Two-Color Rado Numbers for the Equations $2x_1 + 2x_2 + c = x_3$ and $2x_1 + 2x_2 + 2x_3 + c = x_4$

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Abstract

For every integer c, let $r(2,2,c)$ be the least integer n such that for every 2-coloring of the set $\{1,2,\ldots,n\}$ there exists a monochromatic solution to the equation

$$2x_1 + 2x_2 + c = x_3.$$

Secondly, for every integer c, let $r(2,2,2,c)$ be the least integer n such that for every 2-coloring of the set $\{1,2,\ldots,n\}$ there exists a monochromatic solution to the equation

$$2x_1 + 2x_2 + 2x_3 + c = x_4.$$

In this paper, exact values are found for $r(2,2,c)$ and $r(2,2,2,c)$.

Note: The major work for this paper occurred when the second author was an undergraduate student at South Dakota State University under the direction of the first author.

Introduction

Let $\mathbb{N}$ represent the set of natural numbers and let $[a,b]$ denote the set $\{n \in \mathbb{N} \mid a \leq n \leq b\}$. A function $\Delta : [1,n] \to [0,k-1]$ is referred to as a *k-coloring* of the set $[1,n]$. Given a k-coloring Δ and a system L of linear equations in m variables, a solution $(x_1, x_2, \ldots, x_m)$ to L is *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, I. Schur [19] proved that for every $k \geq 2$, there exists a least integer $n = S(k)$ such that for every k-coloring of the set $[1,n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_3.$$

The integers $S(k)$ are called *Schur numbers*. It is known that $S(2) = 5$, $S(3) = 14$ and $S(4) = 45$, but no other Schur numbers are known [20].

In 1933, R. Rado generalized the concept of Schur numbers to arbitrary systems of linear equations. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution under every k-coloring of the natural numbers [11, 12, 13]. For a given system L of linear equations, the least integer n, provided that it exists, such that for every k-coloring of the set $[1, n]$ there exists a monochromatic solution to L is called the k-color *Rado number* (or k-color *generalized Schur number*) for the system L. If such an integer n does not exist, then the k-color Rado number for the system L is infinite. In recent years, the exact Rado numbers for several families of equations have been found [3, 7, 8, 10, 14, 15, 17, 18].

In 1982, A. Beutelspacher and W. Brestovansky [2] considered the equation

$$x_1 + x_2 + \cdots + x_{m-1} = x_m.$$

They were able to show that the 2-color Rado number for this equation is $m^2 - m - 1$ for every integer $m \geq 3$.

S.A. Burr and S. Loo [4] were able to prove that for every integer $c \geq 0$, the equation

$$x_1 + x_2 + c = x_3$$

has a 2-color Rado number equal to $4c + 5$.

More recently, these two results were taken together and generalized. For every integer $m \geq 3$, the 2-color Rado numbers for the equation

$$x_1 + x_2 + \cdots + x_{m-1} + c = x_m$$

were found for all positive integers c by D. Schaal [16] and for most negative integers c by W. Kosek and D. Schaal [9]. A. Baer, B. Mammenga and C. Spicer [1] recently found the 2-color Rado numbers for all the remaining values of c.

In 2008, S. Guo and Z.W. Sun [5] confirmed a conjecture originally presented in [6] by finding the 2-color Rado numbers for the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m$$

for every integer $m \geq 3$ and for all natural numbers $a_1, a_2, \ldots, a_{m-1}$.

In this paper, we establish the 2-color Rado numbers for the following families of equations for all integer values of c:

$$2x_1 + 2x_2 + c = x_3$$

and

$$2x_1 + 2x_2 + 2x_3 + c = x_4.$$

We use the following notation.

Definition 1. For every integer c, let $L(2,2,c)$ represent the system consisting of the single equation

$$2x_1 + 2x_2 + c = x_3$$

and let $r(2,2,c)$ represent the 2-color Rado number for $L(2,2,c)$. Also let $L(2,2,2,c)$ represent the system consisting of the single equation

$$2x_1 + 2x_2 + 2x_3 + c = x_4$$

and let $r(2,2,2,c)$ represent the 2-color Rado number for $L(2,2,2,c)$.

Main Results

Theorem 1. *For every integer c,*

$$r(2,2,c) = \begin{cases} 11c + 34 & \text{for } c \geq -3 \\ \left\lceil \frac{1-11c}{34} \right\rceil + \phi_1(c) & \text{for } c \leq -4 \end{cases}$$

where

$$\phi_1(c) = \begin{cases} 4 & \text{for } c = -4 \\ 1 & \text{for } c \in \{-14, -11, -8, -5\} \\ 0 & \text{for } c \text{ otherwise.} \end{cases}$$

Proof. Let an integer c be given. If $c = -3$, it is clear that $r(2,2,c) = 1$ since $x_1 = x_2 = x_3 = 1$ would be a solution to $L(2,2,c)$ and obviously monochromatic. We will now consider the two cases of $c > -3$ and $c < -3$.

Case 1: Assume $c > -3$.

Lower Bound: We will first show that

$$r(2,2,c) \geq 11c + 34$$

by exhibiting a 2-coloring of the interval $[1, 33+11c]$ that avoids a monochromatic solution to $L(2,2,c)$.

Let $\Delta : [1, 33 + 11c] \rightarrow [0,1]$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \leq x \leq 3 + c \\ 1 & \text{for } 4 + c \leq x \leq 15 + 5c \\ 0 & \text{for } 16 + 5c \leq x \leq 33 + 11c. \end{cases}$$

It is easy to verify that Δ avoids a monochromatic solution to $L(2,2,c)$. We may therefore conclude

$$r(2,2,c) \geq 11c + 34.$$

Upper Bound: We will next show that

$$r(2,2,c) \leq 11c + 34$$

by proving that every 2-coloring of the interval $[1, 11c+34]$ must contain a monochromatic solution to $L(2,2,c)$.

Let $\Delta : [1, 11c+34] \to [0,1]$ be an arbitrary coloring. Without loss of generality, we may assume

$$\Delta(1) = 0.$$

Because $x_1 = x_2 = 1$ and $x_3 = 4 + c$ is a solution to $L(2,2,c)$, we know that if $\Delta(4+c) = 0$, then we have a monochromatic solution to $L(2,2,c)$. Therefore we may assume

$$\Delta(4+c) = 1.$$

Now because $x_1 = x_2 = 4+c$ and $x_3 = 16+5c$ is a solution to $L(2,2,c)$, we know that if $\Delta(16+5c) = 1$, then we have a monochromatic solution to $L(2,2,c)$. Therefore we may assume

$$\Delta(16+5c) = 0.$$

Next because $x_1 = 1, x_2 = 7 + 2c$ and $x_3 = 16 + 5c$ is a solution to $L(2,2,c)$, we know that if $\Delta(7+2c) = 0$, then we have a monochromatic solution to $L(2,2,c)$. Therefore we may assume

$$\Delta(7+2c) = 1.$$

Now because $x_1 = 4 + c, x_2 = 7 + 2c$ and $x_3 = 22 + 7c$ is a solution to $L(2,2,c)$, we know that if $\Delta(22+7c) = 1$, then we have a monochromatic solution to $L(2,2,c)$. Therefore we may assume

$$\Delta(22+7c) = 0.$$

Next because $x_1 = 1, x_2 = 10 + 3c$ and $x_3 = 22 + 7c$ is a solution to $L(2,2,c)$, we know that if $\Delta(10+3c) = 0$, then we have a monochromatic solution to $L(2,2,c)$. Therefore we may assume

$$\Delta(10+3c) = 1.$$

Finally, because $x_1 = 1$, $x_2 = 16 + 5c$ and $x_3 = 11c + 34$ is a solution to $L(2,2,c)$, it is true that if $\Delta(11c+34) = 0$, then a monochromatic solution to $L(2,2,c)$ exists. Similarly, because $x_1 = 7 + 2c$, $x_2 = 10 + 3c$ and $x_3 = 11c+34$ is also a solution to $L(2,2,c)$, it is also true that if $\Delta(11c+34) = 1$, then a monochromatic solution to $L(2,2,c)$ exists. Thus for both values of $\Delta(11c+34)$ we have a monochromatic solution to $L(2,2,c)$. We may conclude

$$r(2,2,c) \leq 11c + 34.$$

Because we have previously shown $r(2,2,c) \geq 11c + 34$, we have now proven that for $c > -3$,

$$r(2,2,c) = 11c + 34.$$

Case 2: Assume $c < -3$. There exists a unique $s \in \mathbb{N}$ and a unique $t \in [0,33]$ such that $c = -34s + 3t$. Note that for every $t \in [0,33]$,

$$\left\lceil \frac{1-33t}{34} \right\rceil = \frac{1-33t}{34} + \frac{33-t}{34} = 1 - t.$$

Therefore,

$$\left\lceil \frac{1-11c}{34} \right\rceil = \left\lceil \frac{1-11(-34s+3t)}{34} \right\rceil = 11s + \left\lceil \frac{1-33t}{34} \right\rceil = 11s + 1 - t.$$

Lower Bound: The lower bound for all $c < -3$ results from the proof of the following claim.

Claim 1. *For all integers $c < -3$, $r(2,2,c) \geq \lceil \frac{1-11c}{34} \rceil$.*

Proof of Claim 1. Let an integer $c < -3$ be given and let $c = -34s + 3t$ for $s \in \mathbb{N}$ and $t \in [0,33]$. We will show that

$$r(2,2,c) \geq \left\lceil \frac{1-11c}{34} \right\rceil = 11s + 1 - t$$

by exhibiting a 2-coloring of the interval $[1, 11s-t]$ that avoids a monochromatic solution to $L(2,2,c)$.

Let $c' = s - 3$. Because $s \in \mathbb{N}$, it is clear that $c' > -3$. Therefore it is known from the proof of Case 1 that

$$r(2,2,c') = 11c' + 34 = 34 + 11(s-3) = 11s + 1.$$

Therefore there exists some coloring $\Delta' : [1, 11s] \to [0,1]$ that avoids a monochromatic solution to $L(2,2,c')$. Let $\Delta : [1, 11s-t] \to [0,1]$ be defined by

$$\Delta(x) = \Delta'(11s + 1 - t - x).$$

We will now show that Δ avoids a monochromatic solution to $L(2,2,c)$.

Let (x_1, x_2, x_3) be a solution to $L(2,2,c)$ for $x_1, x_2, x_3 \in [1, 11s - t]$. That is

$$2x_1 + 2x_2 + c = x_3.$$

For every $i \in [1,3]$, define $y_i \in [1, 11s]$ as

$$y_i = 11s + 1 - t - x_i.$$

Note that (y_1, y_2, y_3) is a solution to $L(2,2,c')$ because

$$\begin{aligned} 2y_1 + 2y_2 + c' &= 2(11s + 1 - t - x_1) + 2(11s + 1 - t - x_2) + s - 3 \\ &= 11s + 1 - t - (2x_1 + 2x_2 - 34s + 3t) \\ &= 11s + 1 - t - (2x_1 + 2x_2 + c) \\ &= 11s + 1 - t - x_3 \\ &= y_3. \end{aligned}$$

Since (y_1, y_2, y_3) is a solution to $L(2,2,c')$ and Δ' admits no monochromatic solutions to $L(2,2,c')$, we know (y_1, y_2, y_3) is not monochromatic in Δ'. That is

$$\Delta'(y_i) \neq \Delta'(y_j) \text{ for some } i, j \in [1,3] \text{ with } i \neq j.$$

Since $\Delta'(y_i) = \Delta'(11s + 1 - t - x_i) = \Delta(x_i)$, we have

$$\Delta(x_i) = \Delta'(y_i) \neq \Delta'(y_j) = \Delta(x_j).$$

Therefore the solution (x_1, x_2, x_3) to $L(2,2,c)$ is not monochromatic in Δ.

Because $\Delta : [1, 11s - t] \to [0,1]$ avoids a monochromatic solution to $L(2,2,c)$, we may conclude

$$r(2,2,c) \geq 11s + 1 - t$$

and the proof of Claim 1 is complete.

We will now need the following definition.

Definition 2. For integers $c < -3$, for natural numbers s and for $t \in [0,33]$, the set E_1 contains all $c = -34s + 3t$ for which $s = 1$ or $s = 2$.

By this definition, $E_1 = \{-68, -65, -62, \ldots, -38, -35, -34, -32, -31, -29, -28, \ldots, -5, -4\}$ and has a cardinality of 33.

Upper Bound: An upper bound for $c < -3$ and $c \notin E_1$ results from the proof of the following claim.

Claim 2. *For integers $c < -3$, if $c \notin E_1$, then $r(2,2,c) \leq \lceil \frac{1-11c}{34} \rceil$.*

Proof of Claim 2. Let an integer $c < -3$ be given such that $c \notin E_1$ and let $c = -34s + 3t$ for $s \in \mathbb{N}$ and $t \in [0,33]$. We will show that

$$r(2,2,c) \leq \left\lceil \frac{1 - 11c}{34} \right\rceil = 11s + 1 - t$$

by proving that every 2-coloring of the interval $[1, 11s + 1 - t]$ must contain a monochromatic solution to $L(2,2,c)$.

Let $c' = s - 6$. Because $c \notin E_1$, we have $s \geq 3$, and it is clear that $c' \geq -3$. Therefore it is known from the proof of Case 1 that

$$r(2,2,c') = 11c' + 34 = 34 + 11(s-6) = 11s - 32.$$

It follows that every coloring $\Delta' : [1, 11s - 32] \to [0,1]$ must contain a monochromatic solution to $L(2,2,c')$. Let $\Delta : [1, 11s + 1 - t] \to [0,1]$ be an arbitrary coloring and let $\Delta' : [1, 11s - 32] \to [0,1]$ be defined by

$$\Delta'(y) = \Delta(11s + 2 - t - y).$$

We will now show that Δ contains a monochromatic solution to $L(2,2,c)$.

Let (y_1, y_2, y_3) be a solution to $L(2,2,c')$ that is monochromatic in Δ' with $y_1, y_2, y_3 \in [1, 11s - 32]$. That is

$$2y_1 + 2y_2 + c' = y_3$$

and

$$\Delta'(y_1) = \Delta'(y_2) = \Delta'(y_3).$$

For every $i \in [1,3]$, define $x_i \in [1, 11s + 1 - t]$ as

$$x_i = 11s + 2 - t - y_i.$$

We will next show that (x_1, x_2, x_3) is a solution to $L(2,2,c)$ that is monochromatic in Δ.

Algebraically,

$$\begin{aligned}
2x_1 + 2x_2 + c &= 2(11s + 2 - t - y_1) + 2(11s + 2 - t - y_2) - 34s + 3t \\
&= 11s + 2 - t - (2y_1 + 2y_2 + s - 6) \\
&= 11s + 2 - t - (2y_1 + 2y_2 + c') \\
&= 11s + 2 - t - y_3 \\
&= x_3.
\end{aligned}$$

Therefore (x_1, x_2, x_3) is a solution to $L(2,2,c)$. Also for $i, j \in [1,3]$,

$$\Delta(x_i) = \Delta(11s+2-t-y_i) = \Delta'(y_i) = \Delta'(y_j) = \Delta(11s+2-t-y_j) = \Delta(x_j),$$

so (x_1, x_2, x_3) is monochromatic in Δ.

Because $\Delta : [1, 11s + 1 - t] \to [0,1]$ contains a monochromatic solution to $L(2,2,c)$, we may conclude

$$r(2,2,c) \leq 11s + 1 - t$$

and the proof of Claim 2 is complete.

Because we previously showed $r(2,2,c) \geq 11s+1-t$ for all $c < -3$, we have now proven that for $c < -3$ when $c \notin E_1$,

$$r(2,2,c) = 11s + 1 - t = \left\lceil \frac{1-11c}{34} \right\rceil.$$

For integers $c < -3$ when $c \in E_1$, the exact value of $r(2,2,c)$ has been determined by a computer program. The computer results are displayed in the table below. The function $\phi_1 : (-\infty, -4] \to \mathbb{Z}$ is the difference between $r(2,2,c)$ and the lower bound determined by Claim 1. The integers $c \in E_1$ for which $\phi_1(c) = 0$ have been omitted from the table.

c	$r(2,2,c)$	$\lceil \frac{1-11c}{34} \rceil$	$\phi_1(c)$
-4	6	2	4
-5	3	2	1
-8	4	3	1
-11	5	4	1
-14	6	5	1

The proof of Theorem 1 is complete. □

Theorem 2. *For every integer c,*

$$r(2,2,2,c) = \begin{cases} 15c+76 & \text{for } c \geq -5 \\ \left\lceil \frac{1-15c}{76} \right\rceil + \phi_2(c) & \text{for } c \leq -6 \end{cases}$$

where

$$\phi_2(c) = \begin{cases} 8 & \text{for } c = -6 \\ 5 & \text{for } c = -7 \\ 3 & \text{for } c = -12 \\ 2 & \text{for } c \in \{-18, -13, -8\} \\ 1 & \text{for } c \in \{-44, -39, -34, -29, -28, -24, -23, -19, -14, -9\} \\ 0 & \text{for } c \text{ otherwise.} \end{cases}$$

Proof. Let an integer c be given. If $c = -5$, it is clear that $r(2,2,2,c) = 1$ since $x_1 = x_2 = x_3 = x_4 = 1$ would be a solution to $L(2,2,2,c)$ and obviously monochromatic. We will now consider the two cases of $c > -5$ and $c < -5$.

Case 1: Assume $c > -5$.

Lower Bound: We will first show that

$$r(2,2,2,c) \geq 15c + 76$$

by exhibiting a 2-coloring of the interval $[1, 75+15c]$ that avoids a monochromatic solution to $L(2,2,2,c)$.

Let $\Delta : [1, 75+15c] \to [0,1]$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{for } 1 \le x \le 5+c \\ 1 & \text{for } 6+c \le x \le 35+7c \\ 0 & \text{for } 36+7c \le x \le 75+15c. \end{cases}$$

It is easy to verify that Δ avoids a monochromatic solution to $L(2,2,2,c)$. We may therefore conclude

$$r(2,2,2,c) \ge 15c + 76.$$

Upper Bound: We will next show that

$$r(2,2,2,c) \le 15c + 76$$

by proving that every 2-coloring of the interval $[1, 15c+76]$ must contain a monochromatic solution to $L(2,2,2,c)$.

Let $\Delta : [1, 15c+76] \to [0,1]$ be an arbitrary coloring. Without loss of generality, we may assume

$$\Delta(1) = 0.$$

Because $x_1 = x_2 = x_3 = 1$ and $x_4 = 6+c$ is a solution to $L(2,2,2,c)$, we know that if $\Delta(6+c) = 0$, then we have a monochromatic solution to $L(2,2,2,c)$. Therefore we may assume

$$\Delta(6+c) = 1.$$

Now because $x_1 = x_2 = x_3 = 6+c$ and $x_4 = 36+7c$ is a solution to $L(2,2,2,c)$, we know that if $\Delta(36+7c) = 1$, then we have a monochromatic solution to $L(2,2,2,c)$. Therefore we may assume

$$\Delta(36+7c) = 0.$$

Next because $x_1 = x_2 = 1$, $x_3 = 16+3c$ and $x_4 = 36+7c$ is a solution to $L(2,2,2,c)$, we know that if $\Delta(16+3c) = 0$, then we have a monochromatic solution to $L(2,2,2,c)$. Therefore we may assume

$$\Delta(16+3c) = 1.$$

Finally, because $x_1 = x_2 = 1$, $x_3 = 36+7c$ and $x_4 = 15c+76$ is a solution to $L(2,2,2,c)$, it is true that if $\Delta(15c+76) = 0$, then a monochromatic solution to $L(2,2,2,c)$ exists. Similarly, because $x_1 = 6+c$, $x_2 = x_3 = 16+3c$ and $x_4 = 15c+76$ is also a solution to $L(2,2,2,c)$, it is also true that if $\Delta(15c+76) = 1$, then a monochromatic solution to $L(2,2,2,c)$ exists.

Thus for both values of $\Delta(15c+76)$ we have a monochromatic solution to $L(2,2,2,c)$. We may conclude

$$r(2,2,2,c) \leq 15c+76.$$

Because we have previously shown $r(2,2,2,c) \geq 15c+76$, we have now proven that for $c > -5$,

$$r(2,2,2,c) = 15c+76.$$

Though Case 1 of Theorem 2 is significantly different than Case 1 of Theorem 1, Case 2 of Theorem 2 is very similar to Case 2 of Theorem 1. For this reason, we will omit many of the details in the proof of Case 2 of Theorem 2.

Case 2: Assume $c < -5$. There exists a unique $s \in \mathbb{N}$ and a unique $t \in [0,75]$ such that $c = -76s+5t$. Note that for every $t \in [0,75]$,

$$\left\lceil \frac{1-75t}{76} \right\rceil = \frac{1-75t}{76} + \frac{75-t}{76} = 1-t.$$

Therefore,

$$\left\lceil \frac{1-15c}{76} \right\rceil = \left\lceil \frac{1-15(-76s+5t)}{76} \right\rceil = 15s + \left\lceil \frac{1-75t}{76} \right\rceil = 15s+1-t.$$

Lower Bound: The lower bound for all $c < -5$ results from the proof of the following claim.

Claim 3. *For all integers $c < -5$, $r(2,2,2,c) \geq \lceil \frac{1-15c}{76} \rceil$.*

Proof of Claim 3. Let an integer $c < -5$ be given and let $c = 76s+5t$ for $s \in \mathbb{N}$ and $t \in [0,75]$. We will show that

$$r(2,2,2,c) \geq \left\lceil \frac{1-15c}{76} \right\rceil = 15s+1-t$$

by exhibiting a 2-coloring of the interval $[1, 15s-t]$ that avoids a monochromatic solution to $L(2,2,2,c)$.

Let $c' = s-5$. Because $s \in \mathbb{N}$, it is clear that $c' > -5$. Therefore it is known from the first case of this Theorem 2 proof that

$$r(2,2,2,c') = 15c'+76 = 76+15(s-5) = 15s+1.$$

Therefore there exists some coloring $\Delta' : [1,15s] \to [0,1]$ that avoids a monochromatic solution to $L(2,2,2,c')$. Let $\Delta : [1,15s-t] \to [0,1]$ be defined by

$$\Delta(x) = \Delta'(15s+1-t-x).$$

Let (x_1, x_2, x_3, x_4) be a solution to $L(2,2,2,c)$ for $x_1, x_2, x_3, x_4 \in [1, 15s-t]$. For every $i \in [1,4]$, define $y_i \in [1, 15s]$ such that

$$y_i = 15s + 1 - t - x_i.$$

It can be shown that (y_1, y_2, y_3, y_4) is a solution to $L(2,2,2,c')$.

Since (y_1, y_2, y_3, y_4) is a solution to $L(2,2,2,c')$ and Δ' admits no monochromatic solutions to $L(2,2,2,c')$, we know (y_1, y_2, y_3, y_4) is not monochromatic in Δ'. Therefore the solution (x_1, x_2, x_3, x_4) to $L(2,2,2,c)$ is not monochromatic in Δ.

Because $\Delta : [1, 15s - t] \to [0,1]$ avoids a monochromatic solution to $L(2,2,2,c)$, we may conclude

$$r(2,2,2,c) \geq 15s + 1 - t.$$

The proof of Claim 3 is complete.

We will now need the following definition.

Definition 3. For integers $c < -5$, for natural numbers s and for $t \in [0, 75]$, the set E_2 contains all $c = -76s + 5t$ for which $s \leq 4$.

By this definition, $E_2 = \{-304, -299, -294, \ldots, -234, -229, -228, -224, -223, -219, -218, \ldots, -159, -158, -154, -153, -152, -149, -148, -147, -144, -143, -142, \ldots, -84, -83, -82, -79, -78, -77, -76, -74, -73, -72, -71, -69, -68, -67, -66, \ldots, -9, -8, -7, -6\}$ and has a cardinality of 150.

Upper Bound: An upper bound for $c < -5$ and $c \notin E_2$ results from the proof of the following claim.

Claim 4. *For integers $c < -5$, if $c \notin E_2$, then $r(2,2,2,c) \leq \lceil \frac{1-15c}{76} \rceil$.*

Proof of Claim 4. Let an integer $c < -5$ be given such that $c \notin E_2$ and let $c = -76s + 5t$ for $s \in \mathbb{N}$ and $t \in [0, 75]$. We will show that

$$r(2,2,2,c) \leq \left\lceil \frac{1 - 15c}{76} \right\rceil = 15s + 1 - t$$

by proving that every 2-coloring of the interval $[1, 15s + 1 - t]$ must contain a monochromatic solution to $L(2,2,2,c)$.

Let $c' = s - 10$. Because $c \notin E_2$, we have $s \geq 5$, and it is clear that $c' \geq -5$. Therefore it is known from the first case of this Theorem 2 proof that

$$r(2,2,2,c') = 15c' + 76 = 76 + 15(s - 10) = 15s - 74.$$

It follows that every coloring $\Delta' : [1, 15s - 74] \to [0, 1]$ must contain a monochromatic solution to $L(2, 2, 2, c')$. Let $\Delta : [1, 15s + 1 - t] \to [0, 1]$ be an arbitrary coloring and let $\Delta' : [1, 15s - 74] \to [0, 1]$ be defined by

$$\Delta'(y) = \Delta(15s + 2 - t - y).$$

We will now show that Δ contains a monochromatic solution to $L(2, 2, 2, c)$.

Let (y_1, y_2, y_3, y_4) be a solution to $L(2, 2, 2, c')$ for $y_1, y_2, y_3, y_4 \in [1, 15s - 74]$ that is monochromatic in Δ'. For every $i \in [1, 4]$, define $x_i \in [1, 15s + 1 - t]$ as

$$x_i = 15s + 2 - t - y_i.$$

It can be shown that (x_1, x_2, x_3, x_4) is a solution to $L(2, 2, 2, c)$. Also for $i, j \in [1, 4]$,

$$\Delta(x_i) = \Delta(15s+2-t-y_i) = \Delta'(y_i) = \Delta'(y_j) = \Delta(15s+2-t-y_j) = \Delta(x_j),$$

so (x_1, x_2, x_3, x_4) is monochromatic in Δ.

Because $\Delta : [1, 15s + 1 - t] \to [0, 1]$ contains a monochromatic solution to $L(2, 2, 2, c)$, we may conclude

$$r(2, 2, 2, c) \leq 15s + 1 - t.$$

The proof of Claim 4 is complete.

Because we previously showed $r(2, 2, 2, c) \geq 15s + 1 - t$ for all $c < -5$, we have now proven that for $c < -5$ when $c \notin E$,

$$r(2, 2, 2, c) = 15s + 1 - t = \left\lceil \frac{1 - 15c}{76} \right\rceil.$$

For integers $c < -5$ when $c \in E_2$, the exact value of $r(2, 2, 2, c)$ has been determined by a computer program. The computer results are displayed in the table below. The function $\phi_2 : (-\infty, -6] \to \mathbb{Z}$ is the difference between $r(2, 2, 2, c)$ and the lower bound determined by Claim 3. The integers $c \in E_2$ for which $\phi_2(c) = 0$ have been omitted from the table.

c	$r(2,2,2,c)$	$\lceil \frac{1-15c}{76} \rceil$	$\phi_2(c)$
-6	10	2	8
-7	7	2	5
-8	4	2	2
-9	3	2	1
-12	6	3	3
-13	5	3	2
-14	4	3	1
-18	6	4	2
-19	5	4	1
-23	6	5	1
-24	6	5	1
-28	7	6	1
-29	7	6	1
-34	8	7	1
-39	9	8	1
-44	10	9	1

The proof of Theorem 2 is complete. □

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