1 vertices of a tree T in $\mathcal{S}$ isomorphic to T_0^∞, (itself present in the 4th row of this table), with the two vertices that follow each 4-hole being the vertices of degree 3 in T. These data insure that $\mathcal{Y}$ is $\mathcal{I}$-UH.

Of the 24 2-paths in a copy σ^i of $L(Q_3)$ in $\mathcal{Y}$, 12 are in the partition $\Omega(\sigma^i)$ of σ^i. The other 12 form a different edge partition $\Omega'(\sigma^i) \neq \Omega(\sigma^i)$ of σ^i. The family of 2-paths in all of the $\Omega'(\sigma^i)$ s reassembles, by means of unions of those of its members having a common degree-2 vertex, as a family $\mathcal{J}$ of 306 copies of $K_{1,4}$.

A generating-set table for $\mathcal{J}$ representing 18 copies of $K_{1,4}$ is shown subsequently, with the remaining copies of $K_{1,4}$ obtained from those 18 by uniform addition of $r \in \mathbf{Z}_{17}$ to all indexes $i \in \mathbf{Z}_{17}$ of vertices Σ_i and subgraphs σ^i_j, where $j = 1, 2, 3$ stands for black, dark gray and light gray, respectively. This generating-set table has each entry starting with a vertex Σ_0 of degree 4 in a copy of $K_{1,4}$ in $\mathcal{J}$ followed by 4 parenthesized expressions, each containing as its central entry a neighbor Σ' of Σ_0 flanked by two subgraphs σ^i_j to which the edge $\Sigma_0\Sigma'$ belongs, so that each participating σ^i appears repeated twice — with 2 different colors j, j', as σ^i_j and $\sigma^i_{j'}$ — once before a right parenthesis and once after the subsequent left parenthesis, the first of the 4 left parentheses considered subsequent to the last right parenthesis, in a mod 4 fashion:

A_0	$(c^b_3\ A_3\ d^1_2)$	$(d^1_1\ E_1\ c^1_1)$	$(c^1_2\ B_2\ e^8_3)$	$(e^8_1\ C_1\ c^b_1)$
A_0	$(f^8_3\ C_4\ d^2_1)$	$(d^2_2\ D_0\ d^f_3)$	$(d^f_1\ C_d\ f^9_2)$	$(f^9_3\ F_0\ f^8_2)$
A_0	$(d^g_3\ A_c\ e^6_3)$	$(e^6_1\ C_g\ e^9_1)$	$(e^9_2\ B_f\ c^g_3)$	$(c^g_2\ E_g\ d^g_1)$
B_0	$(e^6_1\ B_d\ a^0_3)$	$(a^0_2\ B_4\ e^b_1)$	$(e^b_3\ C_9\ f^0_3)$	$(f^0_2\ C_8\ e^6_3)$
B_0	$(e^7_3\ A_f\ d^0_3)$	$(d^0_2\ A_2\ e^a_3)$	$(e^a_2\ B_g\ c^0_3)$	$(c^0_2\ B_1\ e^7_2)$
B_0	$(a^4_2\ D_2\ c^1_1)$	$(c^1_2\ F_9\ a^d_1)$	$(a^d_3\ D_f\ c^g_1)$	$(c^g_3\ F_8\ a^4_1)$
C_0	$(d^f_3\ D_0\ d^2_2)$	$(d^2_3\ A_1\ f^9_1)$	$(f^9_3\ F_0\ f^8_2)$	$(f^8_1\ A_g\ d^f_2)$
C_0	$(e^7_2\ A_d\ e^2_2)$	$(e^2_1\ B_9\ a^d_3)$	$(a^d_1\ E_d\ f^d_1)$	$(f^d_3\ C_5\ e^7_3)$
C_0	$(d^f_1\ B_8\ a^4_2)$	$(a^4_1\ E_4\ f^4_1)$	$(f^4_2\ C_c\ e^a_3)$	$(e^a_2\ A_4\ d^f_2)$
D_0	$(b^c_1\ F_f\ b^1_1)$	$(b^1_2\ E_d\ d^f_2)$	$(d^f_1\ B_f\ a^f_1)$	$(a^f_2\ D_b\ b^c_3)$
D_0	$(a^d_1\ F_9\ c^1_2)$	$(c^1_3\ A_0\ c^g_2)$	$(c^g_3\ F_8\ a^4_1)$	$(a^4_3\ C_0\ a^d_2)$
D_0	$(b^5_3\ D_6\ a^2_3)$	$(a^2_1\ B_2\ d^2_1)$	$(d^2_3\ E_4\ b^g_2)$	$(b^g_1\ F_2\ b^5_1)$
E_0	$(a^0_2\ D_d\ b^e_2)$	$(b^e_2\ E_2\ d^0_3)$	$(d^0_2\ E_f\ b^3_2)$	$(b^3_3\ D_4\ a^0_3)$
E_0	$(b^5_3\ F_1\ c^0_3)$	$(c^0_2\ F_g\ b^c_3)$	$(b^c_1\ E_9\ f^0_2)$	$(f^0_3\ E_8\ b^5_1)$
E_0	$(f^9_1\ A_1\ d^2_3)$	$(d^2_1\ C_4\ f^8_3)$	$(f^8_1\ A_g\ d^f_2)$	$(d^f_1\ C_d\ f^9_2)$
F_0	$(c^g_2\ A_0\ c^1_3)$	$(c^1_1\ D_2\ a^4_2)$	$(a^4_3\ C_0\ a^d_2)$	$(a^d_3\ D_f\ c^g_1)$
F_0	$(b^d_2\ D_8\ b^3_2)$	$(b^3_3\ F_7\ c^8_2)$	$(c^8_1\ B_8\ f^8_1)$	$(f^8_3\ E_g\ b^d_1)$
F_0	$(f^9_2\ E_1\ b^4_1)$	$(b^4_2\ D_9\ b^e_2)$	$(b^e_3\ F_a\ c^9_3)$	$(c^9_1\ B_9\ f^9_1)$

Here, a copy of $K_{1,4}$ with degree-4 vertex Σ_i has its degree-1 vertices as those of a binary tree of $\mathcal{S}$ with depth 2 and whose root is one of the 3 neighbors of Σ_i. Thus, there are 3 such copies of $K_{1,4}$. As a result, in contrast to the fact mentioned above that $\mathcal{Y}$ is $\mathcal{I}$-UH, now any homomorphism between members of $\mathcal{J}$ preserving the order of presentation of the degree-1 vertices in corresponding copies of $K_{1,4}$, as in the table above (with the expressed parenthetical behavior with respect to the σ^i_j s), extends to an automorphism of $\mathcal{Y}$. On the other hand, each copy σ of $L(Q_3)$ in $\mathcal{Y}$ intersects 8 other copies of $L(Q_3)$ in a triangle each, and 12 other copies of $L(Q_3)$, each in a 2-path of $\Omega(\sigma)$ and one more vertex at distance 2 from the ends of the 2-path.

The graph $\mathcal{I}'$ generated by the (diagonal) chords of the 4-cycles of $\mathcal{I}$ coincides with $\mathcal{S}_2$. On the other hand, by expressing the copies of $K_{1,4}$ in $\mathcal{J}$ as $u(v)(w)(x)(y)$, (for example the copy of K_4 in the first line of the last table as $A_0(A_3)(E_1)(B_2)(C_1)$), we consider the graph $\mathcal{J}'$ generated by the corresponding 4-cycles (v, w, x, y). Then $\mathcal{J}'$ coincides with $\mathcal{S}_4$. We obtain the following final result.

Theorem 7.1 $\mathcal{Y} = \mathcal{S}_3$.

Proof. This is obtained from the Biggs-Smith association scheme, as follows. As $\mathcal{I}' = \mathcal{S}_2$ and $\mathcal{J}' = \mathcal{S}_4$, and because $\mathcal{S}$ has girth 9 and $\mathcal{Y}$ was constructed from the family $(\mathcal{C}_9)_3$ of distance-3 digraphs of directed 9-cycles in the set $\mathcal{C}_9$ of 136 directed 9-cycles in Section 3, taking into account the discussion previous to the statement, we arrive at

$$K_{102} = \mathcal{S} \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 = \mathcal{S} \cup \mathcal{I}' \cup \mathcal{Y} \cup \mathcal{J}',$$

and so $\mathcal{Y} = \mathcal{S}_3$. □

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