New examples of maximal partial line spreads in $\mathrm{PG}(4,q)$

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Abstract

We construct a class of maximal partial line spreads in $\mathrm{PG}(4,q)$, that we call *q-added* maximal partial line spreads. We obtain them by depriving a line spread of a hyperplane of some lines and adding $q+1$ pairwise skew lines not of the hyperplane for each removed line. We do it in a theoretical way for every value of q, and by a computer search for $q \leq 16$. More precisely we prove that for every q there are q-added MPS of size $q^2 + kq + 1$, for every integer $k = 1, \ldots, q-1$, while by a computer search we get larger cardinalities.

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1 Introduction

A *partial line spread* $\mathcal{F}$ in $\mathrm{PG}(4,q)$, the projective space of dimensions four over the Galois field $\mathrm{GF}(q)$ of order q, is a set of pairwise skew lines. We say that $\mathcal{F}$ is *maximal* if it cannot be extended to a larger partial line spread. A *line spread* in $\mathrm{PG}(3,q)$, generally in $\mathrm{PG}(n,q)$, n odd, is a set of pairwise skew lines covering the space.

Maximal partial line spreads (from now on MPS) in $\mathrm{PG}(4,q)$ have been investigated by several authors, but little is known about them (see [4], [5]). The smallest examples are the spreads in hyperplanes (of size q^2+1).

A. Beutelspacher determined the largest examples, which are of size q^3+1, and also found examples inside the interval $\left[q^2+1, q^2+q\sqrt{q}-\sqrt{q}\right]$ (see [1]). Afterwards, we only have a density result by J. Eisfeld, L. Storme and P. Sziklai [3], precisely the interval $[q^3-q+3, q^3+1]$.

Here we construct a particular class of MPS in $\mathrm{PG}(4,q)$. We do it for every value of q by using theoretical methods, and for $q \leq 16$ by a computer search.

We start from a line spread $\mathcal{F}_{\mathcal{H}}$ of a hyperplane $\mathcal{H}$ of $\mathrm{PG}(4,q)$. We begin by depriving $\mathcal{F}_{\mathcal{H}}$ of a line $r_1 \in \mathcal{H}$. Starting from $\mathcal{F}_{\mathcal{H}}\setminus\{r_1\}$ we construct a new maximal partial line spread $\mathcal{F}'_{\mathcal{H}}$, by adding $q+1$ pairwise skew lines not of $\mathcal{H}$ and covering the line r_1. Afterwards we deprive $\mathcal{F}'_{\mathcal{H}}$ of a line $r_2 \in \mathcal{H}$ and add $q+1$ pairwise skew lines not of $\mathcal{H}$, meeting r_2 and not meeting the previous added lines, obtaining a MPS $\mathcal{F}''_{\mathcal{H}}$, and so on.

We call $(\mathcal{F}_{\mathcal{H}},q)$*-added maximal partial line spread*, or briefly *q-added maximal partial line spread*, the MPS obtained in this way. We prove that this construction can be repeated $q-1$ times, for every value of q. So we get q-added MPS of size q^2+kq+1, for every integer $k=1,\dots,q-1$.

By a computer search the previous construction can be repeated for a larger number of times, for $q \le 16$. More precisely, we construct all the q-added MPS with size between q^2+q+1 and $q^2+k(q)q+1$, where $k(3)=4$, $k(4)=7$, $k(5)=10$, $k(7)=19$, $k(8)=26$, $k(9)=33$, $k(11)=46$, $k(13)=62$ and $k(16)=87$.

2 A geometric construction of the q-added maximal partial line spreads

We start by proving the following lemma.

Lemma 2.1. *In* $\mathrm{PG}(4,q)$, *q a prime power, let S be a hyperplane and X a point of S. Let $\mathcal{L}$ be a set of lines not of S, not through X, not two of them meeting outside S, and such that $|\mathcal{L}| < q^2$. Then there is a line through X not of S and skew to every line of $\mathcal{L}$.*

Proof. Through the point X there are $\theta_3-\theta_2=q^3$ (where $\theta_r=\sum_{i=0}^{r} q^i$) lines not of S and therefore having only the point X in common with S. Let L be the following point set:

$$L=\bigcup_{\ell\in\mathcal{L}} \ell - S.$$

Evidently, we have:

$$|L| = q\,|\mathcal{L}|. \tag{2.1}$$

Assume that every line through X and not of S meets some lines of $\mathcal{L}$. Since there are q^3 lines having only the point X in common with S and since such lines can meet $\bigcup_{\ell\in\mathcal{L}}\ell$ only at points of L, we have

$$|L| \ge q^3. \tag{2.2}$$

By (2.1) and (2.2) we get

$$|\mathcal{L}| \ge q^2. \tag{2.3}$$

The inequality (2.3) is a contradiction, since $|\mathcal{L}| < q^2$. The contradiction proves that there is a line through X, not of S and not meeting any line of $\mathcal{L}$. Thus the lemma is proved.

□

Now let $\mathcal{F}$ be a spread of a hyperplane S and let $r_1, r_2, \ldots, r_{q-1}$ be $q-1$ lines of $\mathcal{F}$. By using Lemma 2.1 we find $q+1$ mutually skew lines, $r_1^1, r_1^2, \ldots, r_1^{q+1}$, not of S and covering r_1. The line set

$$\mathcal{F}_1 = (\mathcal{F} - \{r_1\}) \bigcup \{r_1^1, r_1^2, \ldots, r_1^{q+1}\}$$

is a q-added MPS of size q^2+q+1. By using Lemma 2.1 we find $q+1$ mutually skew lines, $r_2^1, r_2^2, \ldots, r_2^{q+1}$ not of S, covering r_2 and not meeting $r_1^1, r_1^2, \ldots, r_1^{q+1}$. The line set

$$\mathcal{F}_2 = (\mathcal{F}_1 - \{r_2\}) \bigcup \{r_2^1, r_2^2, \ldots, r_2^{q+1}\}$$

is a q-added MPS of size q^2+2q+1. Lemma 2.1 allows us to construct q-added MPS up to the cardinality $q^2+(q-1)q+1$, by covering all the lines $r_1, r_2, \ldots, r_{q-1}$.

Consequently we have proved the following theorem.

Theorem 2.1. *In* PG$(4,q)$, *q a prime power, there are q-added maximal partial line spreads of size q^2+kq+1, for every integer $k = 1, 2, \ldots, q-1$.*

3 Computer search of q-added maximal partial line spreads in PG$(4,q)$, q a prime

In this section we give some information about the construction of q-added MPS by a computer search and about the tests to verify the correctness of the results.

3.1 The algorithm

The construction of q-added MPS proceeds similarly to the theoretical construction (see Section 2).

More precisely, the program deprives the initial spread of a line and adds a line, not of the hyperplane, skew with the remaining lines of the initial spread. At this point, the program eliminates all the lines meeting the lines of the obtained partial spread. Similarly, the program adds other q lines up to obtaining the first q-added MPS. At this point, the program deletes another line of the initial spread, and finds the second q-added MPS, and so on. The program stops when it finds a not q-added MPS.

As spreads of the hyperplane we choose the following.

In the case $q = p^h$, we use the following spread (see [6], 17.3.3).

Let $q = p^h$, with $h > 1$ and let $x^{p+1} + bx - c$ be a polynomial without roots in $\mathrm{F} = \mathrm{GF}(q)$. Then the set

$$\{\langle(1,0,0,0),(0,1,0,0)\rangle\} \cup \{\langle(z,y,1,0),(cy^p, z^p + by^p, 0, 1)\rangle \mid\mid (y,z) \in \mathrm{F}^2\}$$

is a spread of $\mathrm{PG}(3,q)$.

In the case q a prime, with $\gcd(q+1,3)=3$, we use either the spread obtained by A. A. Bruen and J. W. P. Hirschfeld and formed by tangent lines, imaginary chords and imaginary axes of a twisted cubic (see [2]), or a spread obtained by a computer search. In the case q a prime, but not with $\gcd(q+1,3)=3$, we use a spread obtained by a computer search (see [7, 8]).

3.2 Result check

We verify the correctness of our construction by several tests.

In particular, to verify the construction of the Plücker coordinates and the writing of the incidence line conditions, we write a (very simple) program that calculates, for every line ℓ, the number $n(\ell)$ of the lines meeting ℓ. We do it entirely for $q=2,3,4,5,7,8$ and partially for $q=9,11,13,16$ and we always find $n(\ell)=\left(q^3+q^2+q\right)(q+1)+1$. The number $n(\ell)$ has been calculated approximately one million of times.

Concerning the correctness of the obtained maximal partial spreads, we do the following tests.

First we check that the q^2+1 lines that we use as an initial partial spread, form a set of pairwise skew lines and so a spread, which trivially is a maximal partial spread of $\mathrm{PG}(4,q)$.

Then we test our largest q-added MPS by checking that its lines are pairwise skew and that all the added lines are not of the hyperplane.

We also test the "test program". We include some set $\mathcal{L}$ of lines of $\mathrm{PG}(4,q)$ and the program calculates the number $n(\ell)$, for each line of $\mathcal{L}$, and the number of lines of $\mathrm{PG}(4,q)$ meeting a line of $\mathcal{L}$, obtaining always correct results.

In addition to this we remark that the program never gives results against the theory. In particular, in the case $\mathrm{PG}(4,2)$, for which there is a complete characterization of the MPS, the program constructs MPS of size 7 and 9, according to the above characterization which asserts that in $\mathrm{PG}(4,2)$ the only cardinalities for the MPS are 5, 7 and 9.

4 Results

In this paper we obtain MPS $\mathcal{F}$ of cardinality q^2+kq+1, with k integer, which assumes all the values from 1 to the maximum value $k_{\max}$. In the following table we report the values of q, $k_{\max}$ and the minimum and the maximum values of $|\mathcal{F}|$.

Table 1

q	$k_{\max}$	$\lvert\mathcal{F}_{\min}\rvert = q^2 + q + 1$	$\lvert\mathcal{F}_{\max}\rvert = q^2 + k_{\max}q + 1$
3	4	13	22
4	7	21	45
5	10	31	76
7	19	57	183
8	26	73	273
9	33	91	379
11	46	133	628
13	62	183	976
16	87	273	1649

Some of the values above are already obtained by theoretic ways.

We show an example of the Plücker coordinates of a maximal partial spread, precisely the maximal partial spread of size 76 for $q = 5$. In the first group we report the remaining lines of the spread of a hyperplane (the Bruen-Hirschfeld spread in $PG(3,5)$). In the second group we report the added lines.

(1, 4, 0, 0, 0, 2, 0, 3, 0, 0) (1, 4, 1, 0, 1, 2, 0, 2, 0, 0)
(1, 4, 2, 0, 2, 2, 0, 4, 0, 0) (1, 4, 3, 0, 3, 2, 0, -1, 0, 0)
(1, 4, 4, 0, 4, 2, 0, -3, 0, 0) (1, 1, 0, 0, 0, 3, 0, 3, 0, 0)
(1, 1, 1, 0, 1, 3, 0, 2, 0, 0) (1, 1, 2, 0, 2, 3, 0, -1, 0, 0)
(1, 1, 3, 0, 3, 3, 0, -1, 0, 0) (1, 1, 4, 0, 4, 3, 0, -3, 0, 0)
(1, 3, 0, 0, 0, 4, 0, 2, 0, 0) (1, 3, 1, 0, 1, 4, 0, 1, 0, 0)
(1, 3, 2, 0, 2, 4, 0, 3, 0, 0) (1, 3, 3, 0, 3, 4, 0, 3, 0, 0)
(1, 3, 4, 0, 4, 4, 0, -4, 0, 0) (0, 0, 0, 0, 0, 0, 0, 1, 0, 0)

(1, 0, 0, 1, 0, 0, 0, 0, 0, 0) (1, 0, 1, 1, 1, 0, 0, -1, -1, 0)
(1, 2, 2, 1, 2, 0, 0, -4, -2, 0) (1, 0, 3, 1, 3, 0, 0, -4, -3, 0)
(1, 2, 4, 1, 4, 0, 0, -1, -4, 0) (1, 2, 1, 1, 0, 1, 0, 2, 0, -1)
(1, 4, 2, 1, 1, 1, 0, 2, -1, -1) (1, 2, 1, 2, 2, 1, 0, 0, -4, -2)
(1, 3, 2, 2, 3, 1, 0, -3, -1, -2) (1, 4, 0, 2, 4, 1, 0, 4, -3, -2)
(1, 0, 1, 0, 0, 0, 1, 0, 0, 1) (1, 0, 0, 0, 1, 0, 1, 0, 0, 0)
(1, 2, 0, 1, 2, 0, 1, 0, 0, 0) (1, 1, 0, 2, 3, 0, 1, 0, 0, 0)
(1, 2, 0, 3, 4, 0, 1, 0, 0, 0) (1, 0, 4, 4, 0, 1, 1, 0, 0, 0)

(1, 0, 0, 4, 1, 1, 1, 0, -4, -4)	(1, 1, 2, 1, 2, 1, 1, -3, -1, 1)
(1, 4, 3, 2, 3, 1, 1, 0, -2, 1)	(1, 4, 4, 3, 4, 1, 1, -2, -3, 1)
(1, 1, 1, 2, 0, 2, 1, 2, 1, -3)	(1, 1, 4, 4, 1, 2, 1, -2, -3, -4)
(1, 0, 2, 0, 2, 2, 1, -4, 0, 2)	(1, 1, 4, 1, 3, 2, 1, 0, -2, 2)
(1, 1, 3, 3, 4, 2, 1, 0, -1, -3)	(1, 2, 4, 4, 0, 3, 1, 1, 2, -3)
(1, 3, 1, 1, 1, 3, 1, 3, 2, -2)	(1, 4, 3, 3, 2, 3, 1, 1, -2, -1)
(1, 0, 3, 0, 3, 3, 1, -4, 0, 3)	(1, 3, 2, 3, 4, 3, 1, 1, -4, -2)
(1, 3, 4, 4, 0, 4, 1, 2, 3, -2)	(1, 2, 0, 0, 1, 4, 1, 3, 2, 0)
(1, 3, 3, 1, 2, 4, 1, 1, 1, -1)	(1, 0, 1, 2, 3, 4, 1, -3, -1, -2)
(1, 0, 4, 0, 4, 4, 1, -1, 0, 4)	(1, 2, 4, 2, 0, 0, 2, 0, 4, 3)
(1, 3, 4, 3, 1, 0, 2, -4, 3, 3)	(1, 3, 2, 1, 2, 0, 2, -4, 4, 4)
(1, 2, 1, 0, 3, 0, 2, -3, 4, 2)	(1, 0, 0, 2, 4, 0, 2, 0, -3, 0)
(1, 2, 0, 4, 0, 1, 2, 2, 4, -4)	(1, 1, 2, 2, 1, 1, 2, -1, 0, 2)
(1, 1, 1, 4, 2, 1, 2, -1, -1, -2)	(1, 2, 3, 1, 3, 1, 2, -2, 1, 0)
(1, 2, 4, 0, 4, 1, 2, -4, 4, 3)	(1, 3, 0, 3, 0, 2, 2, 1, 1, -1)
(1, 2, 3, 4, 1, 2, 2, 1, 0, -2)	(1, 0, 4, 1, 2, 2, 2, -3, -2, 1)
(1, 2, 2, 3, 3, 2, 2, -2, 0, -2)	(1, 3, 0, 1, 4, 2, 2, 1, 2, -2)
(1, 2, 2, 0, 0, 3, 2, 1, 4, 4)	(1, 0, 2, 4, 2, 3, 2, -4, -3, -3)
(1, 1, 0, 3, 3, 3, 2, 3, -2, -4)	(1, 0, 1, 3, 1, 4, 2, -1, -3, 0)
(1, 0, 2, 2, 2, 4, 2, -4, -4, -4)	(1, 2, 3, 0, 1, 0, 3, -3, 1, 4)
(1, 0, 2, 1, 3, 0, 3, -1, -3, 1)	(1, 3, 4, 1, 0, 2, 3, 1, 4, 0)
(1, 4, 2, 4, 3, 2, 3, 2, 0, -2)	(1, 4, 1, 3, 0, 4, 3, 1, 2, -4)

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