A characterization of independent domination critical graphs with a cutvertex

N. Ananchuen,* †
Department of Mathematics, Faculty of Science,
Silpakorn University, Nakorn Pathom 73000, Thailand
Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd.,
Bangkok 10400, Thailand
e-mail: ananchuen_n@su.ac.th

W. Ananchuen,*
School of Liberal Arts, Sukhothai Thammathirat Open University,
Pakkred, Nonthaburi 11120, Thailand
e-mail: watcharaphong.ana@stou.ac.th

Abstract

Let $i(G)$ denote the minimum cardinality of an independent dominating set for G. A graph G is k-i-critical if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. In this paper, we show that if G is a connected k-i-critical graph, for $k \geq 3$, with a cutvertex u, then the number of components of $G - u$, $\omega(G - u)$, is at most $k - 1$ and there are at most two non-singleton components. Further, if $\omega(G - u) = k - 1$, then a characterization of such graphs is given.

keyword: independent domination, critical, cutvertex
MSC2000: 05C69

1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. S is independent if no two vertices of S are adjacent. The number of components of G is denoted by $\omega(G)$. A vertex v of G is a cutvertex if $\omega(G - v) > \omega(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G, denoted by $N_G(v)$, is the set of all vertices of

*This research is supported by the Thailand Research Fund grant #BRG5480014.
†Corresponding author.

$V(G) - \{v\}$ which are adjacent to v. $N_G[v]$ is denoted the closed neighborhood of v, i.e., $N_G[v] = N_G(v) \cup \{v\}$. The non-neighborhood of v in G denoted by $\overline{N}_G(v)$ is $V(G) - N_G[v]$. For $S \subseteq V(G)$, $N_G(v) \cap S$ and $N_G[v] \cap S$ are denoted by $N_S(v)$ and $N_S[v]$, respectively. Further, $N_G(S)$ denotes $\bigcup_{x \in S} N_G(x)$ and $N_G[S]$ denotes $\bigcup_{x \in S} N_G[x]$. For simplicity, if S is a subgraph of G, then we also denote $N_G(v) \cap V(S)$ and $N_G[v] \cap V(S)$ by $N_S(v)$ and $N_S[v]$, respectively.

For subsets S and T of $V(G)$, we say that S dominates T, denoted by $S \succ T$, if $T \subseteq N_G[S]$. If $S \succ T$ where $S = \{s\}$, then we write $s \succ T$ instead of $\{s\} \succ T$. Further, if $T = V(H)$ where H is a subgraph of G, then we also write $S \succ H$ instead of $S \succ V(H)$ and we say that S is a *dominating set* for H. Thus S is a dominating set for G if each vertex of $V(G)$ is either in S or adjacent to some vertex of S. The minimum cardinality of a dominating set for G is called the *domination number* of G and it is denoted by $\gamma(G)$.

For a subgraph H of G, if $S \succ H$ and S is independent, then we say that S is an *independent dominating set* for H and denote this by $S \succ_i H$. Thus S is an independent dominating set for G if $S \succ_i G$. The minimum cardinality of an independent dominating set for G is called the *independent domination number* of G and is denoted by $i(G)$. Observe that for any graph G, $\gamma(G) \leq i(G)$ and if $\gamma(G) = 1$, then $i(G) = 1$.

In 1994, Ao [4] introduced the so called concept of "independent domination critical". A graph G is *k-i-critical* if $i(G) = k$, but $i(G + uv) < k$ for any pair of non-adjacent vertices u and v of G. It is easy to see that the only 1-i-critical graphs are K_n for some positive integer n. Ao [4] proved that G is 2-i-critical if and only if $\overline{G} \cong \bigcup_{i=1}^{n} K_{1,r_i}$ for some positive integers r_i and n. The problem that arises is that of characterizing connected k-i-critical graphs for $k \geq 3$. Up to the present, there has been no characterization of such graphs. Further, there are not many known results concerning their properties especially when $k \geq 4$. This might be because the problem gets increasingly harder as k gets larger. So it makes sense to add some additional hypothesis in order to investigate connected k-i-critical graphs for $k \geq 3$. In this paper, we concentrate on connected k-i-critical graphs, for $k \geq 3$, with a cutvertex. We provide some necessary conditions for such graphs. In fact, we show that if G is a connected k-i-critical graph, for $k \geq 3$, with a cutvertex u, then $\omega(G - u) \leq k - 1$ and $G - u$ has at most two non-singleton components. Further, if $\omega(G - u) = k - 1$, then a characterization of such graphs is given.

We conclude this section by pointing out that graph domination is an area of graph theory that has applications in many other fields. Most of the applications occur in the optimal location of public facilities such as police stations and hospitals. Other areas of application include the design of communications networks and the placing of monitoring devices in electrical networks. The reader interested in further study of the application of graph domination is urged to consult [9, 10]. The study of criticality is not limited to independent domination. Research has been done researching criticality and related concepts for such domination parameters as: (ordinary) domination number, total domination number, connected domination number and others. Just as can be seen in the study of k-i-critical graphs, the study of criticality of the other domination parameters becomes progressively more difficult the higher the domination, total domination or connected domination number becomes. The reader interested in studying criticality in terms of other domination parameters should consult [1, 2, 3, 5, 6, 7, 8, 11, 12, 13, 14].

2 Preliminary results

In this section we state some results that we make use of in establishing our main results. We begin with some terminology. For a pair of non-adjacent vertices u and v of G, I_{uv} denotes a minimum independent dominating set for $G + uv$.

Lemma 2.1. *Let G be a connected k-i-critical graph and let u and v be non-adjacent vertices of G. Then $|I_{uv}| = k-1$, $|I_{uv} \cap \{u, v\}| = 1$ and $I_{uv} \cap (N_G(u) \cup N_G(v)) = \emptyset$.*

Proof. Clearly, $|I_{uv}| \leq k-1$ and $|I_{uv} \cap \{u, v\}| \leq 1$ since I_{uv} is independent. If $I_{uv} \cap \{u, v\} = \emptyset$, then I_{uv} is also an independent dominating set for G. But this contradicts the fact that $i(G) = k$. Hence, $|I_{uv} \cap \{u, v\}| = 1$. Without loss of generality, we may assume that $I_{uv} \cap \{u, v\} = \{u\}$. If there is a vertex of $I_{uv} - \{u\}$, say z, such that $zv \in E(G)$, then I_{uv} is also an independent dominating set for G, again a contradiction. Hence, $I_{uv} \cap (N_G(u) \cup N_G(v)) = \emptyset$ since $u \in I_{uv}$ and I_{uv} is independent. Finally, if $|I_{uv}| \leq k-2$, then $I_{uv} \cup \{v\}$ is an independent dominating set of size at most $k-1$ for G. This contradicts the fact that $i(G) = k$ and completes the proof of our lemma. □

The next result provides an upper bound of the diameter of connected 3-i-critical graphs.

Lemma 2.2. *[4]*
The diameter of a connected 3-i-critical graph is at most 3. □

Before we state the last result, we need one more definition. A graph G is k-i-vertex-critical if $i(G) = k$ and for each $u \in V(G)$, $i(G-u) < k$. It is easy to see that if G is k-i-vertex-critical, then $i(G-u) = k-1$.

Lemma 2.3. *[4]*

1. *A graph G is 2-i-critical if and only if $\overline{G} \cong \bigcup_{i=1}^{n} K_{1,r_i}$ for some positive integers r_i and n.*
2. *A graph G is 2-i-vertex-critical if and only if G is isomorphic to a complete graph without a perfect matching.* □

3 Classes of connected k-i-critical graphs

In this section, we provide seven classes of connected k-i-critical graphs with a cutvertex, four of them for $k = 3$, one for $k = 4$ and the another two for $k \geq 4$.

We begin with four classes of connected 3-i-critical graphs.

I. The class $\mathcal{G}_1$

For positive integers n, t and $s \geq 2$, define a graph $G \in \mathcal{G}_1$ of order $2n + 2t + s + 1$ as follows. Set $V(G) = \{u\} \cup X \cup Y \cup Z$ where $|X| = 2n$, $|Y| = 2t$ and $|Z| = s$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $G[Y] = K_{2t}$ - a perfect matching and $G[Z] = K_s$. Further, join u to each vertex of $Y \cup Z$ and finally join each vertex of Y to every vertex of X. This defines the class $\mathcal{G}_1$. Figure 1 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_1$ is 3-i-critical. Note that in our diagram a "double line" denotes the join.

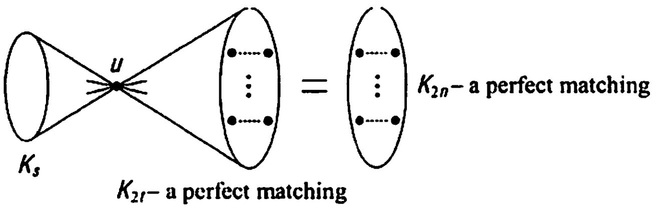

Figure 1: The structure of a graph in the class $\mathcal{G}_1$

II. The class $\mathcal{G}_2$

For positive integers n and $t \geq 2$, define a graph $G \in \mathcal{G}_2$ of order $2n + t + 2$ as follows. Set $V(G) = \{u, c\} \cup X \cup Y$ where $|X| = 2n$, $|Y| = t$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $\overline{G[Y]} = \bigcup_{i=1}^{l} K_{1,r_i}$ such that $t = l + \sum_{i=1}^{l} r_i, t \geq 2l$ and $l \geq 1$. Further, join u to each vertex of $\{c\} \cup Y$ and finally join each vertex of Y to every vertex of X. This defines the class $\mathcal{G}_2$. Figure 2 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_2$ is 3-i-critical.

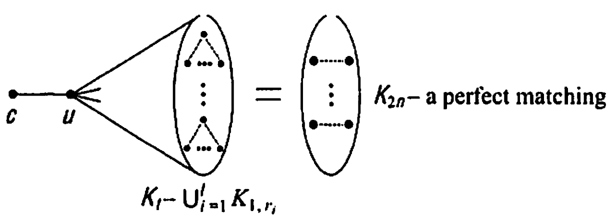

Figure 2: The structure of a graph in the class $\mathcal{G}_2$

III. The class $\mathcal{G}_3$

For positive integers n_1 and $s \geq 2n_1$ and non-negative integer n_2, define a graph $G \in \mathcal{G}_3$ of order $2n_1+2n_2+s+2$ as follows. Set $V(G) = \{u, c\} \cup X_1 \cup X_2 \cup Z$ where $|X_1| = 2n_1$, $|X_2| = 2n_2$ and $|Z| = s$. The edges of G are defined as follows. $G[X_1 \cup X_2] = K_{2n_1+2n_2}$ - $(F_1 \cup F_2)$ where F_i is a perfect matching in

$G[X_i]$ for $1 \leq i \leq 2$ and $G[Z] = K_s$. Further, join u to each vertex of $\{c\} \cup Z$ and join each vertex of Z to every vertex of X_2 (if $X_2 \neq \emptyset$). Finally, we add the set of edges E between Z and X_1. Each vertex of Z is joined to $2n_1 - 1$ vertices of X_1 in such a way that each vertex of X_1 is a non-neighbor of some vertex of Z. This defines the class $\mathcal{G}_3$. Figure 3 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_3$ is 3-i-critical.

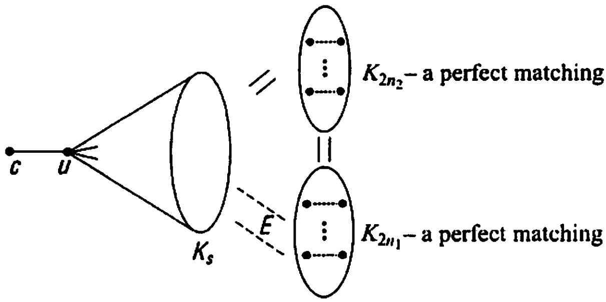

Figure 3: The structure of a graph in the class $\mathcal{G}_3$

IV. The class $\mathcal{G}_4$

For positive integers n_1, $s \geq 2n_1$ and $t \geq 2$ and non-negative integer n_2, define a graph $G \in \mathcal{G}_4$ of order $2n_1 + 2n_2 + t + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X_1 \cup X_2 \cup Y \cup Z$ where $|X_1| = 2n_1$, $|X_2| = 2n_2$, $|Y| = t$ and $|Z| = s$. The edges of G are defined as follows. $G[X_1 \cup X_2] = K_{2n_1+2n_2}$ - $(F_1 \cup F_2)$ where F_i is a perfect matching in $G[X_i]$ for $1 \leq i \leq 2$, $\overline{G[Y]} = \bigcup_{i=1}^{l} K_{1,r_i}$ such that $t = l + \sum_{i=1}^{l} r_i, t \geq 2l$ and $l \geq 1$ and $G[Z] = K_s$. Further, join u to each vertex of $\{c\} \cup Y \cup Z$, join each vertex of Y to every vertex of $Z \cup X_1 \cup X_2$ and join each vertex of Z to every vertex of X_2 (if $X_2 \neq \emptyset$). Finally, we add the set of edges E between Z and X_1 as defined in the class $\mathcal{G}_3$. This defines the class $\mathcal{G}_4$. Figure 4 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_4$ is 3-i-critical.

Our next class is a class of connected 4-i-critical graphs with a cutvertex.

V. The class $\mathcal{G}_5$

For positive integers n, t and $s \geq 2$, define a graph $G \in \mathcal{G}_5$ of order $2n + 2t + s + 2$ as follows. Set $V(G) = \{u, c\} \cup X \cup Y \cup Z$ where $|X| = 2n$, $|Y| = 2t$ and $|Z| = s$. The edges of G are defined as follows. $G[X] = K_{2n}$ - a perfect matching, $G[Y] = K_{2t}$ - a perfect matching and $G[Z] = K_s$. Further, join u to each vertex of $Y \cup \{c\}$ and $s - 1$ vertices of Z and finally join each vertex of Y to every vertex of X. This defines the class $\mathcal{G}_5$. Figure 5 illustrates our construction. It is not difficult to show that a graph $G \in \mathcal{G}_5$ is 4-i-critical.

We conclude this section by constructing two classes of connected k-i-critical

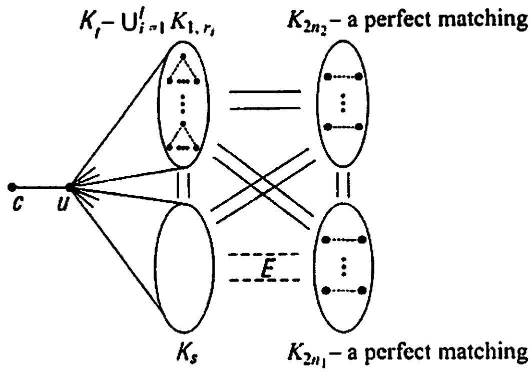

Figure 4: The structure of a graph in the class $\mathscr{G}_4$

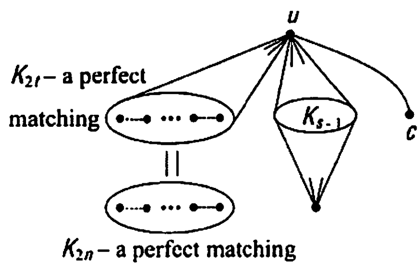

Figure 5: The structure of a graph in the class $\mathscr{G}_5$

graphs with a cutvertex for $k \geq 4$.

VI. The class $\mathscr{G}_6$

For positive integers s, n, and k where $s \geq n \geq k-1 \geq 3$, define a graph $G \in \mathscr{G}_6$ of order $s+n+k-1$ as follows. Set $V(G) = \{u\} \cup X \cup Z \cup W$ where $|X| = n$, $|Z| = s$ and $|W| = k-2$. The edges of G are defined as follows. $G[X]$ is both $(k-1)$-i-critical and $(k-1)$-i-vertex-critical, $G[Z] = K_s$ and G[W] $= \overline{K}_{k-2}$. Further, join u to each vertex of $W \cup Z$, and then add the set of edges E between Z and X as similar as defined in the class $\mathscr{G}_3$. That is, each vertex of Z is joined to $n-1$ vertices of X in such a way that each vertex of X is a non-neighbor of some vertex of Z. This defines the class $\mathscr{G}_6$. Figure 6 illustrates our construction. It is easy to see that $i(G) = k$. To show that G is k-i-critical, we have to establish I_{xy} for each pair of non-adjacent vertices x and y. We distinguish four cases according to x.

Case 1: $x \in W$.

If $y \in Z$, then $I_{xy} = (W - \{x\}) \cup \{y, \overline{y}\}$ where $\{\overline{y}\} = \overline{N}_X(y)$. Similarly, if $y \in X$, then $I_{xy} = W \cup \{\overline{y}\}$ where $\overline{y} \in \overline{N}_Z(y)$. We now assume that $y \in W - \{x\}$. Choose $z \in Z$. Then $I_{xy} = (W - \{y\}) \cup \{z, \overline{z}\}$ where $\{\overline{z}\} = \overline{N}_X(z)$.

Case 2: $x = u$.

Clearly, $y \in X$. Since $G[X]$ is $(k-1)$-i-vertex critical, there is an independent dominating set of size $k-2$, say D_y, such that D_y dominates $G[X] - y$. Then $I_{xy} = D_y \cup \{u\}$.

Case 3: $x \in Z$.

By symmetry, we may assume that $y \in X$. Then $I_{xy} = W \cup \{x\}$.

Case 4: $x \in X$.

Again, by symmetry, we may assume that $y \in X$. Since $G[X]$ is $(k-1)$-i-critical, there is an independent dominating set of size $k-2$, say D_{xy}, such that D_{xy} dominates $G[X] + xy$. Then $I_{xy} = D_{xy} \cup \{u\}$.

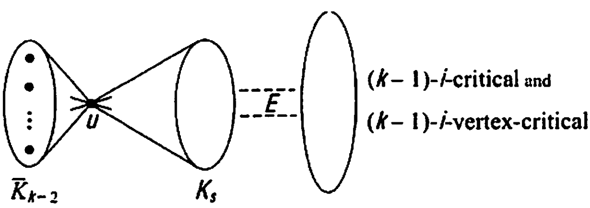

Figure 6: The structure of a graph in the class $\mathscr{G}_6$

VI. The class $\mathscr{G}_7$

For positive integers s, n, k and t where $s \geq n \geq k-1 \geq 3$ and $t \geq 2$, define a graph $G \in \mathscr{G}_7$ of order $s+n+k+t-1$ as follows. Set $V(G) = \{u\} \cup X \cup Y \cup Z \cup W$ where $|X| = n$, $|Y| = t$, $|Z| = s$ and $|W| = k-2$. The edges of G are defined as follows. $G[X]$ is both $(k-1)$-i-critical and $(k-1)$-i-vertex-critical, $\overline{G[Y]} = \bigcup_{i=1}^{l} K_{1,r_i}$ such that $t = l + \sum_{i=1}^{l} r_i, t \geq 2l$ and $l \geq 1$, $G[Z] = K_s$ and $G[W] = \overline{K}_{k-2}$. Further, join u to each vertex of $W \cup Y \cup Z$, join each vertex of Y to every vertex of $Z \cup X$. Finally, we add the set of edges E between Z and X as similar as defined in the class $\mathscr{G}_6$. This defines the class $\mathscr{G}_7$. Figure 7 illustrates our construction. Observe that $i(G) = k$. To show that G is k-i-critical, we have to establish I_{xy} for each pair of non-adjacent vertices x and y. If $\{x, y\} \cap Y = \emptyset$, then I_{xy} can be found by applying similar arguments as in the class $\mathscr{G}_6$. So we may suppose that $x \in Y$. Then $x \in V(K_{1,r_j})$ where K_{1,r_j} is an induced subgraph of $\overline{G[Y]}$ for some $1 \leq j \leq l$. It is easy to see that $y \in Y \cup W$. If $y \in Y$, then $y \in V(K_{1,r_j})$ and thus either $I_{xy} = W \cup \{x\}$ or $I_{xy} = W \cup \{y\}$. So we now consider $y \in W$. Since $x \in V(K_{1,r_j})$, there is a vertex $x_1 \in V(K_{1,r_j})$ such that $xx_1 \notin E(G)$. Then $I_{xy} = (W - \{y\}) \cup \{x, x_1\}$.

Note that for a positive integer $k \geq 2$, $\overline{K}_{k-1}$ is both $(k-1)$-i-critical and $(k-1)$-i-vertex-critical. Hence, $\mathscr{G}_6 \neq \emptyset$ and $\mathscr{G}_7 \neq \emptyset$.

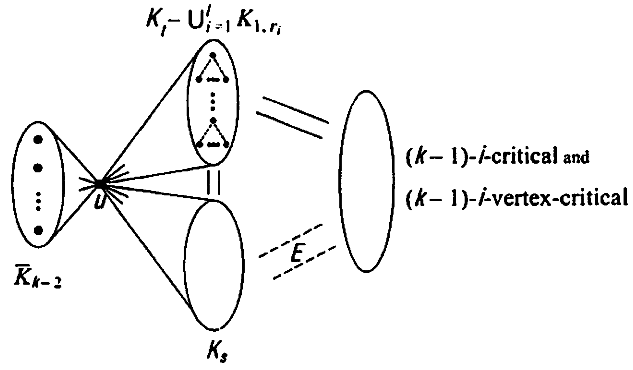

Figure 7: The structure of a graph in the class $\mathscr{G}_7$

4 The main results

Lemma 4.1. *For a positive integer $k \geq 3$, let G be a connected k-i-critical graph containing u as a cutvertex. Then*

1. *If $\omega(G-u) \geq k-1$, then $G-u$ contains at least one non-singleton and at most two non-singleton components.*

2. *$\omega(G-u) \leq k-1$.*

Proof. For $1 \leq i \leq t$, let $C_1, C_2, \ldots, C_t$ be the components of $G-u$. It is easy to see that $G-u$ must contain at least one non-singleton component since G is connected and $i(G) = k \geq 3$.

(1) Suppose to the contrary that $G-u$ contains $C_j, C_{j'}$ and $C_{j''}$ as non-singleton components where $\{j, j', j''\} \subseteq \{1, 2, \ldots, t\}$. Choose $x \in N_{C_j}(u)$ and $y \in N_{C_{j'}}(u)$. Consider $G+xy$. By Lemma 2.1, $|I_{xy}| = k-1$, $|I_{xy} \cap \{x, y\}| = 1$ and $u \notin I_{xy}$. We may suppose without loss of generality that $x \in I_{xy}$. Since $u \notin I_{xy}$, $I_{xy} \cap V(C_i) \neq \emptyset$ for $1 \leq i \leq t$ and $i \notin \{j, j'\}$. Further, $I_{xy} \cap V(C_{j'}) \neq \emptyset$ since $V(C_{j'}) - \{y\} \neq \emptyset$. Consequently, $t = k-1$ and $|I_{xy} \cap V(C_i)| = 1$ for $1 \leq i \leq t$. Thus $x \succ C_j$. Put $\{w\} = I_{xy} \cap V(C_{j''})$. Then $w \succ C_{j''}$. Now consider $G+xw$. By Lemma 2.1, $|I_{xw}| = k-1$, $|I_{xw} \cap \{x, w\}| = 1$ and $u \notin I_{xw}$. If $x \notin I_{xw}$, then $I_{xw} \cap V(C_j) = \emptyset$ since $I_{xw} \cap (N_G(x) \cup N_G(w)) = \emptyset$ and $x \succ C_j$. But then no vertex of I_{xw} dominates $V(C_j) - \{x\}$, a contradiction. Hence, $x \in I_{xw}$. By similar arguments, $w \in I_{xw}$. But this contradicts the fact that $|I_{xw} \cap \{x, w\}| = 1$. Hence, $G-u$ contains at most two non-singleton components. This proves (1).

(2) Suppose to the contrary that $\omega(G-u) = t \geq k$. By (1), we may assume that $|V(C_1)| \geq 1$ and $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq t$. Since G is connected, u is adjacent to every vertex of $\bigcup_{i=3}^{t} V(C_i)$. Now choose $a \in N_{C_1}(u)$ and $b \in N_{C_2}(u)$. Consider $G+ab$. By Lemma 2.1, $|I_{ab}| = k-1$, $|I_{ab} \cap \{a, b\}| = 1$

and $u \notin I_{ab}$. We may assume without loss of generality that $a \in I_{ab}$. Since $u \notin I_{ab}$, $I_{ab} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq t$. Then $t = k$ and $V(C_2) = \{b\}$ since $|I_{ab}| = k - 1$. Clearly, $bu \in E(G)$ since G is connected. Further, $\overline{N}_{C_1}(u) \neq \emptyset$ otherwise $u \succ G$. Let $c \in \overline{N}_{C_1}(u)$. We now consider $G + bc$. By Lemma 2.1, $|I_{bc}| = k - 1$, $|I_{bc} \cap \{b, c\}| = 1$ and $u \notin I_{bc}$. Then $I_{bc} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq t$ and $V(C_1) \cup V(C_2)$ must be dominated, in $G + bc$, by one of element in $\{b, c\}$ since $|I_{bc}| = k - 1$ and $t = k$. It is easy to see that $I_{bc} \cap \{b, c\} = \{c\}$ since $a \in V(C_1) - \{c\}$. Thus $c \succ C_1$. But then $\{c, u\} \succ_i G$, a contradiction since $i(G) = k \geq 3$. This proves (2) and completes the proof of our lemma. □

Lemma 4.2. *For a positive integer $k \geq 3$, let G be a connected k-i-critical graph containing u as a cutvertex. Suppose $C_1, C_2, \ldots, C_{k-1}$ are the components of $G - u$ where $|V(C_1)| \geq 2$, $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$. If $x \in V(C_1)$ and $y \in V(C_2)$ where $\{x, y\} \cap N_G(u) \neq \emptyset$, then $u \notin I_{xy}$, $|I_{xy} \cap V(C_1)| \geq 1$ and $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$. Further, if $|V(C_2)| \geq 2$, then $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$.*

Proof. Consider $G + xy$. By Lemma 2.1, $|I_{xy}| = k - 1$, $|I_{xy} \cap \{x, y\}| = 1$ and $u \notin I_{xy}$. Then $I_{xy} \cap V(C_i) \neq \emptyset$ for $3 \leq i \leq k - 1$. Because $|V(C_i)| = 1$ for $3 \leq i \leq k - 1$, $|I_{xy} \cap V(C_i)| = 1$ for $3 \leq i \leq k - 1$. Since $|I_{xy}| = k - 1$ and $\omega(G - u) = k - 1$, it follows that $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$. Further, $|I_{xy} \cap V(C_1)| \geq 1$ because $|V(C_1)| \geq 2$. We now suppose that $|V(C_2)| \geq 2$. If $|I_{xy} \cap V(C_1)| = 2$, then no vertex of I_{xy} dominates $C_2 - y$, a contradiction. Hence, $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$. This proves our lemma. □

Lemma 4.3. *For a positive integer $k \geq 3$, let G be a connected k-i-critical graph containing u as a cutvertex. Suppose $C_1, C_2, \ldots, C_{k-1}$ are the components of $G - u$ where $|V(C_1)| \geq 2$, $|V(C_2)| \geq 1$ and $|V(C_i)| = 1$ for $3 \leq i \leq k-1$. Then either $i(C_1) = 1$ and $i(C_2) = 2$ or $i(C_1) = 2$ and $i(C_2) = 1$. Further, if $i(C_1) = 2$ and $i(C_2) = 1$, then C_2 is complete.*

Proof. We first suppose that $|V(C_2)| = 1$. Clearly, $i(C_1) \geq 2$ otherwise $i(G) \leq k - 1$. We need only show that $i(C_1) = 2$. Put $\{y\} = V(C_2)$. Then $yu \in E(G)$. Since $i(G) = k \geq 3$, $\overline{N}_{C_1}(u) \neq \emptyset$. Let $x \in \overline{N}_{C_1}(u)$. Consider $G + xy$. By Lemma 4.2, $|I_{xy} \cap (V(C_1) \cup V(C_2))| = 2$ and $|I_{xy} \cap V(C_1)| \geq 1$. By Lemma 2.1, $|I_{xy} \cap \{x, y\}| = 1$. If $x \in I_{xy}$, then $y \notin I_{xy}$ and thus $|I_{xy} \cap V(C_1)| = 2$ since $V(C_2) = \{y\}$. If $y \in I_{xy}$, then $x \notin I_{xy}$ and thus the only vertex of $I_{xy} \cap V(C_1)$, say w, dominates $V(C_1) - \{x\}$ since $|V(C_1)| \geq 2$ and $wx \notin E(G)$. In either case, $i(C_1) = 2$.

We next suppose that $|V(C_2)| \geq 2$. Choose $a \in N_{C_1}(u)$ and $b \in N_{C_2}(u)$. Consider $G + ab$. By Lemma 4.2, $|I_{ab} \cap V(C_1)| = 1$ and $|I_{ab} \cap V(C_2)| = 1$. Put $\{a'\} = I_{ab} \cap V(C_1)$ and $\{b'\} = I_{ab} \cap V(C_2)$. If $a' = a$, then $b' \neq b$ by Lemma 2.1. Thus $a \succ C_1$ and $b' \succ C_2 - b$. Consequently, $i(C_1) = 1$ and $\{b, b'\} \succ_i C_2$. If $i(C_2) = 1$, then $i(G) \leq k - 1$, a contradiction. Hence, $i(C_2) = 2$. Similarly, if $a' \neq a$, then $b' = b$ and thus $i(C_1) = 2$ and $i(C_2) = 1$ as required.

We now suppose that $i(C_1) = 2$, $i(C_2) = 1$ and endeavor to show that C_2 is complete. In order to finish the proof of our lemma, we need the following claim.

Claim: *If $y \in N_{C_2}(u)$, then $y \succ C_2$.*

If $|V(C_2)| = 1$, then our claim follows immediately. So we may assume that $|V(C_2)| \geq 2$. Choose $x \in N_{C_1}(u)$. Consider $G + xy$. By Lemma 4.2, $|I_{xy} \cap V(C_1)| = 1$ and $|I_{xy} \cap V(C_2)| = 1$. Further, $u \notin I_{xy}$. If $I_{xy} \cap V(C_1) = \{x\}$, then $x \succ C_1$, contradicting the fact that $i(C_1) = 2$. Hence, $I_{xy} \cap V(C_1) \neq \{x\}$. Consequently, $I_{xy} \cap V(C_2) = \{y\}$ by Lemma 2.1 and thus $y \succ C_2$. This settles our claim.

We are now ready to finish our proof. Suppose to the contrary that C_2 is not complete. Then there exist non-adjacent vertices v and w of $V(C_2)$. By the above claim, $\{v, w\} \subseteq \overline{N}_{C_2}(u)$. Let $x \in N_{C_1}(u)$. Consider $G + xv$. By Lemma 4.2, $|I_{xv} \cap V(C_1)| = 1$ and $|I_{xv} \cap V(C_2)| = 1$. Further, $u \notin I_{xv}$. If $I_{xv} \cap V(C_2) = \{v\}$, then no vertex of I_{xv} dominates w, a contradiction. Hence, $I_{xv} \cap V(C_2) \neq \{v\}$. Therefore, $I_{xv} \cap V(C_1) = \{x\}$ by Lemma 2.1. Clearly, $x \succ C_1$. But this contradicts the fact that $i(C_1) = 2$. Hence, C_2 is complete as required. This completes the proof of our lemma. □

Theorem 4.4. *For a positive integer $k \geq 3$, let G be a connected k-i-critical graph containing u as a cutvertex and $\omega(G-u) = k-1$. Suppose $G-u$ contains exactly two non-singleton components, say C_1 and C_2, where $i(C_1) = 2$ and $i(C_2) = 1$. Then*

1. *$G[N_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching. Further, each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$.*
2. *$G[\overline{N}_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching.*
3. *$3 \leq k \leq 4$. Further, if $k = 3$, then G is isomorphic to a graph in the class $\mathscr{G}_1$ and if $k = 4$, then G is isomorphic to a graph in the class $\mathscr{G}_5$.*

Proof. Let W be a set of all vertices in the singleton components of $G - u$. Clearly, $|W| = k - 3 \geq 0$ since $\omega(G - u) = k - 1$. Further, $W \subseteq N_G(u)$ since G is connected. By our hypothesis that $i(C_1) = 2$ and $i(C_2) = 1$ together with Lemma 4.3, C_2 is complete. It then follows that $\overline{N}_{C_1}(u) \neq \emptyset$ since $i(G) = k \geq 3$.

Claim 1: *If $x \in N_{C_1}(u)$, $y \in V(C_2)$, then $I_{xy} \cap V(C_1) = \{x'\}$ where $x' \in N_{C_1}(u) - \{x\}$ and $I_{xy} \cap V(C_2) = \{y\}$.*

Consider $G+xy$. By Lemma 4.2, $u \notin I_{xy}$ and $|I_{xy} \cap V(C_1)| = |I_{xy} \cap V(C_2)| = 1$. Suppose to the contrary that $x \in I_{xy}$. By Lemma 2.1, $I_{xy} \cap V(C_2) = \{y'\}$ where $y' \neq y$. Then $I_{xy} \cap N_G(y) \neq \emptyset$ since C_2 is complete. But this contradicts Lemma 2.1. Hence, $x \notin I_{xy}$. Therefore, $y \in I_{xy}$ and thus $I_{xy} \cap V(C_2) = \{y\}$. Put $\{x'\} = I_{xy} \cap V(C_1)$. Clearly, $x' \neq x$ and $x' \succ C_1 - x$. Suppose to the contrary that $x' \in \overline{N}_{C_1}(u)$. Then $\{x', u\} \succ_i G - \overline{N}_{C_2}(u)$. If $\overline{N}_{C_2}(u) = \emptyset$, then

$i(G) = 2 < k$, a contradiction. Hence, $\overline{N}_{C_2}(u) \neq \emptyset$. But then $3 \le i(G) = k \le 3$ since C_2 is complete. Thus $k = 3$. Clearly, the distance between x' and y' is greater than 3 where $y' \in \overline{N}_{C_2}(u)$. But this contradicts Lemma 2.2. Hence, $x' \in N_{C_1}(u) - \{x\}$ as required. This settles our claim.

(1) Let $x \in N_{C_1}(u)$ and $y \in V(C_2)$. Consider $G + xy$. By Claim 1, $I_{xy} \cap V(C_1) = \{x'\}$ for some $x' \in N_{C_1}(u) - \{x\}$. Clearly, $x' \succ C_1 - x$. Now consider $G + x'y$. Again, by Claim 1, $I_{x'y} \cap V(C_1) = \{x''\}$ for some $x'' \in N_{C_1}(u) - \{x'\}$. Since $x' \succ C_1 - x$, $x'' = x$ by Lemma 2.1. Then $x'' = x \succ C_1 - x'$. If $|N_{C_1}(u)| = 2$, we are done. So suppose $|N_{C_1}(u)| \ge 3$. Let $w \in N_{C_1}(u) - \{x, x'\}$. Consider $G + wy$. By Claim 1, $I_{wy} \cap V(C_1) = \{w'\}$ for some $w' \in N_{C_1}(u) - \{w\}$. By Lemma 2.1, $w' \notin \{x, x'\}$ since $w \in N_G(x) \cap N_G(x')$. Clearly, $w' \succ C_1 - w$. Now consider $G + w'y$. By similar arguments, $I_{w'y} \cap V(C_1) = \{w\}$ and $w \succ C_1 - w'$. Continuing in this fashion, $G[N_{C_1}(u)]$ is isomorphic to a complete graph without a perfect matching. From our argument, it is clear also that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$. This proves (1).

Claim 2: *For $x \in \overline{N}_{C_1}(u)$ and $y \in N_{C_2}(u)$, $I_{xy} \cap V(C_1) = \{x'\}$ where $x' \in \overline{N}_{C_1}(u) - \{x\}$ and $I_{xy} \cap V(C_2) = \{y\}$.*

By similar arguments as in the proof of Claim 1 and the fact that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$, by (1), our claim follows.

(2) By applying similar arguments as in the proof of (1) together with Claim 2 and the fact that each vertex of $N_{C_1}(u)$ dominates $\overline{N}_{C_1}(u)$, by (1), (2) follows.

(3) Clearly, $u \succ N_{C_1}(u) \cup N_{C_2}(u) \cup W$. We first suppose that $\overline{N}_{C_2}(u) = \emptyset$. Then $V(C_2) \subseteq N_G(u)$ and thus $i(G) = k = 3$ since $i(G[\overline{N}_{C_1}(u)]) = 2$ by (2). Hence, G is isomorphic to the graph in the class $\mathscr{G}_1$.

We now suppose that $\overline{N}_{C_2}(u) \neq \emptyset$. Let $y \in \overline{N}_{C_2}(u)$. Since C_2 is complete, $\{u, y\} \succ_i N_{C_1}(u) \cup V(C_2) \cup W$. Thus, by (2), $3 \le i(G) = k \le 4$. By Lemma 2.2, $k \neq 3$. Hence, $k = 4$ and $|W| = 1$. We next show that $|\overline{N}_{C_2}(u)| = 1$. Suppose this is not the case. Let $y' \in \overline{N}_{C_2}(u) - \{y\}$. Consider $G + uy'$. By Lemma 2.1, $|I_{uy'}| = 3$, $|I_{uy'} \cap \{u, y'\}| = 1$. We first suppose that $I_{uy'} \cap \{u, y'\} = \{u\}$. Then $I_{uy'} \cap V(C_2) = \emptyset$ since $I_{uy'} \cap (N_G(u) \cup N_G(y')) = \emptyset$ and C_2 is complete. But then no vertex of $I_{uy'}$ dominates y, a contradiction. Hence, $I_{uy'} \cap \{u, y'\} = \{y'\}$. Then $I_{uy'} \cap W = \emptyset$ by Lemma 2.1 since the only vertex of W is adjacent to u. But then no vertex of $I_{uy'}$ dominates W, again a contradiction. Hence, $|\overline{N}_{C_2}(u)| = 1$. It follows by (1) and (2) that G is isomorphic to the graph in the class $\mathscr{G}_5$. This completes the proof of our theorem. □

Theorem 4.5. *For a positive integer $k \ge 3$, let G be a connected k-i-critical graph with a cutvertex u where $\omega(G - u) = k - 1$. Suppose $G - u$ contains exactly one non-singleton component, say C. Then*

1. *$G[\overline{N}_C(u)]$ is $(k-1)$-i-critical and $(k-1)$-i-vertex-critical.*

2. *$N_C(u)$ consists of two disjoint sets, say Y and Z, where each vertex of Y dominates $\overline{N}_C(u)$ and $Z = N_C(u) - Y$. Further, if $Y \neq \emptyset$, then $G[Y]$ is 2-i-critical and if $Z \neq \emptyset$, then $G[Z]$ is complete and for each vertex $z \in Z$, there is exactly one vertex of $\overline{N}_C(u)$, say z', such that $z \succ C - z'$. Moreover, if $k \geq 4$, each vertex of $\overline{N}_C(u)$ is not adjacent to at least one vertex of Z and thus $Z \neq \emptyset$ and $|Z| \geq |\overline{N}_C(u)|$.*

Proof. Let W be a set of all vertices in the singleton components of $G - u$. Clearly, $|W| = k - 2 \geq 1$ since $\omega(G - u) = k - 1$. Further, $W \subseteq N_G(u)$ since G is connected. Note that $N_G(u) = W \cup N_C(u)$. It then follows that $\overline{N}_C(u) \neq \emptyset$ since $i(G) = k \geq 3$. Put $X = \overline{N}_C(u)$. It is easy to see that $i(G[X]) \geq k - 1 \geq 2$. Then the following claims follow by Lemma 2.1 and the fact that $N_G(u) = W \cup N_C(u)$.

Claim 1: *For $v \in X$, $u \in I_{uv}$ and $I_{uv} - \{u\} \subseteq X - \{v\}$.*

Claim 2: *If $u \in I_{vw}$ for some $v, w \in X$, then $I_{vw} - \{u\} \subseteq X$.*

Claim 3: *If $u \notin I_{vw}$ for some $v, w \in V(C)$, then either $I_{vw} = \{v\} \cup W$ or $I_{vw} = \{w\} \cup W$.*

(1) Since $i(G[X]) \geq k-1 \geq 2$, there exist non-adjacent vertices $x_1, x_2 \in X$. We first show that $G[X]$ is $(k-1)$-i-critical. Consider $G + x_1x_2$. By Lemma 2.1, $|I_{x_1x_2}| = k - 1$ and $|I_{x_1x_2} \cap \{x_1, x_2\}| = 1$. We may assume without loss of generality that $I_{x_1x_2} \cap \{x_1, x_2\} = \{x_1\}$. We first suppose that $u \notin I_{x_1x_2}$. Then, by Claim 3, $I_{x_1x_2} = \{x_1\} \cup W$. This implies that $x_1 \succ C - x_2$ and thus $\{x_1, x_2\} \succ_i G[X]$. Consequently, $k = 3$ and $G[X]$ is 2-i-critical.

We now suppose that $u \in I_{x_1x_2}$. By our Claim 2, $I_{x_1x_2} - \{u\} \subseteq X$ and then $I_{x_1x_2} - \{u\} \succ_i X - \{x_2\}$. Hence, $(I_{x_1x_2} - \{u\}) \cup \{x_2\} \succ_i G[X]$ and thus $i(G[X]) = k - 1$. Consequently, G is $(k-1)$-i-critical.

By the above argument, $i(G[X]) = k-1$. We next show that $G[X]$ is $(k-1)$-i-vertex-critical. Let $x \in X$. Consider $G + ux$. By Lemma 2.1, $|I_{ux}| = k - 1$ and by Claim 1, $u \in I_{ux}$ and $I_{ux} - \{u\} \subseteq X - \{x\}$. Then $I_{ux} - \{u\} \succ_i X - \{x\}$. Thus $i(G[X - \{x\}]) \leq k - 2$. If $i(G[X - \{x\}]) < k - 2$, then $i(G[X]) < k - 1$, a contradiction. Hence, $G[X]$ is $(k-1)$-i-vertex-critical as required. This proves (1).

(2) We first suppose that $Z \neq \emptyset$. Let $z \in Z$. Then there exists $x \in X$ such that $zx \notin E(G)$. Consider $G + zx$. By Lemma 2.1, $u \notin I_{zx}$. By Claim 3, either $I_{zx} = \{x\} \cup W$ or $I_{zx} = \{z\} \cup W$. If $I_{zx} = \{x\} \cup W$, then $x \succ C - z$ and thus $x \succ X$. But this contradicts (1) since $k \geq 3$. Hence, $I_{zx} = \{z\} \cup W$. Then $z \succ C - x$. Consequently, $G[Z]$ is complete.

We next suppose $Y \neq \emptyset$. Note that each vertex of Y dominates $Z \cup X$. If there is a vertex $y \in Y$ such that $y \succ Y$, then $\{y\} \cup W \succ_i G$. But this contradicts the fact that $i(G) = k$. Hence, for each vertex $y \in Y$, there exists a vertex $y' \in Y - \{y\}$ such that $yy' \notin E(G)$. Consider $G + yy'$. By Lemma 2.1, $u \notin I_{yy'}$. We may assume by Claim 3 that $I_{yy'} = \{y\} \cup W$. Then $y \succ C - y'$ and thus $y \succ Y - y'$. Hence, $\{y, y'\} \succ_i G[Y]$ and $y \succ G[Y] + yy'$. Therefore, $G[Y]$ is 2-i-critical.

We now suppose that $k \geq 4$. By the definition of Y, each vertex of X is adjacent to every vertex of Y. Suppose to the contrary that there exists a vertex $x \in X$ such that x is adjacent to every vertex of Z. Then $x \succ Y \cup Z$. Let $w \in W$. Consider $G+xw$. By Lemma 2.1, $|I_{xw}| = k-1$, $|I_{xw} \cap \{x, w\}| = 1$ and $I_{xw} \cap (\{u\} \cup Y \cup Z) = \emptyset$. Since $W \subseteq N_G(u)$, $W - \{w\} \subseteq I_{xw} - \{x, w\}$. Because $|W| = k-2$ and $|I_{xw} \cap \{x, w\}| = 1$, it follows that $|I_{xw} - (\{x, w\} \cup W)| = 1$. Put $\{a\} = I_{xw} - (\{x, w\} \cup W)$. Clearly, $a \in X - \{x\}$. Consequently, $\{a, x\} \succ X$. But this contradicts the fact that $i(G[X]) = k - 1 \geq 3$. Hence, each vertex of X is not adjacent to at least one vertex of Z. It then follows that $|Z| \geq |X|$. This proves (2) and completes the proof of our theorem. □

We are now ready to establish our characterizations.

Theorem 4.6. *For a positive integer $k \geq 4$, if G is a connected k-i-critical graph with a cutvertex u where $\omega(G - u) = k - 1$, then $G \in \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$.*

Proof. By Lemma 4.1(1), $G - u$ contains at least one non-singleton and at most two non-singleton components. If $G - u$ contains exactly two non-singleton components, then $k = 4$ and G is isomorphic to a graph in the class $\mathcal{G}_5$ by Theorem 4.4(3). We now suppose that $G - u$ contains C as the only non-singleton component. Let Y and Z be defined as in Theorem 4.5. Then $Z \neq \emptyset$ and $G[Z]$ is complete. It is easy to see that, by Theorem 4.5, if $Y = \emptyset$, then G is isomorphic to a graph in the class $\mathcal{G}_6$ and if $Y \neq \emptyset$, then G is isomorphic to a graph in the class $\mathcal{G}_7$. This completes the proof of our theorem □

We now consider connected 3-i-critical graphs. Let Z and X be defined as in Theorem 4.5. Note that a graph in the class $\mathcal{G}_2$ shows that Z is empty. Further, a graph in the class $\mathcal{G}_3 \cup \mathcal{G}_4$ shows that there might exist a vertex of X which is adjacent to every vertex of Z. Our next theorem provides a characterization of connected 3-i-critical graphs containing a cutvertex.

Theorem 4.7. *If G is a connected 3-i-critical graph containing a cutvertex, then $G \in \mathcal{G}_i$, defined in Section 3, for some $1 \leq i \leq 4$.*

Proof. Let u be a cutvertex of G. By Lemma 4.1, $G - u$ contains exactly two components, at least one of them is non-singleton. If both components of $G - u$ are non-singleton, then G is isomorphic to a graph in the class $\mathcal{G}_1$ by Theorem 4.4(3). We may now suppose that $G - u$ contains exactly one non-singleton component. Let C_1 and C_2 be the components of $G - u$ where $|V(C_1)| = 1$ and $|V(C_2)| \geq 2$. Clearly, $\overline{N}_{C_2}(u) \neq \emptyset$. Put $V(C_1) = \{c\}$ and $X = \overline{N}_{C_2}(u)$. Then, by Lemma 2.3(2) and Theorem 4.5(1), $G[X]$ is isomorphic to a complete graph without a perfect matching.

By Theorem 4.5(2), $N_{C_2}(u)$ consists of two disjoint sets, say Y and Z, where each vertex of Y dominates X and $Z = N_{C_2}(u) - Y$. Further if $Y \neq \emptyset$, then $G[Y]$ is 2-i-critical and if $Z \neq \emptyset$, then $G[Z]$ is complete and for each vertex $z \in Z$, there is exactly one vertex of X, say z', such that $z \succ C_2 - z'$. Consequently, $N_G[z] = V(G) - \{c, z'\}$.

Claim 1: *Let $x_1, x_2 \in X$ where $x_1x_2 \notin E(G)$. If $x_1 \succ Z$, then $x_2 \succ Z$.*

By the definition of Y, $x_1 \succ Z \cup Y = N_{C_2}(u)$. Then $N_G[x_1] = V(C_2) - \{x_2\}$ since $G[X]$ is isomorphic to a complete graph without a perfect matching. Consider $G + cx_1$. Let $\{z\} = I_{cx_1} - \{c, x_1\}$. Then $\{c, x_1, z\}$ is independent by Lemma 2.1. Thus $z = x_2$ since $N_G[c] = \{c, u\}$ and $N_G[x_1] = V(C_2) - \{x_2\}$. If $I_{cx_1} = \{x_1, x_2\}$, then no vertex of I_{cx_1} is adjacent to u, a contradiction. Hence, $I_{cx_1} = \{c, x_2\}$. Since c is not adjacent to any vertex of C_2, $x_2 \succ C_2 - x_1$. Hence, $x_2 \succ Z$. This settles our claim.

Claim 2: *If $z \in Z$ and $zx \notin E(G)$ for some $x \in X$, then the only non-neighbor of x in X is not adjacent to some vertex of $Z - \{z\}$.*

This claim follows by Claim 1 and the fact that for each vertex $z \in Z$, there is exactly one vertex of X, say z', $N_G[z] = V(G) - \{c, z'\}$.

Let $X_1 = \{x \in X | xz \notin E(G) \text{ for some } z \in Z\}$. It is easy to see that, by the definition of Z, if $Z \neq \emptyset$, then $X_1 \neq \emptyset$. Now put $X_2 = X - X_1$. Then the following claims follow by Claims 1 and 2.

Claim 3: *For $x_1, x_2 \in X$ where $x_1x_2 \notin E(G)$, if $x_1 \in X_i$ for $1 \leq i \leq 2$, then $x_2 \in X_i$. Consequently, $|X_1|$ and $|X_2|$ are both even.*

Claim 4: *If $Z \neq \emptyset$, then $2 \leq |X_1| \leq |Z|$.*

By Lemma 2.3, $G[Y]$ is isomorphic to $K_t - \bigcup_{i=1}^{l} K_{1,r_i}$ for some positive integers t, l and r_i. Clearly, $t = l + \sum_{i=1}^{l} r_i$ and $t \geq 2l \geq 2$. It is now easy to see that if $Y \neq \emptyset$ but $Z = \emptyset$, then G is isomorphic to a graph in the class $\mathscr{G}_2$ and if $Z \neq \emptyset$ but $Y = \emptyset$, then G is isomorphic to a graph in the class $\mathscr{G}_3$. Finally, if $Y \neq \emptyset$ and $Z \neq \emptyset$, then G is isomorphic to a graph in the class $\mathscr{G}_4$. This completes the proof of our theorem. □

We conclude our paper by pointing out that Ao, in [4], gave a theorem which described 3-i-critical graphs with a cutvertex but her theorem does not depict how such graphs exactly look like while our Theorem 4.7 provides the explicit structure. Moreover, our proof is much more easier and shorter.

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions which helped to improve our paper.

References

[1] N. Ananchuen and M.D. Plummer: *Some results related to the toughness of 3-domination-critical graphs.* Discrete Math., **272**(2003), 5–15. (Erratum: **306**(2006), 292.)

[2] N. Ananchuen and M.D. Plummer: *Matching properties in domination critical graphs.* Discrete Math., **277**(2004), 1–13.

[3] N. Ananchuen and M.D. Plummer: *Some results related to the toughness of 3-domination-critical graphs II.* Utilitas Math., **70**(2006), 11–32.

[4] S. Ao: *Independent domination critical graphs.* Master Thesis, University of Victoria, 1994.

[5] X-G. Chen, L. Sun and D-X. Ma: *Connected domination critical graphs.* Appl. Math.Lett., **17**(2004), 503–507.

[6] Y. Chen and F. Tian: *A new proof of Wojcicka's conjecture.* Discrete Appl. Math., **127**(2003), 545–554.

[7] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi: *Total domination in graphs.* Networks, **10**(1980), 211–219.

[8] O. Favaron and M.A. Henning: *Total domination in claw-free graphs with minimum degree 2.* Discrete Math., **308**(2008), 3213–3219.

[9] P. Gupta: *Domination in graph with application.* Indian Journal of Research., **2**(2013), 115–117.

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater: *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[11] M.A. Henning: *A survey of selected recent results on total domination in graphs.* Discrete Math., **309**(2009), 32–63.

[12] M.A. Henning and A. Yeo: *Total domination and matching numbers in claw-free graphs.* Electron. J. Comb., **13**(2006) # R 59, 28 pp.

[13] P. Kaemawichanurat and N. Ananchuen: *On 4-γ_c-critical graphs with cutvertices.* Utilitas Math., **82**(2010), 253–268.

[14] D.P. Sumner and P. Blitch: *Domination critical graphs.* J.Combin. Theory Ser.B, **34**(1983), 65–76.