On Characterizations of Trees Having Large $(2, 0)$-Chromatic Numbers

Eric Andrews, Chira Lumduanhom and Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008-5248, USA

Abstract

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_2$ be a vertex coloring of G where $c(v) \neq 0$ for at least one vertex v of G. Then the coloring c induces a new coloring $\sigma : V(G) \to \mathbb{Z}_2$ of G defined by $\sigma(v) = \sum_{u \in N[v]} c(u)$ where $N[v]$ is the closed neighborhood of v and addition is performed in $\mathbb{Z}_2$. If $\sigma(v) = 0 \in \mathbb{Z}_2$ for every vertex v in G, then the coloring c is called a modular monochromatic $(2, 0)$-coloring of G. A graph G having a modular monochromatic $(2, 0)$-coloring is a monochromatic $(2, 0)$-colorable graph. The minimum number of vertices colored 1 in a modular monochromatic $(2, 0)$-coloring of G is the $(2, 0)$-chromatic number $\chi_{(2,0)}(G)$ of G. A monochromatic $(2, 0)$-colorable graph G of order n is $(2, 0)$-extremal if $\chi_{(2,0)}(G) = n$. It is known that a tree T is $(2, 0)$-extremal if and only if every vertex of T has odd degree. In this work, we characterize all trees of order n having $(2, 0)$-chromatic number $n - 1, n - 2$ or $n - 3$ and investigate the structures of connected graphs having the large $(2, 0)$-chromatic numbers.

Key Words: modular monochromatic coloring, $(2, 0)$-chromatic number, $(2, 0)$-minimal and $(2, 0)$-extremal trees.

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1 Introduction

For an integer $k \geq 2$ and a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ be a coloring where $c(v) \neq 0$ for at least one vertex v of G and adjacent vertices may be assigned the same color. Then the coloring c induces a new coloring $\sigma : V(G) \to \mathbb{Z}_k$ of the graph G defined by

$$\sigma_c(v) = \sum_{u \in N[v]} c(u) \tag{1}$$

where $N[v]$ is the closed neighborhood of v (consisting of v and the vertices in the open neighborhood $N(v)$ of v) and addition is performed in $\mathbb{Z}_k$. The number $\sigma_c(v)$ is called the *color sum of a vertex* v with respect to the coloring c. (We also write $\sigma(v)$ for $\sigma_c(v)$ if the coloring c under consideration is clear.) If $\sigma_c(u) = \sigma_c(v)$ for every two vertices u and v in G, then the coloring c is called a *(modular) monochromatic k-coloring*. For a given integer t with $0 \le t \le k-1$, a monochromatic k-coloring c of G is said to be of *type* t if the induced vertex coloring σ has the property that $\sigma(v) = t$ for each vertex v of G. Such a coloring is also referred to as a *modular monochromatic (k,t)-coloring* or simply *monochromatic (k,t)-coloring*. A graph G is *monochromatic (k,t)-colorable* or *(k,t)-colorable* if G has a monochromatic (k,t)-coloring for some integers k and t with $0 \le t \le k-1$. These concepts were introduced and studied in [1] inspired by the Lights Out Puzzle (also see [3]).

To illustrate these concepts, Figure 1 shows two vertex colorings c' and c'' of a graph G, where the color of a vertex assigned by c' or c'' is placed within the vertex and the color sum of a vertex is placed next to the vertex. The coloring c' is a monochromatic 2-coloring since $\sigma_{c'}(v) = 1$ for each $v \in V(G)$ while c'' is a monochromatic 2-coloring since $\sigma_{c''}(v) = 0$ for each $v \in V(G)$. Thus, c' is a monochromatic $(2,1)$-coloring; while c'' is a monochromatic $(2,0)$-coloring. Hence the graph G of Figure 1 is both $(2,1)$-colorable and $(2,0)$-colorable. These two examples also illustrate a useful observation.

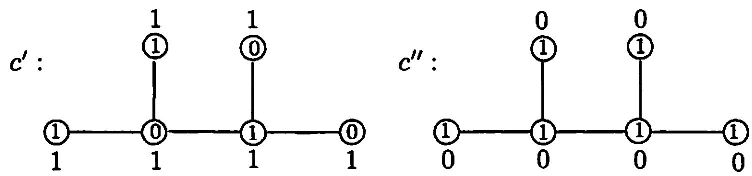

Figure 1: Illustrating monochromatic colorings

Monochromatic $(2,1)$-colorings and $(2,0)$-colorings are closely related to certain domination parameters, namely odd and even dominations in graphs (see [1, 4]). A vertex v of a graph G *dominates* a vertex u if u is in the closed neighborhood $N[v]$ of v. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by some vertex in S. A dominating set S in G is an *odd dominating set* if every vertex of G is dominated by an odd number of vertices of S. In [5] Sutner showed that every graph has an odd dominating set. As a consequence of Sutner's Theorem, it was observed in [1] that every connected graph G is $(2,1)$-colorable. A dominating set S in a graph G is an *even dominating set* if every vertex of G is dominated by an even number of vertices of S and the minimum cardinality of an even dominating set in G is the *even domination number* of G and denoted by $\gamma_e(G)$. It is known that not every graph has an even dominating set. If G is a connected graph of order n such that $\chi_{(2,0)}(G)$ and $\gamma_e(G)$ both exist, then $2 \leq \chi_{(2,0)}(G) \leq \gamma_e(G) \leq n$ and $\chi_{(2,0)}(G)$ and $\gamma_e(G)$ are even. The relationship between (2,0)-chromatic numbers and even domination numbers of graphs was studied. It was shown that (i) for each pair a, b of even integers with $2 \leq a \leq b$, there is a connected graph G such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$ and (ii) there is a connected graph G of order n such that $\chi_{(2,0)}(G) = a$ and $\gamma_e(G) = b$ if and only if $a = b$, or $a \leq \frac{b}{2}$ and $(a, b, n) \neq (2, 4, 4)$ or $\frac{b}{2} < a < b$ and $n \geq \frac{2a+b}{2}$.

A graph G is called an *odd-degree graph* if every vertex of G has odd degree and a (2,0)-colorable graph G of order n is *(2,0)-extremal* if $\chi_{(2,0)}(G) = n$. For example, the tree T of Figure 1 is an odd-degree tree and it can be shown that T is also (2,0)-extremal. In fact, this is the case for trees.

Theorem 1.1 [1] *A (2,0)-colorable tree is (2,0)-extremal if and only if T is an odd-degree tree.*

Theorem 1.1 is, however, not true for connected graphs in general. Although every (2,0)-extremal graph is odd-degree graph, there are odd-degree graphs that are not (2,0)-extremal. For example, $\chi_{(2,0)}(K_n) = 2$ for each even integer $n \geq 4$ and $\chi_{(2,0)}(P) = \chi_{(2,0)}(Q_3) = 4$ for the Petersen graph P and the cube Q_3. In this work, we characterize all trees of order n having (2,0)-chromatic number $n-1, n-2$ or $n-3$ and investigate the structures of connected graphs having the large (2,0)-chromatic numbers. We refer to the books [2] for graph-theoretical notation and terminology not described in this paper. All graphs under consideration here are nontrivial connected graphs.

2 On a Class of Trees

In order to present characterizations of trees of order n having (2,0)-chromatic number $n-1, n-2$ or $n-3$, we first study a special family of trees and establish some preliminary results. The following two observations will be useful for us, the second of which appeared in [1].

Observation 2.1 *Let G be a connected graph and let c be a modular monochromatic (2,0)-coloring of G.*

(a) *If G' is a graph obtained by attaching a connected graph at any vertex of G that is colored 0 by c, then G' is also (2,0)-colorable.*

(b) *If S is any set of vertices of G that are colored 0 by c, then the restriction of c to $G - S$ is a modular monochromatic (2,0)-coloring* *of $G - S$ (where it is possible that the restriction of c assigns the color 0 to every vertex of some component of $G - S$).*

In Observation 2.1(b), if the restriction of a monochromatic $(2,0)$-coloring of G to the subgraph $G - S$ assigns the color 0 to every vertex of some component G' of $G - S$, then this restriction is called a *trivial* monochromatic $(2,0)$-coloring of G'; Otherwise, it is a *nontrivial* monochromatic $(2,0)$-coloring of G', in which case, c assigns the color 1 to at least one vertex of G'.

Proposition 2.2 [1] *If c is a modular monochromatic $(2,0)$-coloring of a connected graph G, then the subgraph of G induced by the vertices colored 1 by c is an odd-degree graph and so the number of vertices colored 1 by c is even.*

A $(2,0)$-colorable tree T is called $(2,0)$-*minimal* if for every end-vertex v of T, the subtree $T - v$ is not $(2,0)$-colorable. For example, every $(2,0)$-colorable path, star and double star is $(2,0)$-minimal. We now describe a class of trees that are closed related to $(2,0)$-minimal trees. For a graph G, let H be a subgraph of G and let v be a vertex of G not belonging to H. The vertex v is *adjacent to H* if v is adjacent to some vertex of H. Let T' be a tree of order $k \geq 1$, where $V(T') = \{v_1, v_2, \ldots, v_k\}$ and $E(T') = \{e_1, e_2, \ldots, e_{k-1}\}$. The *subdivision graph* $S(T')$ of T' is the tree of order $2k - 1$ obtained from T' by replacing each edge e_i $(1 \leq i \leq k-1)$ by the vertex u_i which is joined to the two vertices of T' incident with e_i. A tree T is an *odd-degree subdivision tree* if the vertices $v_1, v_2, \ldots, v_k$ of some subdivision graph $S(T')$ of a tree T' of order k correspond to k pairwise disjoint odd-degree trees $T_1, T_2, \ldots, T_k$ in T and $V(T) - \cup_{i=1}^{k} V(T_i)$ consists of an independent set of $k - 1$ vertices of T each adjacent to exactly two of the trees $T_1, T_2, \ldots, T_k$. In this case, T is referred to as an *odd-degree subdivision tree with respect to odd-degree trees $T_1, T_2, \ldots, T_k$* and each of the vertices $u_1, u_2, \ldots, u_{k-1}$ is called a *subdividing vertex* of T. If F is the odd-degree forest consisting of odd-degree trees $T_1, T_2, \ldots, T_k$, then T is also referred to as an *odd-degree subdivision tree with respect to F*. In particular, if $k = 1$, then an odd-degree subdivision tree is an odd-degree tree. Among the results established on trees is the following.

Theorem 2.3 [1] *For a nontrivial tree T, the following* (1), (2) *and* (3) *are equivalent:*

(1) *T is $(2,0)$-minimal,*

(2) *T is $(2,0)$-colorable and every modular monochromatic $(2,0)$-coloring of T must assign the color 1 to each end-vertex of T,*

(3) *T is an odd-degree subdivision tree.*

The following is a consequence of Theorem 2.3.

Corollary 2.4 [1] *If T is a nontrivial tree having exactly one even vertex, then T is not $(2,0)$-colorable.*

We now determine the monochromatic $(2,0)$-coloring and the $(2,0)$-chromatic number of an arbitrary odd-degree subdivision tree with respect to an odd-degree forest.

Theorem 2.5 *Let F be an odd-degree forest. If T is an odd-degree subdivision tree with respect to F, then the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is the unique monochromatic $(2,0)$-coloring of T and $\chi_{(2,0)}(T) = |V(F)|$.*

Proof. Since the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a monochromatic $(2,0)$-coloring of T, it remains to show that this is the only monochromatic $(2,0)$-coloring of T. We proceed by induction on the number k of components of a forest. By Theorem 1.1, the result holds for $k = 1$. Suppose, for some integer $k \geq 2$, that the statement is true for all odd-degree forests having exactly $k-1$ components. Let F be an odd-degree forest of order $p \geq 2$ having exactly k components, say $T_1, T_2, \ldots, T_k$. Assume, to the contrary, that there is an odd-degree subdivision tree T with respect to F such that T has a monochromatic $(2,0)$-coloring c that assigns the color 0 to some vertex in F. Let S be the set of vertices of T colored 0 by c. We claim that S is an independent set of vertices of degree 2 in T. First, we show that S is an independent set of vertices of T; for otherwise, suppose that $uv \in E(T)$ where $c(u) = c(v) = 0$. Then $T - uv$ consists of two components Q_1 and Q_2. Note that either Q_1 or Q_2 has a vertex colored 1 by c, say the former. Since the restriction c_1 of c to Q_1 is a (nontrivial) modular monochromatic $(2,0)$-coloring of Q_1 by Observation 2.1(b), it follows that c_1 can be extended to a modular monochromatic $(2,0)$-coloring c' of T by assigning the color 0 to all vertices of Q_2. However then c' must assign the color 0 to at least one end-vertex of T, which is a contradiction by Theorem 2.3. Next we show that each vertex of S has degree 2 in T. Let $v \in S$. Since $\sigma(v) = 0$ and $c(v) = 0$, it follows that v is adjacent to an even number of vertices that are colored 1 and so $\deg_T v = d \geq 2$ is even. If $d \geq 4$, then, since T is not an odd-degree tree and $T - v$ consists of d nontrivial components, all vertices in $d-2$ of these components can be recolored 0 producing a modular monochromatic $(2,0)$-coloring in which at least $d-2$ end-vertices are colored 0, which is a contradiction by Theorem 2.3. Therefore, S is an independent set of vertices of degree 2 in T, as claimed.

Suppose that U is the set of subdividing vertices of T where then $|U| = k-1$ and $\deg_T u = 2$ for each $u \in U$. We may assume, without loss of generality, that T_1 is an end-tree of T, that is, T_1 is adjacent to exactly one vertex $u \in U$. Suppose that u is adjacent to $w \in V(T_1)$ and $x \in V(T_j)$ where $j \neq 1$, say $j = 2$. We consider two cases, according to whether $c(w) = 0$ or $c(w) = 1$.

Case 1. $c(w) = 0$. Then $c(u) = 1$ by (1) and $\deg_T w = 2$ by (2). Since $w \in V(T_1)$ and $uw \in E(T)$, it follows that w is an end-vertex of T_1, say w is adjacent w' in T_1. Hence $T_1 - w$ is a tree with exactly one even vertex (namely, w'). By Corollary 2.4 then, $T_1 - w$ is not $(2,0)$-colorable. On the other hand, since $\sigma(w) = 0$ and $c(u) = 1$, it follows that $c(w') = 1$. Because $c(w) = 0$ and w is an end-vertex of T_1, the restriction of c to $T_1 - w$ is a nontrivial monochromatic $(2,0)$-coloring of $T_1 - w$ by Observation 2.1(b). Hence $T_1 - w$ is $(2,0)$-colorable, which is a contradiction.

Case 2. $c(w) = 1$. We claim that $c(u) = 0$, for otherwise, suppose that $c(u) = 1$. Since $\sigma(u) = 0$ and $c(w) = 1$, it follows that $c(x) = 0$ and $\deg_T x = 2$ by (2). Let T^* be the tree obtained from T_1 by adding the pendant edge uw. Then T^* has exactly one even vertex (namely w) and so T^* is not $(2,0)$-colorable by Corollary 2.4. On the other hand, since $c(w) = 1$ and $c(u) = 1$, the restriction of c to T^* is a nontrivial monochromatic $(2,0)$-coloring of T^* by Observation 2.1(b), which is a contradiction. Thus, $c(u) = 0$, as claimed. By (1) then, $c(x) = c(w) = 1$ and the restriction of c to T_1 is a nontrivial monochromatic $(2,0)$-coloring of T_1. Since T_1 is an odd-degree tree, $c(z) = 1$ for all $z \in V(T_1)$ and so c does not assign the color 0 to any vertex of T_1. Hence c must assign the color 0 to some vertex in $V(F) - V(T_1)$, namely some vertex in T_i for some $i \in \{2, 3, \dots, k\}$. Let $T' = T - (\{u\} \cup V(T_1))$. Then T' is an odd-degree subdivision tree with respect to the odd-degree forest F' having exactly $k-1$ components $T_2, T_3, \dots, T_k$. By the induction hypothesis, the coloring that assigns the color 1 to each vertex of F' and the color 0 to the remaining vertices of T' is the unique monochromatic $(2,0)$-coloring of T'. On the other hand, since $c(x) = 1$ and $c(u) = 0$, the restriction c' of c to T' is a nontrivial monochromatic $(2,0)$-coloring of T'. However, c' assigns the color 0 to some vertex of F', which is a contradiction. Thus, Case 2 cannot occur.

Therefore, the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is the unique monochromatic $(2,0)$-coloring of T and so $\chi_{(2,0)}(T) = |V(F)|$. ∎

In [1], it was shown that if T is an odd-degree subdivision tree with respect to an odd-degree forest, then every monochromatic $(2,0)$-coloring of T must assign the color 1 to each end-vertices T. Hence Theorem 2.5 improves this known result.

For an odd-degree forest F, let $ods(F)$ be the set of all odd-degree subdivision trees with respect to F and let $\mathcal{P}_{ods}(F)$ be the set of trees

T such that either $T \in ods(F)$ or T is obtained from some $T_o \in ods(F)$ by adding pendant edges at one or more subdividing vertices of T_o The following observation will be useful to us.

Observation 2.6 *If uv is a pendant edge in a $(2,0)$-colorable graph G, then $c(u) = c(v)$ for every modular monochromatic $(2,0)$-coloring c of G.*

Theorem 2.7 *Let F be an odd-degree forest. If $T \in \mathcal{P}_{ods}(F)$, then the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is the unique monochromatic $(2,0)$-coloring of T and $\chi_{(2,0)}(T) = |V(F)|$.*

Proof. Since the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a monochromatic $(2,0)$-coloring of T, it remains to show that this is the only monochromatic $(2,0)$-coloring of T. We proceed by induction on the number k of components of a forest. By Theorem 1.1, the result holds for $k = 1$. Suppose, for some integer $k \geq 2$, that the statement is true for all odd-degree forests having exactly $k-1$ components. Let F be an odd-degree forest having exactly k components, say $T_1, T_2, \ldots, T_k$ and let $T \in \mathcal{P}_{ods}(F)$. We may assume, without loss of generality, that T_1 is an end-tree of T, that is, T_1 is adjacent to exactly one subdividing vertex u of T. Then either $\deg_T u = 2$ or there are ℓ pendant edges of T at u, say $uv_1, uv_2, \ldots, uv_\ell$ be the pendant edges of T at u. Suppose that u is adjacent to $w \in V(T_1)$ and $x \in V(T_j)$ where $j \neq 1$, say $j = 2$. Let c be a monochromatic $(2,0)$-coloring of T. We consider two cases.

Case 1. $c(u) = 0$. By Observation 2.6 then, $c(v_i) = 0$ for each i with $1 \leq i \leq \ell$. Let $X = \{u, v_1, v_2, \ldots, v_\ell\}$. Then $T - X$ has exactly two components, namely T_1 and $H = T - (V(T_1) \cup X)$. Since $c(u) = \sigma(u) = 0$, it follows that $c(w) = c(x)$. Furthermore, either the restriction c_{T_1} of c to T_1 or the restriction c_H of c to H is a nontrivial monochromatic $(2,0)$-coloring of T_1 or H, respectively. First, suppose that c_{T_1} is a nontrivial monochromatic $(2,0)$-coloring of T_1. Since T_1 is an odd-degree tree, $c(z) = 1$ for each $z \in V(T_1)$ by Theorem 1.1. Thus $c(w) = 1$ and $c(x) = 1$, which implies that c_H is a nontrivial monochromatic $(2,0)$-coloring of H as well. Let F' be the odd-degree forest consisting of the $k-1$ components $T_2, T_3, \ldots, T_k$. Then $H \in \mathcal{P}_{ods}(F')$. By the induction hypothesis, the coloring that assigns the color 1 to each vertex of F' and the color 0 to the remaining vertex of H is the unique monochromatic $(2,0)$-coloring of H. Hence c must assign the color 1 to at least $|V(F')|$ vertices of H. This implies that c must assign the color 1 to at least $|V(F')| + |V(T_1)| = |V(F)|$ vertices of T and so $\chi_{(2,0)}(T) = |V(F)|$. Furthermore, the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a unique monochromatic $(2,0)$-coloring of T. Next, suppose that c_H is a nontrivial monochromatic $(2,0)$-coloring of H. Again, by the induction hypothesis, $c(x) = 1$ and $c(w) = 1$. Now apply an argument similar to the one above shows that $\chi_{(2,0)}(T) = |V(F)|$ and the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a unique monochromatic $(2,0)$-coloring of T.

Case 2. $c(u) = 1$. In what follows, we show that this case is impossible. Then $c(v_i) = 1$ for $1 \leq i \leq \ell$ by Observation 2.6. Let $N_{T_1}(w) = \{w_1, w_2, \ldots, w_t\}$ be the set of neighbors of w in T_1, where t is an odd integer. Next, we consider two subcases, according to whether $c(w) = 0$ or $c(w) = 1$.

Subcase 2.1. $c(w) = 0$. Since $\sigma(w) = 0$ and $c(u) = 1$, it follows that $c(w_i) = 1$ for some $i \in \{1, 2, \ldots, t\}$, say $c(w_1) = 1$. Now $\sigma(w_1) = c(w) = 0$ implies that w_1 is not an end-vertex of T_1. Let Q_1 be the component of $T_1 - w$ that contains w_1. Then Q_1 is a nontrivial tree with exactly one even vertex (namely w_1) and so Q_1 is not $(2,0)$-colorable by Corollary 2.4. However, the restriction of c to Q_1 is a nontrivial monochromatic $(2,0)$-coloring of Q_1, which is impossible.

Subcase 2.2. $c(w) = 1$. Since $\sigma(w) = 0$, $c(u) = 1$ and $\deg_{T_1} w = t$ is odd, there is at least one $i \in \{1, 2, \ldots, t\}$ such that $c(w_i) = 0$, say $c(w_1) = 0$. Since $\sigma(w_1) = c(w_1) = 0$ and $c(w) = 1$, it follows that w_1 is adjacent to some vertex $w_{1,1}$ in T_1 for which $c(w_{1,1}) = 1$. Then $w_{1,1}$ cannot be an end-vertex of T_1. Let $Q_{1,1}$ be the component of $T_1 - w_1$ that contains $w_{1,1}$. Then $Q_{1,1}$ is a nontrivial tree with exactly one even vertex (namely $w_{1,1}$) and so $Q_{1,1}$ is not $(2,0)$-colorable. On the other hand, the restriction of c to $Q_{1,1}$ is a nontrivial monochromatic $(2,0)$-coloring of $Q_{1,1}$, which is impossible.

Hence, Case 2 is impossible and so only Case 1 can occur. Therefore, $\chi_{(2,0)}(T) = |V(F)|$ and the coloring that assigns the color 1 to each vertex of F and the color 0 to the remaining vertices of T is a unique monochromatic $(2,0)$-coloring of T. ∎

3 On Nearly $(2,0)$-Extremal Graphs

A $(2,0)$-colorable graph G of order n is said to be *nearly* $(2,0)$*-extremal* if $\chi_{(2,0)}(G) = n - 1$. Thus, if G is a nearly $(2,0)$-extremal graph of order n, then n is odd. We first characterize all nearly $(2,0)$-extremal trees. To simplify the notation, if F is an odd-degree forest having exactly k components and T is an odd-degree subdivision tree with respect to F, then T is referred to as an *ods-k-tree with respect to F* (or simply an *ods-k-tree*). Thus, each ods-k-tree has exactly $k - 1$ subdividing vertices. In particular, an ods-1-tree is an odd-degree tree and so has no subdividing vertices.

Theorem 3.1 *Let T be a $(2,0)$-colorable tree of order $n \geq 3$. Then $\chi_{(2,0)}(T) = n-1$ if and only if T is an ods-2-tree.*

Proof. First, suppose that T is a nontrivial tree of order n with $\chi_{(2,0)}(T) = n-1$. Let there be given a monochromatic $(2,0)$-coloring c of T such that $c(v) = 0$ and $c(x) = 1$ for all $x \in V(T) - \{v\}$. Then $\deg v$ is even. We claim that $\deg v = 2$. For otherwise, suppose that $\deg v = s \geq 4$ and let $N(v) = \{v_1, v_2, \ldots, v_s\}$. Let T_i $(1 \leq i \leq s)$ be the component of $T - v$ containing v_i. Then the coloring c' defined by $c'(x) = c(x)$ if $x \in V(T_1) \cup V(T_2)$ and $c'(x) = 0$ for the remaining vertices of T is a monochromatic $(2,0)$-coloring of T such that fewer than $n-1$ vertices of G are colored 1, which is impossible. Thus, $\deg v = 2$, as claimed. For each $x \neq v$ in T, since $c(x) = 1$ and $\sigma(x) = 0$, it follows that x is adjacent to an odd number of vertices colored 1. Thus $\deg v_1$ and $\deg v_2$ are even and $\deg x$ is odd for all $x \in V(T) - \{v, v_1, v_2\}$. Therefore, T_1 and T_2 are odd-degree trees.

For the converse, suppose that T consists of two disjoint odd-degree trees T_1 and T_2 and a vertex v of degree 2 that is adjacent to T_1 and T_2. Suppose that v is adjacent to $v_i \in V(T_i)$ for $i = 1, 2$. Note that the coloring that assigns the color 1 to each vertex in T_1 and T_2 and the color 0 to the vertex v is a monochromatic $(2,0)$-coloring of T, implying that $\chi_{(2,0)}(T) \leq n-1$. Assume, to the contrary, that $\chi_{(2,0)}(T) \leq n-2$. Let c be a minimum monochromatic $(2,0)$-coloring of T. Then c assigns the color 0 to at least two vertices of T and so there is $w \neq v$ such that $c(w) = 0$.

We claim that $c(v) = 0$. If this is not the case, then $c(v) = 1$. Since $\sigma(v) = 0$, either $c(v_1) = 1$ and $c(v_2) = 0$ or $c(v_1) = 0$ and $c(v_2) = 1$, say the former. Let $T' = T - V(T_2)$. Then T' is a nontrivial tree with exactly one even vertex, namely v_1. Thus T' is not $(2,0)$-colorable by Corollary 2.4. On the other hand, since $c(v_1) = 1$ and $c(v_2) = 0$, the restriction of c to T' is a monochromatic $(2,0)$-coloring of T' by Observation 2.1(b). However then, T' is $(2,0)$-colorable, which is a contradiction. Thus, as claimed, $c(v) = 0$.

By Observation 2.1(b) and Theorem 2.3, it can be shown that no two adjacent vertices can be colored 0 (see the proof of Theorem 4.6 in [1]). Hence $c(v_1) = c(v_2) = 1$. We may assume, without loss of generality, that $w \in V(T_1) - \{v_1\}$. Since $c(v) = 0$, the restriction c_1 of the coloring c to the subtree T_1 is a nontrivial monochromatic $(2,0)$-coloring of T_1 by Observation 2.1(b) and c_1 assigns the color 0 to the vertex w in T_1. This is impossible since $\chi_{(2,0)}(T_1) = |V(T_1)|$ by Theorem 1.1 and the only monochromatic $(2,0)$-coloring of T_1 must assign the color 1 to every vertex of T_1. Therefore, $\chi_{(2,0)}(T) \geq n-1$ and the result follows. ∎

As is the case of $(2,0)$-extremal trees (Theorem 1.1), Theorem 3.1 is not true for connected graphs in general. By Proposition 2.2, if G is a connected graph of order $n \geq 3$ with $\chi_{(2,0)}(G) = n-1$, then n must be odd and there is a monochromatic $(2,0)$-coloring c of G such that $c(v) = 0$ for exactly one vertex v of G and $G - v$ is an odd-degree graph. Thus $\deg_G v$ is even and so G must contain at least three even vertices. By Theorem 3.1, if T is a $(2,0)$-colorable tree of order n with $\chi_{(2,0)}(T) = n-1$, then T has exactly three even vertices. However, this is not the case for connected graphs in general. In fact, there are connected $(2,0)$-colorable graphs G of order n with $\chi_{(2,0)}(G) = n-1$ such that G has a large number of even vertices. For example, for each pair k, ℓ of integers where $k \geq 1$ and $\ell \geq 0$, let G be the graph obtained from $H = K_{2,2k}$ with partite sets $\{u, v\}$ and $X = \{x_1, x_2, \ldots, x_{2k}\}$ by adding $2\ell+1$ pendant edges vv_i for $1 \leq i \leq 2\ell+1$ at the vertex v of H. Then G has $2k+1$ even vertices (namely each vertex in $\{u\} \cup X$). The order of G is $n = 2 + 2k + 2\ell + 1$. We claim that $\chi_{(2,0)}(G) = n-1$. Since the coloring that assigns the color 0 to u and the color 1 to all vertices in $V(G) - \{u\}$ is a monochromatic $(2,0)$-coloring, $\chi_{(2,0)}(G) \leq n-1$. Let c be a minimum monochromatic $(2,0)$-coloring of G. First, suppose that $c(v) = 0$. Then $c(v_i) = 0$ for $1 \leq i \leq 2\ell+1$ by Observation 2.6. Since c must assign the color 1 to at least two vertices of G by Proposition 2.2, at least one vertex in X is colored 1 by c, say $c(x_1) = 1$. Because $\sigma(x_1) = 0$ in $\mathbb{Z}_2$, it follows that $c(u) = 1$. Now, the fact that $\sigma(x_j) = 0$ for $2 \leq j \leq 2k$ and $c(v) = 0$ implies that $c(x_j) = 1$ for each j with $2 \leq j \leq 2k$. However then, $\sigma(u) = 1$ in $\mathbb{Z}_2$, which is impossible. Next, suppose that $c(v) = 1$. Thus $c(v_i) = 1$ for $1 \leq i \leq 2\ell+1$ by Observation 2.6. If $c(u) = 0$, then $c(x_j) = 1$ for $1 \leq j \leq 2k$ and so c must assign the color 1 to $n-1$ vertices of G, as desired. Hence we may assume that $c(u) = 1$. Since $\sigma(u) = 0$, it follows that $c(x_j) = 1$ for some j with $1 \leq j \leq 2k$. However then $\sigma(x_j) = 1$ in $\mathbb{Z}_2$, which is impossible. Therefore, the coloring that assigns the color 0 to u and the color 1 to all vertices in $V(G) - \{u\}$ is the only monochromatic $(2,0)$-coloring of G and so $\chi_{(2,0)}(G) = n-1$, as claimed.

There is another possible interesting feature of connected graphs G of order $n \geq 3$ with $\chi_{(2,0)}(G) = n-1$; that is, if c is a minimum modular monochromatic $(2,0)$-coloring of G such that $c(v) = 0$ for exactly one vertex v in G, then it is possible that $\chi_{(2,0)}(G - v)$ is significantly smaller than $\chi_{(2,0)}(G)$. In fact, it can be shown that if $H = C_n \square K_2$ is the Cartesian product of the n-cycle C_n and K_2 and G is the graph of order $2n+1$ obtained from H by adding a new vertex v and joining v to two adjacent vertices on a copy of C_n in H, then $\chi_{(2,0)}(G) = 2n$ and $\chi_{(2,0)}(G - v) = n$.

4 On Trees Having Large $(2, 0)$-Chromatic Numbers

In this section, we characterize all trees of order n having $(2, 0)$-chromatic number $n - 2$ or $n - 3$.

Theorem 4.1 *Let T be a $(2, 0)$-colorable tree of order $n \geq 4$. Then $\chi_{(2,0)}(T) = n - 2$ if and only if either T is an ods-3-tree or T is obtained from an ods-2-tree T' by adding a pendant edge at the subdividing vertex of T'.*

Proof. By Theorem 2.7, it remains to show that if T is a $(2, 0)$-colorable tree of order $n \geq 4$ with $\chi_{(2,0)}(T) = n - 2$, then either T is an ods-3-tree or T is obtained from an ods-2-tree T' by adding a pendant edge at the subdividing vertex of T'. Let c be a minimum monochromatic $(2, 0)$-coloring of T such that $c(u) = c(v) = 0$ for two distinct vertices u and v in T and $c(x) = 1$ for all $x \in V(T) - \{u, v\}$. Since $\sigma(u) = \sigma(v) = 0$, each of u and v is adjacent to an even number of vertices colored 1 by c (it is possible that u or v is adjacent to no vertex colored 1). First, we claim that each of u and v is adjacent to at most two vertices colored 1 by c. For otherwise, we may assume that u is adjacent to at least four vertices colored 1 by c. Let $\{u_1, u_2, \ldots, u_s\}$ be the set of the neighbors of u colored 1 by c, where then $s \geq 4$. Now let Q_j be the component of $T - u$ that contains u_j for $1 \leq j \leq s$. Then the coloring c' defined by $c'(x) = c(x)$ if $x \in V(Q_1) \cup V(Q_2)$ and $c'(x) = 0$ otherwise is a monochromatic $(2, 0)$-coloring of T that assigns the color 1 to fewer than $n - 2$ vertices of T, which is a contradiction. Thus, as claimed, each of u and v is adjacent to at most two vertices colored 1 by c. Next, we consider two cases, according to the adjacency of u and v.

Case 1. $uv \notin E(T)$. Then every vertex adjacent to u or v is colored 1 and so $\deg_T u = \deg_T v = 2$ by the claim above. Since $uv \notin E(T)$, it follows that $T - \{u, v\}$ consists of three components, say T_1, T_2 and T_3, and each of u and v is adjacent to exactly two of T_1, T_2 and T_3. Furthermore, $F_B = T - \{u, v\}$ is an odd forest and so each T_i is an odd-degree tree for $i = 1, 2, 3$. Thus T is an ods-3-tree with respect to $T - \{u, v\}$.

Case 2. $uv \in E(T)$. Since each of u and v is adjacent to at most two vertices colored 1 by c, it follows that $1 \leq \deg_T v, \deg_T u \leq 3$. Because the order of T is at least 4, we may assume that $\deg_T u = 3$. We show that $\deg_T v = 1$, for otherwise, $T - uv$ consists of two nontrivial components Q_u and Q_v containing u and v, respectively. Then the coloring c^* defined by $c^*(x) = c(x)$ if $x \in V(Q_u)$ and $c^*(x) = 0$ otherwise is a monochromatic $(2, 0)$-coloring of T that assigns the color 1 to fewer than $n - 2$ vertices of

T, which is a contradiction. Therefore, $\deg_T v = 1$. Since $F_B = T - \{u, v\}$ is an odd-degree forest consisting of two components T_1 and T_2. Each T_i ($i = 1, 2$) is an odd-degree tree. Thus $T' = T - v$ is an ods-2-tree (with respect to $T - \{u, v\}$) and T is obtained by adding the pendant edge uv at the subdividing vertex u of T'. ■

Theorem 4.2 *Let T be a $(2,0)$-colorable tree of order $n \geq 5$. Then $\chi_{(2,0)}(T) = n - 3$ if and only if T satisfies one of the following*

(1) *T is a subdivision graph $S(K_{1,3})$ of $K_{1,3}$,*

(2) *T is an ods-4-tree,*

(3) *T is obtained from an ods-3-tree T' by adding a pendant edge at one subdividing vertex of T',*

(4) *T is obtained from an ods-2-tree T' by adding two pendant edges at the subdividing vertex of T'.*

Proof. Since $\chi_{(2,0)}(S(K_{1,3})) = 4$ and Theorem 2.7, it remains to show that if T is a $(2,0)$-colorable tree of order $n \geq 5$ with $\chi_{(2,0)}(T) = n - 3$, then T satisfies one of (1) – (4). We may assume that $T \neq S(K_{1,3})$. Let c be a minimum monochromatic $(2,0)$-coloring of T such that $c(u) = c(v) = c(w) = 0$ for three distinct vertices u, v and w in T and $c(x) = 1$ for all $x \in V(T) - \{u, v, w\}$. Since $\sigma(u) = \sigma(v) = \sigma(w) = 0$, each of u, v and w is adjacent to an even number of vertices colored 1 by c (it is possible that u or v is adjacent to no vertex colored 1). First, we claim that each of u, v and w is adjacent to at most two vertices colored 1 by c. For otherwise, we may assume that u is adjacent to at least four vertices colored 1 by c. Let $\{u_1, u_2, \ldots, u_s\}$ be the set of the neighbors of u colored 1 by c, where then $s \geq 4$. Now let Q_j be the component of $T - u$ that contains u_j for $1 \leq j \leq s$. For each j with $1 \leq j \leq s$, since $c(u) = 0$, $c(u_j) = 1$ and $\sigma(u_j) = 0$, each u_j must be adjacent to some vertex colored 1 in Q_j and so Q_j is a nontrivial tree. Thus $|V(Q_1) \cup V(Q_2)| \leq n - 5$. The coloring c' defined by $c'(x) = c(x)$ if $x \in V(Q_1) \cup V(Q_2)$ and $c'(x) = 0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most $n - 5$ vertices of T, which is a contradiction. Thus, as claimed, each of u, v and w is adjacent to at most two vertices colored 1 by c. The subtree $T[\{u, v, w\}]$ induced by $\{u, v, w\}$ is either $\overline{K}_3$, $P_2 \cup K_1$ (the union of P_2 and K_1) or P_3. We consider these three cases.

Case 1. $T[\{u, v, w\}] = \overline{K}_3$. Then each vertex in $\{u, v, w\}$ is only adjacent to vertices colored 1 by c and so $\deg_T u = \deg_T v = \deg_T w = 2$ by the claim above. Since $\{u, v, w\}$ is an independent set, the forest $F = T - \{u, v, w\}$ consists of four components, say T_1, T_2, T_3 and T_4 and each vertex in $\{u, v, w\}$ is adjacent to exactly two of T_1, T_2, T_3 and T_4. Furthermore, $F_B = F$ is an odd-degree forest and so each T_i is an odd-degree tree for $i = 1, 2, 3, 4$. Thus T is an ods-4-tree with respect to the odd-degree forest F whose subdividing vertices are u, v, w.

Case 2. $T[\{u, v, w\}] = P_2 \cup K_1$, *say* $uv \in E(T)$ *and* $uw, vw \notin E(T)$. Since each of u, v and w is adjacent to at most two vertices colored 1 by c, it follows that (i) $1 \leq \deg_T v, \deg_T u \leq 3$ and $\deg_T v$ and $\deg_T u$ are odd and (ii) $\deg_T w = 2$. Because T is connected, at least one of $\deg_T v$ and $\deg_T u$ is 3, say $\deg_T u = 3$. Next, we claim that $\deg_T v = 1$, for otherwise, $\deg_T v = 3$. Let Q_u and Q_v be the two components of $T - uv$ that contains u and v, respectively. Note that each of Q_u and Q_v contains at least five vertices. The coloring c^* defined by $c^*(x) = c(x)$ if $x \in V(Q_u)$ and $c^*(x) = 0$ otherwise is a monochromatic $(2, 0)$-coloring of T that assigns the color 1 to at most $n-6$ vertices of T, which is a contradiction. Therefore, $\deg_T v = 1$. Since (a) uv is a pendant edge of T, (b) $\deg_T u = 3$ and (c) $\deg_T w = 2$, it follows that $F_B = T - \{u, v, w\}$ is an odd-degree forest consisting of three components, say T_1, T_2 and T_3. Each T_i $(i = 1, 2, 3)$ is an odd-degree tree. Thus $T' = T - v$ is an ods-3-tree (with respect to $T - \{u, v, w\}$) having two subdividing vertices u and w and T is obtained by adding the pendant edge uv at the subdividing vertex u of T'.

Case 3. $T[\{u, v, w\}] = (v, u, w)$. Since each of u, v, w is adjacent to at most two vertices colored 1 by c, it follows that (i) $1 \leq \deg_T v, \deg_T w \leq 3$ and each of $\deg_T v$ and $\deg_T w$ is odd and (ii) $2 \leq \deg_T u \leq 4$ and $\deg_T u$ is even.

First, we claim that $\deg_T u = 4$, for otherwise, $\deg_T u = 2$. Since the order of T is at least 5, at least one $\deg_T v$ and $\deg_T w$ is 3, say $\deg_T v = 3$. This then will implies that $\deg_T w = 1$. To see this, suppose that $\deg_T w = 3$. Let Q_u and Q_v be the two components of $T - uv$ that contains u and v, respectively. Then Q_u contains at least six vertices. The coloring c^* defined by $c^*(x) = c(x)$ if $x \in V(Q_v)$ and $c^*(x) = 0$ otherwise is a monochromatic $(2, 0)$-coloring of T that assigns the color 1 to at most $n - 7$ vertices of T, which is a contradiction. Therefore, $\deg_T w = 1$. Let v_1 and v_2 be the two neighbors of v that are colored 1. Since $c(v_i) = 1$ for $i = 1, 2$, it follows that v_i is adjacent to an odd number of vertices colored 1. Let Q_1, Q_2 and Q_3 be the three components of $T - v$ where $v_i \in V(Q_i)$ for $i = 1, 2$ and $u \in V(Q_3)$. We may assume that $|V(Q_1)| \leq |V(Q_2)|$. If $T \neq S(K_{1,3})$, it follows that $|(V(Q_2)| \geq 4$. The coloring c^* defined by $c^*(x) = 1$ if $x \in V(Q_1) \cup \{u, w\}$ and $c(x) = 0$ otherwise is a monochromatic $(2, 0)$-coloring of T that assigns the color 1 to at most $n - 4$ vertices of T, which is a contradiction. Therefore, $\deg_T u = 4$, as claimed.

Next, we claim that $\deg_T v = \deg_T w = 1$. Let u_1 and u_2 be the two neighbors of u that are colored 1 by c. Furthermore, T_1, T_2, T_3, T_4 be the four components of $T-u$ where $u_i \in V(T_i)$ for $i=1,2$, $v \in V(T_3)$ and $w \in V(T_4)$. Since the restriction of c to T_i for $i=1,2$ is a nontrivial monochromatic $(2,0)$-coloring of T_i, it follows that T_i is an odd-degree tree. If $|V(T_3) \cup V(T_4)| \geq 3$, then the coloring c' defined by $c^*(x)=1$ if $x \in V(T_1) \cup V(T_2)$ and $c(x)=0$ otherwise is a monochromatic $(2,0)$-coloring of T that assigns the color 1 to at most $n-4$ vertices of T, which is a contradiction. Thus $T_3 = T_4 = K_1$ and so $\deg_T v = \deg_T w = 1$, as claimed.

Let $F = T - \{u, v, w\}$ be an odd-degree forest consisting of T_1 and T_2 and let T' be the ods-2-tree (with respect to F) with the subdividing vertex u. Then T is obtained by adding the two pendant edges uv and uw at the subdividing vertex u of T'. ■

According to Theorems 1.1 and 3.1, 4.1 and 4.2, for each $a \in \{0,1,2,3\}$, if T is a $(2,0)$-colorable tree of order n and $T \neq S(K_{1,3})$ such that $\chi_{(2,0)}(T) = n-a$, then $T \in \mathcal{P}_{ods}(F)$ for some odd-degree forest F having b components for some $b \in \{1,2,3,4\}$. This, however, is not the case if $a \geq 4$. For example, let T' be an ods-2-tree consisting of two odd-degree trees T_1, T_2, and the subdividing vertex u and let $P = (v_1, v_2, v_3)$ be a path of order 3. If T is a tree of order n obtained from T' and P by adding the edge uv_1, which is shown in Figure 2(a), or T is the tree obtained from T' and P by adding the edge uv_2, which is shown in Figure 2(b), then $T \notin \mathcal{P}_{ods}(F)$ for any odd-degree forest F. However, the only monochromatic $(2,0)$-coloring of T assigns the color 0 to each vertex in $\{u, v_1, v_2, v_3\}$ and the color 1 to the remaining vertices of T and so $\chi_{(2,0)}(T) = n-4$.

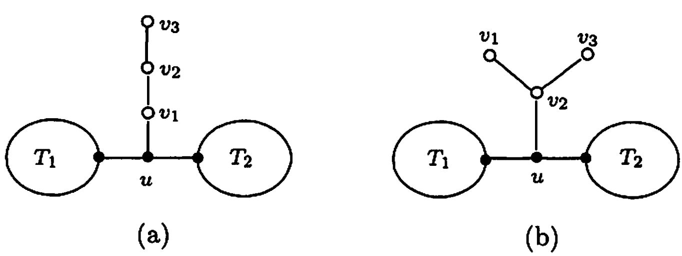

Figure 2: Trees T of order n with $\chi_{(2,0)}(T) = n-4$

It is worthwhile to mention that if T is a tree obtained from an ods-k-tree by adding four or more vertices, then $\chi_{(2,0)}(T)$ can be relatively small. For example, let T' be an ods-2-tree consisting of two odd-degree trees T_1, T_2, and the subdividing vertex u and let $H = 2P_2$ where (x_1, x_2) and (y_1, y_2) be the two copies of P_2 in H. If T is obtained from T' and H by adding two edges ux_1 and uy_1, then $\chi_{(2,0)}(T) = 4$ (see Figure 3). In fact, the coloring that assigns the color 1 to each vertex of H and the color 0 to the remaining vertices of T is a minimum monochromatic $(2,0)$-coloring of T.

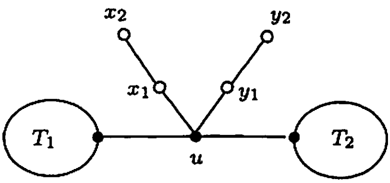

Figure 3: A tree T with $\chi_{(2,0)}(T) = 4$

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