Hamilton-Waterloo Problem: Bipartite case

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Abstract

Given two non-isomorphic bipartite 2-factors F_1 and F_2 of order $4n$, the Bipartite Hamilton-Waterloo Problem (BHWP) asks for a 2-factorization of $K_{2n,2n}$ into α copies of F_1 and β copies of F_2, where $\alpha+\beta=n$ and $\alpha,\beta \geq 1$. We show that the BHWP has solution when F_2 is a refinement of F_1, where no component of F_1 is a C_4 or C_6, except possibly when $\alpha=1$ and either (i) F_2 is a C_4-factor or (ii) F_2 has more than one C_4 with all other components of an order $r \equiv 0 \pmod 4) > 4$ or (iii) F_2 has components with an order $r \equiv 2 \pmod 4$, when n is even. We also show that there does not exist a factorization of $K_{6,6}$ into a single 12-cycle and two C_4-factors.

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1 Introduction

Let G be a graph. Let C_n, K_n and $\overline{K}_n$ denote a cycle, a complete graph and an independent set (or complement of a complete graph) on n vertices respectively. Let $K_{n,n}$ be the complete bipartite graph with partite sets $U=\{u_1,u_2,\ldots,u_n\}$ and $V=\{v_1,v_2,\ldots,v_n\}$. Let $E_k = \{\{u_i,v_j\} \in E(K_{n,n}) : (j-i) \equiv k \pmod n, 1 \leq i,j \leq n\}$, $0 \leq k \leq n-1$ be the set of edges of distance k in $K_{n,n}$. It is clear that each E_k is a 1-factor of $K_{n,n}$ and $\{E_0,E_1,\ldots,E_{n-1}\}$ gives a 1-factorization of $K_{n,n}$. The subgraph of $K_{n,n}$ induced by E_i, E_j and E_k, $0 \leq i \neq j \neq k \leq n-1$ is denoted as $\langle E_i,E_j,E_k\rangle_{n,n}$. A 2-regular subgraph of G, with components $C_{k_1}, C_{k_2},\ldots,C_{k_p}$ is denoted by $[k_1,k_2,\ldots,k_p]$. A cycle with vertices $v_1,v_2,\ldots,v_n$ and edges $\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}$ is denoted as $(v_1,v_2,\ldots,v_n)$. A path with vertices $v_1,v_2,\ldots,v_n$ and edges $\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\}$ is denoted as $\langle v_1,v_2,\ldots,v_n\rangle$. The notation $N_G(v)$ denotes the set of all neighbors of a vertex v in a graph G. A 2-regular spanning subgraph of G is called a *2-factor* of G; In particular, if all its components are isomorphic to C_k, then it is called C_k-factor. A *2-factorization* of G is a partition of G into edge-disjoint 2-factors. For a given $2d$-regular graph G and 2-factors $G_1, G_2, \ldots, G_s$, $s \le d$, the existence of a 2-factorization $\{F_1, F_2, \ldots, F_d\}$ of G such that each $F_i \cong G_j$ for some i and $j, 1 \le i \le d, 1 \le j \le s$, is called the *2-factorization problem* [2].

The 2-factorization problem for the complete graph K_n, in which all the 2-factors are isomorphic to a given 2-factor of K_n, is known as the *Oberwolfach Problem* [11]. Piotrowski [14] has shown that $K_{n,n}$ can be decomposed into copies of any given bipartite 2-factor, except that there does not exist a C_6-factorization of $K_{6,6}$. Liu [13] extended this to the multipartite Oberwolfach problem, where all cycles are of uniform length. A survey of results on this problem can be found in [3]. Let F_1 and F_2 be two non-isomorphic 2-factors of K_n. The *Hamilton-Waterloo Problem* (HWP) [9] asks for a 2-factorization of K_n (respectively $K_n - I$, where I is a 1-factor of K_n, when n even) in which $\alpha(\ge 1)$ 2-factors are isomorphic to F_1 and $\beta(\ge 1)$ 2-factors are isomorphic to F_2, such that $\alpha + \beta = \frac{n-1}{2}$ (respectively $\alpha + \beta = \frac{n-2}{2}$); if such a 2-factorization exists, we say that $(\alpha, \beta) \in HWP(n; F_1, F_2)$ or $HWP(n; F_1, F_2)$ exists. If all the components of F_1 are k-cycles and all the components of F_2 are l-cycles, then we denote the problem by $HWP(n; [k, k, \ldots, k], [l, l, \ldots, l])$. Recently, Bryant, Danziger and Dean [5] have solved the (standard) HWP for bipartite 2-factors. For results on the HWP, see [1, 4, 5, 6, 7, 8, 10, 15, 16, 17, 18].

The Bipartite Hamilton-Waterloo Problem (BHWP) can be stated as follows: Given two non-isomorphic bipartite 2-factors F_1 and F_2 of order $4n$, the Bipartite Hamilton-Waterloo Problem (BHWP) asks for a 2-factorization of $K_{2n,2n}$ into α copies of F_1 and β copies of F_2, where $\alpha + \beta = n$ and $\alpha, \beta \ge 1$. If such a factorization exists, we say that $(\alpha, \beta) \in BHWP(n, n; F_1, F_2)$ or $BHWP(n, n; F_1, F_2)$ exists. If all the components of F_1 are k-cycles and all the components of F_2 are l-cycles, then we denote the problem by $BHWP(n, n; [k, k, \ldots, k], [l, l, \ldots, l])$.

Haggkvist [12] proved that the graph $\langle E_j, E_{j+1} \rangle_{n,n} \otimes \overline{K}_2$ can be factorized into two 2-factors, isomorphic to a given 2-factor of $K_{2n,2n}$. In this paper, first we prove that the graph $\langle E_j, E_{j+1}, E_{j+2} \rangle_{n,n} \otimes \overline{K}_2$, $1 \le j \le n-1$ has a factorization into three 2-factors of which either (i) two of them are isomorphic to a given 2-factor (non isomorphic to a C_4-factor) of $K_{2n,2n}$ and one is a Hamilton cycle, or (ii) one of them is isomorphic to a given 2-factor of $K_{2n,2n}$ and two are Hamilton cycles, or (iii) all of them are isomorphic to a given 2-factor of $K_{2n,2n}$ with components of order divisible by 4 (non isomorphic to a C_4-factor, when n is odd). As a consequence, we show that the BHWP has solution when F_2 is a refinement of F_1, where no component of F_1 is a C_4 or C_6, except possibly when $\alpha = 1$ and either (i) F_2 is a C_4 factor or (ii) F_2 has more than one C_4 with all other components of an order $r \equiv 0 \pmod 4) > 4$ or (iii) F_2 has components with an order $r \equiv 2 \pmod 4$, when n is even. Finally, we show that the BHWP has solution when F_1 is a Hamilton cycle and F_2 has more than one C_4 with all other components of an order $r \equiv 0 \pmod 4$. We also show that there does not exist a factorization of $K_{6,6}$ into a single 12-cycle and two C_4-factors.

2 Preliminaries

The *wreath product* of two graphs G and H is a graph $G \otimes H$ with vertex set $V(G) \times V(H)$, in which (u_1, v_1) is adjacent to (u_2, v_2) whenever (i) $\{u_1, u_2\} \in E(G)$, or (ii) $u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$. The following definition is due to Bryant et.al [5]:

Definition 2.1. *If a 2-regular graph F_2 can be obtained from a 2-regular graph F_1 by replacing each cycle of F_1 with a 2-regular graph on the same vertex set, then F_2 is said to be a refinement of F_1. For example,* $[4, 8^3, 10^2, 12]$ *is a refinement of* $[4, 16, 18, 22]$, *but* $[4, 18^2, 20]$ *is not. Of course, every 2-regular graph of order n is a refinement of an n-cycle.*

In 1985, Haggkvist [12] proved the following:

Lemma 2.1 ([12]). *For a given 2-factor F of $K_{n,n}$, $n \geq 3$, the graph $C_n \otimes \overline{K}_2$ has a 2-factorization $\{H_1, H_2\}$ such that $H_1 \cong H_2 \cong F$.*

Using the notation and terminology from [4] we define a class of graphs J_{2m}, where $m \geq 4$ is even, as follows.

$V(J_{2m}) = \{u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}\} \cup \{v_1, v_2, \ldots, v_m, v_{m+1}, v_{m+2}\}$.
$E(J_{2m}) = \{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\}, \{u_i, v_{i+1}\}, \{v_i, u_{i+1}\} : i = 2, 3, \ldots, m+1\}$
$\cup \{\{u_i, u_{i+3}\}, \{v_i, v_{i+3}\}, \{u_i, v_{i+3}\}, \{v_i, u_{i+3}\} : i = 2, 4, \ldots, m-2\}$
$\cup \{\{u_1, u_3\}, \{v_1, v_3\}, \{u_1, v_3\}, \{v_1, u_3\}\}$, see Figure 2.1.

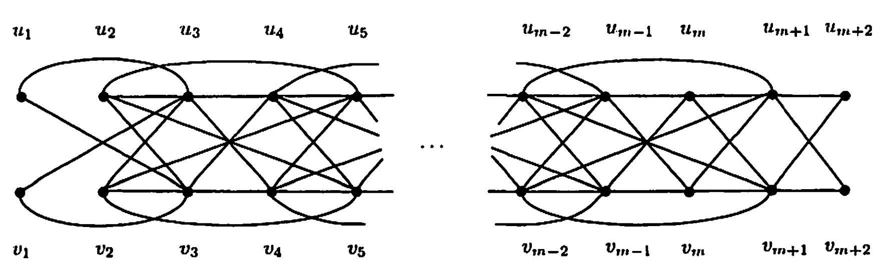


Figure 2.1. The graph J_{2m}

Note that we obtain the J graphs from [4] if we add a pair of isolated vertices, one between u_1 and u_2 and the other between v_1 and v_2. We obtain a new graph $J_{2m} \cdot \{u_1 u_m, u_2 u_{m+2}, v_1 v_m, v_2 v_{m+2}\}$ from J_{2m}, see Figure 2.2, by contracting the vertices u_1 with u_m, u_2 with u_{m+2}, v_1 with v_m and v_2 with v_{m+2} as follows.

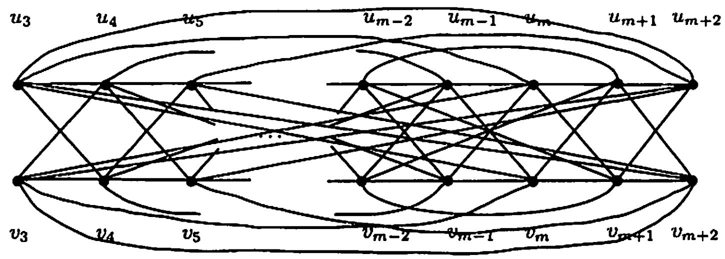


Figure 2.2. The contracted graph $J_{2m} \cdot \{u_1u_m, u_2u_{m+2}, v_1v_m, v_2v_{m+2}\}$

Lemma 2.2. *When m is even*

$$J_{2m} \cdot \{u_1u_m, u_2u_{m+2}, v_1v_m, v_2v_{m+2}\} \cong \langle E_0, E_1, E_2 \rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2.$$

Proof. We relabel the vertices of $\langle E_0, E_1, E_2 \rangle_{\frac{m}{2},\frac{m}{2}}$ as shown in Figure 2.3. Taking the wreath product of this graph with $\overline{K}_2$ makes the simple identification of isomorphism between these two graphs.

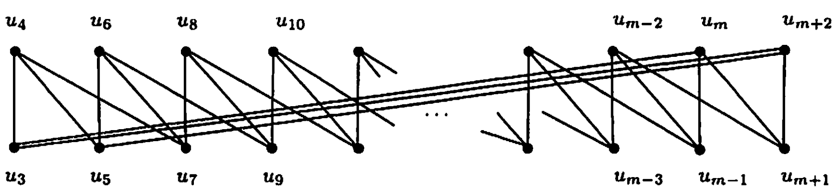


Figure 2.3. The graph $\langle E_0, E_1, E_2 \rangle_{\frac{m}{2},\frac{m}{2}}$

□

We call the vertices $\{u_1, u_2, u_m, u_{m+2}, v_1, v_2, v_m, v_{m+2}\}$ the *end vertices* of J_{2m}, out of which $\{u_1, u_2, v_1, v_2\}$ are called the *left hand end* and $\{u_m, u_{m+2}, v_m, v_{m+2}\}$ are called the *right hand end.*

Definition 2.2. *Let H_1, H_2, H_3 be 2-regular graphs of order $2m$. A decomposition of J_{2m} into $\{H_1, H_2, H_3\}$ satisfying $(p_1), (p_2)$ and (p_3) is denoted by $J_{2m} \to \{H_1, H_2, H_3\}$, where*
$(p_1): V(H_1) = \{u_1, u_2, \dots, u_{m-2}, u_{m-1}, u_{m+1}\} \cup \{v_3, v_4, \dots, v_m, v_{m+1}, v_{m+2}\}$,
$(p_2): V(H_2) = \{u_3, u_4, \dots, u_m, u_{m+1}, u_{m+2}\} \cup \{v_1, v_2, \dots, v_{m-2}, v_{m-1}, v_{m+1}\}$,
$(p_3): V(H_3) = \{u_2, u_3, \dots, u_{m-1}, u_m, u_{m+1}\} \cup \{v_2, v_3, \dots, v_{m-1}, v_m, v_{m+1}\}$.
Thus
H_1 misses the end vertices u_m, u_{m+2}, v_1 and v_2.
H_2 misses the end vertices u_1, u_2, v_m and v_{m+2}.
H_3 misses the end vertices u_1, u_{m+2}, v_1 and v_{m+2}.

We now introduce some notation for decomposition of J_{2m} into specified subgraphs, to get the desired decomposition of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$.

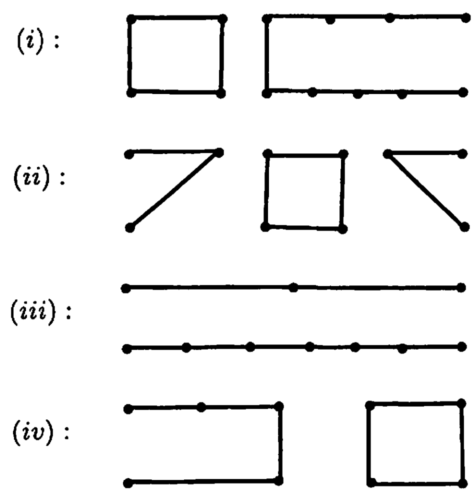


Figure 3.1

Definition 2.3. *1. For $k \geq 0$, $[a_1, a_2, \ldots, a_k, b$ represents a subgraph of J_{2m}, where the first k components are cycles of length $a_1, a_2, \ldots, a_k$ at the left hand end and the last component is a path of length 'b' having both of its end vertices at the right hand end, see Figure* 3.1.(*i*) *for an example of* $[4, 8$. *In particular, if $k = 0$, then $[b$ denotes a path of length 'b' having both of its end vertices at the right hand end.*

2. For $k \geq 0$, $a, b_1, b_2, \ldots, b_k, c$ represents a subgraph of J_{2m}, with cycles of lengths $b_1, b_2, \ldots, b_k$ in the middle, a path of length 'a' having both its end vertices at the left hand end and a path of length 'c' having both its end vertices at the right hand end, see Figure 3.1.(*ii*) *for an example of* $2, 4, 2$.

3. 'a' represents a subgraph of J_{2m}, with two paths each having one end at the left hand end and other end at the right hand end, and contains 'a' edges in total, see Figure 3.1.(*iii*) *for an example where $a = 8$.*

4. For $k \geq 0$, $a, b_1, b_2, \ldots, b_k]$ represents a subgraph of J_{2m}, where the rightmost k components are cycles of length $b_1, b_2, \ldots, b_k$ and the first component is a path of length 'a' having both of its end vertices at the left hand end, see Figure 3.1.(*iv*) *for an example of* $4, 4]$.

A decomposition of J_{2m} into three (not necessarily regular) subgraphs H_1, H_2 and H_3, is denoted as $J_{2m} \mapsto \{H_1; H_2; H_3\}$, where the end vertices of any paths in H_1, H_2 and H_3 are end vertices of J_{2m}. Note that $J_{2m} \to \{H_1, H_2, H_3\}$ denotes the decomposition of J_{2m} into 2-regular subgraphs H_1, H_2 and H_3, whereas $J_{2m} \mapsto \{H_1; H_2; H_3\}$ denotes the decomposition of J_{2m} into subgraphs H_1, H_2 and H_3 with a provision to join some of the end vertices of the components to those of an another J_{2l} to get a larger decomposition of $J_{2(m+l)}$ in a similar manner to that used in [4].

Definition 2.4. *1. $L^{a,b}$ denotes*
$J_{a+b} \mapsto \{[a,b \ ; \ [a,b \ ; \ [a,b\}$

2. *$R^{a,b}$ denotes*
$J_{a+b} \mapsto \{a,b] \ ; \ a,b] \ ; \ a,b]\}$

3. *For $k \geq 0$, $L_1^{a_1,a_2,\ldots,a_k,b}$ denotes*
$J_{a_1+a_2+\cdots+a_k+b} \mapsto \{[a_1,a_2,\ldots,a_k,b \ ; \ [a_1+a_2+\cdots+a_k+b \ ; \ [a_1,a_2,\ldots, a_k,b\}$

4. *For $k \geq 0$, $R_1^{a,b_1,b_2,\ldots,b_k}$ denotes*
$J_{a+b_1+b_2+\cdots+b_k} \mapsto \{a,b_1,b_2,\ldots,b_k] \ ; \ a+b_1+b_2+\cdots+b_k] \ ; \ a,b_1,b_2,\ldots, b_k]\}$

5. *For $k \geq 0$, $C_1^{a,b_1,b_2,\ldots,b_k,c}$ denotes*
$J_{a+b_1+b_2+\cdots+b_k+c} \mapsto \{a,b_1,b_2,\ldots,b_k,c \ ; \ a+b_1+b_2+\cdots+b_k+c \ ; \ a,b_1,b_2,\ldots,b_k,c\}$

6. *P denotes $J_8 \mapsto \{8 \ ; \ 8 \ ; \ 8\}$.*

7. *For $k \geq 0$, $L_2^{a_1,a_2,\ldots,a_k,b}$ denotes*
$J_{a_1+a_2+\cdots+a_k+b} \mapsto \{[a_1,a_2,\ldots,a_k,b \ ; \ [a_1+a_2+\cdots+a_k+b \ ; \ [a_1+a_2+\cdots+a_k+b\}$

8. *For $k \geq 0$, $R_2^{a,b_1,b_2,\ldots,b_k}$ denotes*
$J_{a+b_1+b_2+\cdots+b_k} \mapsto \{a,b_1,b_2,\ldots,b_k] \ ; \ a+b_1+b_2+\cdots+b_k] \ ; \ a+b_1+b_2+\cdots+b_k]\}$

9. *For $k \geq 0$, $C_2^{a,b_1,b_2,\ldots,b_k,c}$ denotes*
$J_{a+b_1+b_2+\cdots+b_k+c} \mapsto \{a,b_1,b_2,\ldots,b_k,c \ ; \ a+b_1+b_2+\cdots+b_k+c \ ; \ a+b_1+b_2+\cdots+b_k+c\}$

10. *For $k \geq 0$, $LR_1^{a,b_1,b_2,\ldots,b_k,c}$ denotes*
$J_{a+b_1+b_2+\cdots+b_k+c} \mapsto \{a,b_1,b_2,\ldots,b_k,c \ ; \ a_1,c_1 \ ; \ a,b_1,b_2,\ldots,b_k,c\}$, *where $a_1+c_1 = a+b_1+b_2+\cdots+b_k+c$.*

11. *For $k \geq 0$, $LR_2^{a,b_1,b_2,\ldots,b_k,c}$ denotes*
$J_{a+b_1+b_2+\cdots+b_k+c} \mapsto \{a,b_1,b_2,\ldots,b_k,c \ ; \ a_1,c_1 \ ; \ a_2,c_2\}$, *where $a_1+c_1 = a_2+c_2 = a+b_1+b_2+\cdots+b_k+c$.*

3 Building blocks for the decomposition of J_{2m}

In this section we provide the building blocks which we put together to get our required decomposition of J_{2m}. We begin with an analogue of Lemma 8 of [4].

Lemma 3.1. *If* $J_{2m} \to \{H_1, H_2, H_3\}$ *and* $J_{2l} \to \{H_1', H_2', H_3'\}$ *then* $J_{2(m+l)} \to \{H_1'', H_2'', H_3''\}$, *where* $H_i'' = H_i \oplus H_i'$, $1 \leq i \leq 3$.

Proof. Consider J_{2m} and J_{2l}, with $V(J_{2m}) = \{u_1, u_2, \ldots, u_m, u_{m+1}, u_{m+2}\} \cup \{v_1, v_2, \ldots, v_m, v_{m+1}, v_{m+2}\}$ and $V(J_{2l}) = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}\} \cup \{y_1, y_2, \ldots, y_l, y_{l+1}, y_{l+2}\}$. If we contract the vertices u_m with x_1, u_{m+2} with x_2, v_m with y_1 and v_{m+2} with y_2, the resulting graph is isomorphic to $J_{2(m+l)}$. From the properties $(p_1), (p_2)$ and (p_3) we observe that H_i and H_i', $1 \leq i \leq 3$ are vertex disjoint in $J_{2(m+l)}$. Let $H_i'' = H_i \oplus H_i'$, $1 \leq i \leq 3$. Hence $J_{2(m+l)} \to \{H_1'', H_2'', H_3''\}$. □

Lemma 3.2. *If* H_1, H_2 *and* H_3 *are given 2-regular graphs of order* $2m$ *and* $J_{2m} \to \{H_1, H_2, H_3\}$, *then* $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$ *has a 2-factorization* $\{H_1, H_2, H_3\}$.

Proof. By the Lemma 2.2, we see that $J_{2m} \cdot \{u_1u_m, u_2u_{m+2}, v_1v_m, v_2v_{m+2}\} \cong \langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$. From the properties $(p_1), (p_2)$ and (p_3), it is clear that $\{H_1, H_2, H_3\}$ of J_{2m} gives a 2-factorization of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$. □

First we present the constructions for $J_{2m} \to \{H_1, H_2, H_3\}$ for smaller values of m as follows.

Lemma 3.3. *The following decompositions exist.*

1. $J_8 \to \{[8], [8], [8]\} = \{[(u_1, u_3, v_4, u_5, v_6, v_5, u_2, v_3)], [(v_1, v_3, u_4, u_5, u_6, v_5, v_2, u_3)], [(u_2, u_3, u_4, v_5, v_4, v_3, v_2, u_5)]\}$
2. $J_8 \to \{[4,4], [4,4], [4,4]\} = \{[(u_1, u_3, u_2, v_3)(u_5, v_6, v_5, v_4)], [(v_1, v_3, v_2, u_3)(u_4, u_5, u_6, v_5)], [(u_2, u_5, v_2, v_5)(u_3, u_4, v_3, v_4)]\}$
3. $J_{12} \to \{[4,8], [4,8], [4,8]\} = \{[(u_1, u_3, u_4, v_3)(u_2, u_5, v_6, v_7, v_8, u_7, v_4, v_5)], [(v_1, v_3, v_4, u_3)(v_2, v_5, u_6, v_7, u_8, u_7, u_4, u_5)], [(u_2, u_3, v_2, v_3)(u_4, v_7, v_4, u_5, u_6, u_7, v_6, v_5]\}$
4. $J_{12} \to \{[6,6], [12], [6,6]\} = \{[(u_1, u_3, u_2, v_5, v_4, v_3)(u_4, u_5, v_6, v_7, v_8, u_7)], [(v_1, v_3, v_2, v_5, u_4, v_7, u_8, u_7, u_6, u_5, v_4, u_3)], [(u_2, u_5, v_2, u_3, u_4, v_3)(v_4, v_7, u_6, v_5, v_6, u_7)]\}$

5. $J_{12} \to \{[6,6],[12],[12]\} = \{[(u_1,u_3,u_2,v_5,v_4,v_3)(u_4,u_5,v_6,v_7,v_8,u_7)],$ $[(v_1,v_3,v_2,u_5,v_4,u_7,u_8,v_7,u_6,v_5,u_4,u_3)],[(u_2,v_3,u_4,v_7,v_4,u_3,v_2,v_5,$ $v_6,u_7,u_6,u_5)]\}$

6. $J_{12} \to \{[4,8],[12],[4,8]\} = \{[(u_1,u_3,u_2,v_3)(u_4,u_5,v_6,v_5,v_4,v_7,v_8,u_7)],$ $[(v_1,v_3,v_2,v_5,u_6,u_5,v_4,u_7,u_8,v_7,u_4,u_3)],[(u_6,u_7,v_6,v_7)(u_2,u_5,v_2,u_3,$ $v_4,v_3,u_4,v_5)]\}$

7. $J_{12} \to \{[4,8],[12],[12]\} = \{[(u_1,u_3,u_2,v_3)(u_4,v_5,v_4,v_7,v_8,u_7,v_6,u_5)],$ $[(v_1,v_3,v_2,v_5,u_6,u_5,v_4,u_7,u_8,v_7,u_4,u_3)],[(u_2,u_5,v_2,u_3,v_4,v_3,u_4,u_7,$ $u_6,v_7,v_6,v_5)]\}$

8. $J_{12} \to \{[12],[12],[12]\} = \{[(u_1,u_3,u_2,u_5,u_4,v_5,v_6,u_7,v_8,v_7,v_4,v_3)],$ $[(v_1,v_3,v_2,v_5,v_4,u_5,u_6,v_7,u_8,u_7,u_4,u_3)],[(u_2,v_5,u_6,u_7,v_4,u_3,v_2,u_5,$ $v_6,v_7,u_4,v_3)]\}$

9. $J_{16} \to \{[8,8],[16],[8,8]\} = \{[(u_1,u_3,u_2,v_5,v_4,u_5,u_4,v_3)(u_6,v_9,v_{10},u_9,$ $v_8,u_7,v_6,v_7)],[(v_1,v_3,v_4,u_7,u_6,v_5,u_4,v_7,u_8,u_9,u_{10},v_9,v_6,u_5,v_2,u_3)],$ $[(u_2,u_5,u_6,u_9,v_6,v_5,v_2,v_3)\ (u_3,u_4,u_7,u_8,v_9,v_8,v_7,v_4)]\}$

10. $J_{16} \to \{[8,8],[16],[16]\} = \{[(u_1,u_3,u_2,v_5,v_4,u_5,u_4,v_3)(u_6,v_9,v_{10},u_9,$ $v_8,u_7,v_6,v_7)],[(v_1,v_3,v_4,u_7,u_6,v_5,u_4,v_7,u_8,v_9,u_{10},u_9,v_6,u_5,v_2,u_3)],$ $[(u_2,u_5,u_6,u_9,u_8,u_7,u_4,u_3,\ v_4,v_7,v_8,v_9,v_6,v_5,v_2,v_3)]\}$

11. $J_{20} \to \{[10,10],[20],[10,10]\} = \{[(u_1,v_3,v_4,v_7,u_8,u_9,u_6,u_5,u_2,u_3)(u_4,$ $u_7,v_8,v_{11},v_{12},\ u_{11},v_{10},v_9,v_6,v_5)],[(v_1,v_3,v_2,v_5,u_6,u_7,u_8,v_9,v_8,u_{11},$ $u_{12},v_{11},u_{10},u_9,v_6,v_7,u_4,u_5,v_4,u_3)],\ [(u_2,v_5,v_4,u_7,v_6,u_5,v_2,u_3,u_4,v_3)$ $(u_6,v_9,u_{10},u_{11},u_8,v_{11},v_{10},u_9,v_8,v_7)]\}$

12. $J_{20} \to \{[10,10],[20],[20]\} = \{[(u_1,v_3,v_4,v_7,u_8,u_9,u_6,u_5,u_2,u_3)(u_4,u_7,$ $v_8,v_{11},v_{12},\ u_{11},v_{10},v_9,v_6,v_5)],[(v_1,v_3,v_2,v_5,u_6,v_9,u_8,v_{11},u_{12},u_{11},u_{10},$ $u_9,v_8,v_7,v_6,u_7,v_4,u_5,u_4,u_3)],\ [(u_2,v_5,u_4,v_7,u_6,u_7,u_8,u_{11},v_8,v_9,u_{10},$ $v_{11},v_{10},u_9,v_6,u_5,v_2,u_3,v_4,v_3)]\}$

Now we present the construction for $J_{2m} \mapsto \{H_1;H_2;H_3\}$ for smaller values of m as follows.

Lemma 3.4. *The following building blocks exist.*

1. $P:\ J_8 \mapsto \{8\ ;\ 8\ ;\ 8\} = \{H_1;H_2;H_3\}$, *where*
   $H_1 = \langle v_1,u_3,u_2,u_5,u_4,v_3,v_4\rangle \cup \langle v_2,v_5,v_6\rangle$,
   $H_2 = \langle u_1,v_3,u_2,v_5,v_4,u_5,v_6\rangle \cup \langle v_2,u_3,u_4\rangle$,
   $H_3 = \langle u_1,u_3,v_4\rangle \cup \langle v_1,v_3,v_2,u_5,u_6,v_5,u_4\rangle$.

2. $L^{4,4}:\ J_8 \mapsto \{[4,4\ ;\ [4,4\ ;\ [4,4\} = \{H_1;H_2;H_3\}$, *where*
   $H_1 = (u_1,u_3,u_4,v_3) \cup \langle v_4,u_5,u_2,v_5,v_6\rangle$,
   $H_2 = (v_1,v_3,v_4,u_3) \cup \langle u_4,v_5,v_2,u_5,v_6\rangle$,
   $H_3 = (u_2,u_3,v_2,v_3) \cup \langle u_4,u_5,u_6,v_5,v_4\rangle$.

3. $L^8 : J_8 \mapsto \{[8\ ;\ [8\ ;\ [8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_4, u_5, u_2, v_3, u_1, u_3, u_4, v_5, v_6 \rangle$,
$H_2 = \langle u_4, v_3, v_1, u_3, v_4, v_5, v_2, u_5, v_6 \rangle$,
$H_3 = \langle u_4, u_5, u_6, v_5, u_2, u_3, v_2, v_3, v_4 \rangle$.

4. $L^{4,8} : J_{12} \mapsto \{[4,8\ ;\ [4,8\ ;\ [4,8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, v_4, v_3) \cup \langle v_6, v_5, u_2, u_5, u_4, u_7, u_6, v_7, v_8 \rangle$,
$H_2 = (v_1, v_3, u_4, u_3) \cup \langle u_6, u_5, v_2, v_5, v_4, v_7, v_6, u_7, v_8 \rangle$,
$H_3 = (u_2, u_3, v_2, v_3) \cup \langle u_6, v_5, u_4, v_7, u_8, u_7, v_4, u_5, v_6 \rangle$.

5. $L^{12} : J_{12} \mapsto \{[12\ ;\ [12\ ;\ [12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_6, u_7, u_6, v_5, v_4, v_3, u_1, u_3, u_2, u_5, u_4, v_7, v_8 \rangle$,
$H_2 = \langle u_6, u_5, v_6, v_7, v_4, u_3, v_1, v_3, v_2, v_5, u_4, u_7, v_8 \rangle$,
$H_3 = \langle u_6, v_7, u_8, u_7, v_4, u_5, v_2, u_3, u_4, v_3, u_2, v_5, v_6 \rangle$.

6. $R^8 : J_8 \mapsto \{8]\ ;\ 8]\ ;\ 8]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, v_5, v_6, u_5, v_4, u_3, v_2 \rangle$,
$H_2 = \langle u_1, u_3, u_2, u_5, u_6, v_5, u_4, v_3, v_2 \rangle$,
$H_3 = \langle u_1, v_3, v_4, v_5, v_2, u_5, u_4, u_3, v_1 \rangle$.

7. $R^{12} : J_{12} \mapsto \{12]\ ;\ 12]\ ;\ 12]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, u_5, v_6, v_5, u_4, v_7, v_8, u_7, v_4, u_3, v_2 \rangle$,
$H_2 = \langle u_1, u_3, u_2, v_5, u_6, v_7, u_8, u_7, u_4, v_3, v_4, u_5, v_2 \rangle$,
$H_3 = \langle u_1, v_3, v_2, v_5, v_4, v_7, v_6, u_7, u_6, u_5, u_4, u_3, v_1 \rangle$.

8. $L_1^{6,2} : J_8 \mapsto \{[6,2\ ;\ [8\ ;\ [6,2\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, u_2, v_5, u_4, v_3) \cup \langle v_4, u_5, v_6 \rangle$,
$H_2 = \langle u_4, u_5, v_2, u_3, v_1, v_3, v_4, v_5, v_6 \rangle$,
$H_3 = (u_2, u_5, u_6, v_5, v_2, v_3) \cup \langle u_4, u_3, v_4 \rangle$.

9. $L_1^{4,4} : J_8 \mapsto \{[4,4\ ;\ [8\ ;\ [4,4\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, v_3, u_2, u_3) \cup \langle v_4, v_5, u_4, u_5, v_6 \rangle$,
$H_2 = \langle u_4, v_3, v_1, u_3, v_4, u_5, v_2, v_5, v_6 \rangle$,
$H_3 = (u_2, v_5, u_6, u_5) \cup \langle u_4, u_3, v_2, v_3, v_4 \rangle$.

10. $L_1^{4,8} : J_{12} \mapsto \{[4,8\ ;\ [12\ ;\ [4,8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, v_4, v_3) \cup \langle v_6, v_7, u_6, u_5, u_2, v_5, u_4, u_7, v_8 \rangle$,
$H_2 = \langle u_6, u_7, v_6, v_5, v_4, u_5, v_2, u_3, v_1, v_3, u_4, v_7, v_8 \rangle$,
$H_3 = (u_7, u_8, v_7, v_4) \cup \langle u_6, v_5, v_2, v_3, u_2, u_3, u_4, u_5, v_6 \rangle$.

11. $L_1^{8,4} : J_{12} \mapsto \{[8,4\ ;\ [12\ ;\ [8,4\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, u_2, u_5, u_6, v_7, u_4, v_3) \cup \langle v_6, v_5, v_4, u_7, v_8 \rangle$,
$H_2 = \langle u_6, v_5, u_4, u_7, v_6, u_5, v_2, v_3, v_1, u_3, v_4, v_7, v_8 \rangle$,
$H_3 = (u_2, v_5, v_2, u_3, u_4, u_5, v_4, v_3) \cup \langle u_6, u_7, u_8, v_7, v_6 \rangle$.

12. $L_1^{10,6}$: $J_{16} \mapsto \{[10,6\ ;\ [16\ ;\ [10,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, u_2, u_5, u_4, v_5, v_6, u_7, v_4, v_3) \cup \langle v_8, v_7, u_6, u_9, u_8, v_9, v_{10}\rangle$,
$H_2 = \langle u_8, u_7, v_8, v_9, u_6, v_5, v_4, v_7, u_4, u_3, v_1, v_3, v_2, u_5, v_6, u_9, v_{10}\rangle$,
$H_3 = (u_2, v_5, v_2, u_3, v_4, u_5, u_6, u_7, u_4, v_3) \cup \langle u_8, v_7, v_6, v_9, u_{10}, u_9, v_8\rangle$.

13. $C_1^{2,4,2}$: $J_8 \mapsto \{2,4,2\ ;\ 8\ ;\ 2,4,2\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, u_4, u_5) \cup \langle v_4, v_5, v_6\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_4\rangle \cup \langle v_2, u_3, v_4, u_5, v_6\rangle$,
$H_3 = \langle u_1, u_3, v_1\rangle \cup (v_2, v_5, u_6, u_5) \cup \langle u_4, v_3, v_4\rangle$.

14. $C_1^{4,4}$: $J_8 \mapsto \{4,4\ ;\ 8\ ;\ 4,4\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_4, u_5, v_2\rangle \cup \langle v_4, v_3, u_2, v_5, v_6\rangle$,
$H_2 = \langle u_1, v_3, u_4\rangle \cup \langle v_2, v_5, v_4, u_3, u_2, u_5, v_6\rangle$,
$H_3 = \langle u_1, u_3, v_2, v_3, v_1\rangle \cup \langle u_4, v_5, u_6, u_5, v_4\rangle$.

15. $C_1^{2,6,4}$: $J_{12} \mapsto \{2,6,4\ ;\ 12\ ;\ 2,6,4\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, v_4, v_5, u_4, u_5) \cup \langle v_6, v_7, u_6, u_7, v_8\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_6\rangle \cup \langle v_2, u_3, u_4, u_7, v_6, u_5, v_4, v_7, v_8\rangle$,
$H_3 = \langle u_1, u_3, v_1\rangle \cup (v_3, v_4, u_7, u_8, v_7, u_4) \cup \langle u_6, u_5, v_2, v_5, v_6\rangle$.

16. $C_1^{4,6,2}$: $J_{12} \mapsto \{4,6,2\ ;\ 12\ ;\ 4,6,2\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_4, u_5, v_2\rangle \cup (u_2, u_3, v_4, u_7, u_6, v_5) \cup \langle v_6, v_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, u_4, v_7, u_6\rangle \cup \langle v_2, v_5, v_4, v_3, u_2, u_5, v_6, u_7, v_8\rangle$,
$H_3 = \langle u_1, v_3, v_2, u_3, v_1\rangle \cup (u_4, u_7, u_8, v_7, v_4, u_5) \cup \langle u_6, v_5, v_6\rangle$.

17. $C_1^{6,6}$: $J_{12} \mapsto \{6,6\ ;\ 12\ ;\ 6,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_4, v_7, v_4, v_3, v_2\rangle \cup \langle v_6, u_5, u_2, v_5, u_6, u_7, v_8\rangle$,
$H_2 = \langle u_1, v_3, u_2, u_3, v_4, u_5, u_6\rangle \cup \langle v_2, v_5, u_4, u_7, v_6, v_7, v_8\rangle$,
$H_3 = \langle u_1, u_3, v_2, u_5, u_4, v_3, v_1\rangle \cup \langle v_6, v_5, v_4, u_7, u_8, v_7, u_6\rangle$.

18. $C_1^{8,4}$: $J_{12} \mapsto \{8,4\ ;\ 12\ ;\ 8,4\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, v_5, v_4, v_3, v_2\rangle \cup \langle v_6, v_7, u_6, u_7, v_8\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_6\rangle \cup \langle v_2, u_3, u_4, u_7, v_6, u_5, v_4, v_7, v_8\rangle$,
$H_3 = \langle u_1, u_3, v_4, u_7, u_8, v_7, u_4, v_3, v_1\rangle \cup \langle u_6, u_5, v_2, v_5, v_6\rangle$.

19. $C_1^{2,10}$: $J_{12} \mapsto \{2,10\ ;\ 12\ ;\ 2,10\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, v_2\rangle \cup \langle v_6, v_5, v_4, v_3, u_2, u_5, u_4, u_7, u_6, v_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, u_2, v_5, u_6\rangle \cup \langle v_2, v_3, u_4, v_7, v_4, u_5, v_6, u_7, v_8\rangle$,
$H_3 = \langle u_1, v_3, v_1\rangle \cup \langle u_6, u_5, v_2, v_5, u_4, u_3, v_4, u_7, u_8, v_7, v_6\rangle$.

20. $C_1^{10,2}$: $J_{12} \mapsto \{10,2\ ;\ 12\ ;\ 10,2\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, v_5, u_4, u_5, u_6, u_7, v_4, u_3, v_2\rangle \cup \langle v_6, v_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, u_2, u_5, v_6, u_7, v_8\rangle \cup \langle v_2, v_5, v_4, v_3, u_4, v_7, u_6\rangle$,
$H_3 = \langle u_1, v_3, v_2, u_5, v_4, v_7, u_8, u_7, u_4, u_3, v_1\rangle \cup \langle u_6, v_5, v_6\rangle$.

21. $C_1^{2,8,6}$: $J_{16} \mapsto \{2,8,6\ ;\ 16\ ;\ 2,8,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, u_4, u_7, u_6, u_5, v_4, v_5) \cup \langle v_8, v_7, v_6, u_9, u_8, v_9, v_{10}\rangle$,
$H_2 = \langle u_1, v_3, u_2, u_5, v_6, v_9, v_8, u_7, u_8\rangle \cup \langle v_2, u_3, v_4, v_7, u_4, v_5, u_6, u_9, v_{10}\rangle$,
$H_3 = \langle u_1, u_3, v_1\rangle \cup (u_4, u_5, v_2, v_5, v_6, u_7, v_4, v_3) \cup \langle u_8, v_7, u_6, v_9, u_{10}, u_9, v_8\rangle$.

22. $C_1^{8,8}$: $J_{16} \mapsto \{8,8\ ;\ 16\ ;\ 8,8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, v_4, u_5, u_4, v_5, u_2, v_3, v_2\rangle \cup \langle v_8, v_9, u_8, v_7, v_6, u_7, u_6, u_9, v_{10}\rangle$,
$H_2 = \langle u_1, v_3, v_4, v_7, v_8, u_9, v_6, u_5, u_2, u_3, u_4, u_7, u_8\rangle \cup \langle v_2, v_5, u_6, v_9, v_{10}\rangle$,
$H_3 = \langle u_1, u_3, v_2, u_5, u_6, v_7, u_4, v_3, v_1\rangle \cup \langle v_8, u_7, v_4, v_5, v_6, v_9, u_{10}, u_9, u_8\rangle$.

23. $C_1^{6,10}$: $J_{16} \mapsto \{6,10\ ;\ 16\ ;\ 6,10\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_4, v_7, v_4, v_3, v_2\rangle \cup \langle v_8, v_9, u_8, u_7, v_6, v_5, u_2, u_5, u_6, u_9, v_{10}\rangle$,
$H_2 = \langle u_1, v_3, u_2, u_3, v_4, u_5, v_6, u_9, v_8, v_7, u_8\rangle \cup \langle v_2, v_5, u_4, u_7, u_6, v_9, v_{10}\rangle$,
$H_3 = \langle u_1, u_3, v_2, u_5, u_4, v_3, v_1\rangle \cup \langle u_8, u_9, u_{10}, v_9, v_6, v_7, u_6, v_5, v_4, u_7, v_8\rangle$.

24. $C_1^{10,6}$: $J_{16} \mapsto \{10,6\ ;\ 16\ ;\ 10,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_4, u_7, u_6, u_5, u_4, u_3, u_2, v_5, v_2\rangle \cup \langle v_8, v_7, v_6, v_9, u_8, u_9, v_{10}\rangle$,
$H_2 = \langle u_1, u_3, v_4, v_7, u_4, v_5, u_6, v_9, v_{10}\rangle \cup \langle v_2, v_3, u_2, u_5, v_6, u_9, v_8, u_7, u_8\rangle$,
$H_3 = \langle u_1, v_3, u_4, u_7, v_6, v_5, v_4, u_5, v_2, u_3, v_1\rangle \cup \langle u_8, v_7, u_6, u_9, u_{10}, v_9, v_8\rangle$.

25. $C_1^{14,2}$: $J_{16} \mapsto \{14,2\ ;\ 16\ ;\ 14,2\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, u_5, u_4, u_7, u_6, u_9, u_8, v_7, v_6, v_5, v_4, u_3, v_2\rangle \cup \langle v_8, v_9, v_{10}\rangle$,
$H_2 = \langle u_1, u_3, u_2, v_5, u_4, v_3, v_4, u_7, v_6, v_9, u_8\rangle \cup \langle v_2, u_5, u_6, v_7, v_8, u_9, v_{10}\rangle$,
$H_3 = \langle u_1, v_3, v_2, v_5, u_6, v_9, u_{10}, u_9, v_6, u_5, v_4, v_7, u_4, u_3, v_1\rangle \cup \langle u_8, u_7, v_8\rangle$.

26. $R_1^{2,6}$: $J_8 \mapsto \{2,6]\ ;\ 8]\ ;\ 2,6]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, v_4, v_5, v_6, u_5)$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_6, u_5, u_4, u_3, v_2\rangle$,
$H_3 = \langle u_1, u_3, v_1\rangle \cup (v_2, u_5, v_4, v_3, u_4, v_5)$.

27. $R_1^{2,6,4}$: $J_{12} \mapsto \{2,6,4]\ ;\ 12]\ ;\ 2,6,4]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, v_2\rangle \cup (u_4, u_5, v_6, v_7, v_8, u_7) \cup (u_2, v_3, v_4, v_5)$,
$H_2 = \langle u_1, u_3, u_2, u_5, v_4, u_7, u_8, v_7, u_6, v_5, u_4, v_3, v_2\rangle$,
$H_3 = \langle u_1, v_3, v_1\rangle \cup (v_2, u_5, u_6, u_7, v_6, v_5) \cup (u_3, u_4, v_7, v_4)$.

28. $R_1^{4,8}$: $J_{12} \mapsto \{4,8]\ ;\ 12]\ ;\ 4,8]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_4, v_5, v_2\rangle \cup (u_2, u_3, u_4, u_7, v_8, v_7, v_6, u_5)$,
$H_2 = \langle u_1, u_3, v_4, u_7, u_8, v_7, u_4, v_3, u_2, v_5, u_6, u_5, v_2\rangle$,
$H_3 = \langle u_1, v_3, v_2, u_3, v_1\rangle \cup (u_4, u_5, v_4, v_7, u_6, u_7, v_6, v_5)$.

29. $R_1^{8,4}$: $J_{12} \mapsto \{8,4]\ ;\ 12]\ ;\ 8,4]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, v_4, u_7, v_8, v_7, u_4, v_3, v_2\rangle \cup (u_2, u_5, v_6, v_5)$,
$H_2 = \langle u_1, v_3, u_2, u_3, u_4, u_7, u_8, v_7, v_4, v_5, u_6, u_5, v_2\rangle$,
$H_3 = \langle u_1, u_3, v_2, v_5, u_4, u_5, v_4, v_3, v_1\rangle \cup (u_6, u_7, v_6, v_7)$.

30. $R_1^{4,6,6}$: $J_{16} \mapsto \{4,6,6] \; ; \; 16] \; ; \; 4,6,6]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_4, v_5, v_2 \rangle \cup (u_2, u_3, u_4, u_7, u_6, u_5) \cup (v_6, v_7, v_8, u_9, v_{10}, v_9)$,
$H_2 = \langle u_1, u_3, v_4, u_7, v_6, u_9, u_{10}, v_9, u_8, v_7, u_6, v_5, u_2, v_3, u_4, u_5, v_2 \rangle$,
$H_3 = \langle u_1, v_3, v_2, u_3, v_1 \rangle \cup (u_4, v_7, v_4, u_5, v_6, v_5) \cup (u_6, v_9, v_8, u_7, u_8, u_9)$.

31. $R_1^{6,10}$: $J_{16} \mapsto \{6,10] \; ; \; 16] \; ; \; 6,10]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_4, u_7, v_6, v_5, v_2 \rangle \cup (u_2, v_3, v_4, v_7, v_8, u_9, v_{10}, v_9, u_6, u_5)$,
$H_2 = \langle u_1, v_3, u_4, u_5, v_6, v_7, u_6, u_9, u_{10}, v_9, u_8, u_7, v_4, v_5, u_2, u_3, v_2 \rangle$,
$H_3 = \langle u_1, u_3, v_4, u_5, v_2, v_3, v_1 \rangle \cup (u_4, v_5, u_6, u_7, v_8, v_9, v_6, u_9, u_8, v_7)$.

32. $R_1^{12,4}$: $J_{16} \mapsto \{12,4] \; ; \; 16] \; ; \; 12,4]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, u_5, v_6, u_9, v_{10}, v_9, u_6, v_5, v_4, u_3, v_2 \rangle \cup (u_4, u_7, v_8, v_7)$,
$H_2 = \langle u_1, u_3, u_2, v_5, v_6, v_9, u_{10}, u_9, u_6, v_7, u_8, u_7, v_4, u_5, u_4, v_3, v_2 \rangle$,
$H_3 = \langle u_1, v_3, v_4, v_7, v_6, u_7, u_6, u_5, v_2, v_5, u_4, u_3, v_1 \rangle \cup (u_8, u_9, v_8, v_9)$.

33. $LR_1^{2,4,4,6}$: $J_{16} \mapsto \{2,4,4,6 \; ; \; 10,6 \; ; \; 2,4,4,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2 \rangle \cup (u_2, u_3, u_4, u_5) \cup (v_4, v_5, v_6, u_7) \cup \langle v_8, v_9, u_8, v_7, u_6, u_9, v_{10} \rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_4, v_7, v_6, u_5, v_4, u_3, v_2 \rangle \cup \langle u_8, u_9, v_8, u_7, u_6, v_9, v_{10} \rangle$,
$H_3 = \langle u_1, u_3, v_1 \rangle \cup (v_2, v_5, u_6, u_5) \cup (v_6, v_9, u_{10}, u_9) \cup \langle u_8, u_7, u_4, v_3, v_4, v_7, v_8 \rangle$.

34. $LR_1^{2,4,10}$: $J_{16} \mapsto \{2,4,10 \; ; \; 8,8 \; ; \; 2,4,10\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2 \rangle \cup (u_2, u_3, u_4, u_5) \cup \langle v_8, u_7, v_6, v_7, v_4, v_5, u_6, u_9, u_8, v_9, v_{10} \rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, v_6, u_5, v_4, u_3, v_2 \rangle \cup \langle u_8, v_7, u_4, u_7, u_6, v_9, v_8, u_9, v_{10} \rangle$,
$H_3 = \langle u_1, u_3, v_1 \rangle \cup (v_6, v_9, u_{10}, u_9) \cup \langle u_8, u_7, v_4, v_3, u_4, v_5, v_2, u_5, u_6, v_7, v_8 \rangle$.

35. $LR_1^{2,4,8,6}$: $J_{20} \mapsto \{2,4,8,6 \, ; \, 10,10 \, ; \, 2,4,8,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2 \rangle \cup (u_2, u_3, u_4, u_5) \cup (v_4, v_5, v_6, v_9, u_6, v_7, v_8, u_7) \cup$
$\langle v_{10}, u_9, u_{10}, v_{11}, u_8, u_{11}, v_{12} \rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_4, v_7, v_6, u_5, v_4, u_3, v_2 \rangle \cup$
$\langle u_{10}, v_9, u_8, u_7, u_6, u_9, v_8, u_{11}, v_{10}, v_{11}, v_{12} \rangle$,
$H_3 = \langle u_1, u_3, v_1 \rangle \cup (v_2, v_5, u_6, u_5) \cup (v_3, v_4, v_7, u_8, u_9, v_6, u_7, u_4) \cup$
$\langle u_{10}, u_{11}, u_{12}, v_{11}, v_8, v_9, v_{10} \rangle$.

36. $LR_1^{2,4,4,8,2}$: $J_{20} \mapsto \{2,4,4,8,2 \; ; \; 4,16 \; ; \; 2,4,4,8,2\} = \{H_1; H_2; H_3\}$,
where
$H_1 = \langle v_1, v_3, v_2 \rangle \cup (u_2, u_5, u_6, v_5) \cup (u_3, u_4, u_7, v_4) \cup (v_6, v_9, v_8, v_7, u_8, u_{11},$
$u_{10}, u_9) \cup \langle v_{10}, v_{11}, v_{12} \rangle$,
$H_2 = \langle u_1, v_3, u_2, u_3, v_2 \rangle \cup$
$\langle u_{10}, v_{11}, u_8, v_9, u_6, v_7, u_4, u_5, v_4, v_5, v_6, u_7, v_8, u_9, v_{10}, u_{11}, v_{12} \rangle$,
$H_3 = \langle u_1, u_3, v_1 \rangle \cup (u_6, u_7, u_8, u_9) \cup (v_8, v_{11}, u_{12}, u_{11}) \cup (v_2, v_5, u_4, v_3, v_4,$
$v_7, v_6, u_5) \cup \langle u_{10}, v_9, v_{10} \rangle$.

37. $LR_1^{2,4,4,8,6}$: $J_{24} \mapsto \{2,4,4,8,6 \; ; \; 14,10 \; ; \; 2,4,4,8,6\} = \{H_1; H_2; H_3\}$,
where
$H_1 = \langle v_1, v_3, v_2 \rangle \cup (u_2, u_3, u_4, u_5) \cup (v_4, v_5, v_6, u_7) \cup (u_6, v_7, v_8, v_9, u_{10}, u_{11},$

$u_8, u_9) \cup \langle v_{12}, v_{13}, u_{12}, v_{11}, v_{10}, u_{13}, v_{14}\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_4, v_7, u_8, v_{11}, u_{10}, u_9, v_6, u_5, v_4, u_3, v_2\rangle \cup$
$\langle u_{12}, u_{13}, v_{12}, u_{11}, v_8, u_7, u_6, v_9, v_{10}, v_{13}, v_{14}\rangle$,
$H_3 = \langle u_1, u_3, v_1)\cup(v_2, v_5, u_6, u_5)\cup(u_{10}, u_{13}, u_{14}, v_{13})\cup(v_3, v_4, v_7, v_6, v_9,$
$u_8, u_7, u_4) \cup \langle u_{12}, u_{11}, v_{10}, u_9, v_8, v_{11}, v_{12}\rangle$.

38. $L_2^{6,2}$: $J_8 \mapsto \{[6,2\ ;\ [8\ ;\ [8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, v_3, u_4, u_5, u_2, u_3) \cup \langle v_4, v_5, v_6\rangle$,
$H_2 = \langle u_4, v_5, v_2, v_3, v_1, u_3, v_4, u_5, v_6\rangle$,
$H_3 = \langle u_4, u_3, v_2, u_5, u_6, v_5, u_2, v_3, v_4\rangle$.

39. $L_2^{4,4}$: $J_8 \mapsto \{[4,4\ ;\ [8\ ;\ [8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, u_4, v_3) \cup \langle v_4, v_5, u_2, u_5, v_6\rangle$,
$H_2 = \langle u_4, u_5, v_4, u_3, v_1, v_3, v_2, v_5, v_6\rangle$,
$H_3 = \langle u_4, v_5, u_6, u_5, v_2, u_3, u_2, v_3, v_4\rangle$.

40. $L_2^{4,8}$: $J_{12} \mapsto \{[4,8\ ;\ [12\ ;\ [12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, v_3, u_2, u_3) \cup \langle v_6, v_7, v_4, v_5, u_4, u_5, u_6, u_7, v_8\rangle$,
$H_2 = \langle u_6, v_5, v_2, v_3, v_1, u_3, v_4, u_5, v_6, u_7, u_4, v_7, v_8\rangle$,
$H_3 = \langle u_6, v_7, u_8, u_7, v_4, v_3, u_4, u_3, v_2, u_5, u_2, v_5, v_6\rangle$.

41. $L_2^{8,4}$: $J_{12} \mapsto \{[8,4\ ;\ [12\ ;\ [12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, u_2, u_5, u_4, v_5, v_4, v_3) \cup \langle v_6, v_7, u_6, u_7, v_8\rangle$,
$H_2 = \langle u_6, u_5, v_6, v_5, v_2, v_3, v_1, u_3, v_4, u_7, u_4, v_7, v_8\rangle$,
$H_3 = \langle u_6, v_5, u_2, v_3, u_4, u_3, v_2, u_5, v_4, v_7, u_8, u_7, v_6\rangle$.

42. $L_2^{10,6}$: $J_{16} \mapsto \{[10,6\ ;\ [16\ ;\ [16\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = (u_1, u_3, u_2, u_5, v_4, v_7, v_6, v_5, u_4, v_3) \cup \langle v_8, v_9, u_8, u_7, u_6, u_9, v_{10}\rangle$,
$H_2 = \langle u_8, u_9, v_8, v_7, u_6, v_5, v_2, u_3, v_1, v_3, v_4, u_7, u_4, u_5, v_6, v_9, v_{10}\rangle$,
$H_3 = \langle u_8, v_7, u_4, u_3, v_4, v_5, u_2, v_3, v_2, u_5, u_6, v_9, u_{10}, u_9, v_6, u_7, v_8\rangle$.

43. $C_2^{2,4,2}$: $J_8 \mapsto \{2,4,2\ ;\ 8\ ;\ 8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, u_4, u_5) \cup \langle v_4, v_5, v_6\rangle$,
$H_2 = \langle u_1, u_3, v_4, u_5, v_6\rangle \cup \langle v_2, v_5, u_2, v_3, u_4\rangle$,
$H_3 = \langle u_1, v_3, v_4\rangle \cup \langle v_1, u_3, v_2, u_5, u_6, v_5, u_4\rangle$.

44. $C_2^{4,4}$: $J_8 \mapsto \{4,4\ ;\ 8\ ;\ 8\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, u_5, v_2\rangle \cup \langle v_4, u_3, u_4, v_5, v_6\rangle$,
$H_2 = \langle u_1, u_3, u_2, v_5, v_4, u_5, v_6\rangle \cup \langle v_2, v_3, u_4\rangle$,
$H_3 = \langle u_1, v_3, v_4\rangle \cup \langle v_1, u_3, v_2, v_5, u_6, u_5, u_4\rangle$.

45. $C_2^{2,6,4}$: $J_{12} \mapsto \{2,6,4\ ;\ 12\ ; 12\ \} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, v_4, v_5, u_4, u_5) \cup \langle v_6, v_7, u_6, u_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, u_4, v_7, v_8\rangle \cup \langle v_2, u_5, v_6, u_7, v_4, v_3, u_2, v_5, u_6\rangle$,
$H_3 = \langle u_1, v_3, u_4, u_7, u_8, v_7, v_4, u_5, u_6\rangle \cup \langle v_1, u_3, v_2, v_5, v_6\rangle$.

46. $C_2^{4,6,2}$: $J_{12} \mapsto \{4,6,2\ ;\ 12\ ;\ 12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, v_3, v_2\rangle \cup (u_4, v_7, v_4, v_5, u_6, u_5) \cup \langle v_6, u_7, v_8\rangle$,
$H_2 = \langle u_1, v_3, u_4, u_7, u_6\rangle \cup \langle v_2, u_3, v_4, u_5, u_2, v_5, v_6, v_7, v_8\rangle$,
$H_3 = \langle u_1, u_3, u_4, v_5, v_2, u_5, v_6\rangle \cup \langle v_1, v_3, v_4, u_7, u_8, v_7, u_6\rangle$.

47. $C_2^{6,6}$: $J_{12} \mapsto \{6,6\ ;\ 12\ ;\ 12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, v_3, v_2\rangle \cup \langle v_6, v_5, u_6, u_7, v_4, v_7, v_8\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_4, u_7, v_8\rangle \cup \langle v_2, u_3, v_4, u_5, v_6, v_7, u_6\rangle$,
$H_3 = \langle u_1, u_3, u_4, v_7, u_8, u_7, v_6\rangle \cup \langle v_1, v_3, v_4, v_5, v_2, u_5, u_6\rangle$.

48. $C_2^{8,4}$: $J_{12} \mapsto \{8,4\ ;\ 12\ ;\ 12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, v_4, v_5, u_6, u_5, u_2, v_3, v_2\rangle \cup \langle v_6, v_7, u_4, u_7, v_8\rangle$,
$H_2 = \langle u_1, v_3, u_4, u_5, v_4, v_7, v_8\rangle \cup \langle v_2, u_3, u_2, v_5, v_6, u_7, u_6\rangle$,
$H_3 = \langle u_1, u_3, u_4, v_5, v_2, u_5, v_6\rangle \cup \langle v_1, v_3, v_4, u_7, u_8, v_7, u_6\rangle$.

49. $C_2^{10,2}$: $J_{12} \mapsto \{10,2\ ;\ 12\ ;\ 12\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, u_2, v_5, u_6, u_7, v_4, u_3, u_4, u_5, v_2\rangle \cup \langle v_6, v_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, u_2, u_5, u_6\rangle \cup \langle v_2, v_3, u_4, v_7, v_4, v_5, v_6, u_7, v_8\rangle$,
$H_3 = \langle u_1, v_3, v_4, u_5, v_6\rangle \cup \langle v_1, u_3, v_2, v_5, u_4, u_7, u_8, v_7, u_6\rangle$.

50. $C_2^{2,8,6}$: $J_{16} \mapsto \{2,8,6\ ;\ 16\ ;\ 16\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, v_4, u_5, u_4, v_7, v_6, v_5) \cup \langle v_8, u_7, u_6, u_9, u_8, v_9, v_{10}\rangle$,
$H_2 = \langle u_1, u_3, u_4, v_5, u_6, v_7, v_8, v_9, v_6, u_9, v_{10}\rangle \cup \langle v_2, u_5, u_2, v_3, v_4, u_7, u_8\rangle$,
$H_3 = \langle u_1, v_3, u_4, u_7, v_6, u_5, u_6, v_9, u_{10}, u_9, v_8\rangle \cup \langle v_1, u_3, v_2, v_5, v_4, v_7, u_8\rangle$.

51. $C_2^{8,8}$: $J_{16} \mapsto \{8,8\ ;\ 16\ ;\ 16\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, v_5, v_4, v_3, v_2\rangle \cup \langle v_8, v_7, v_6, u_9, u_8, u_7, u_6, v_9, v_{10}\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, v_6, u_5, u_6, u_9, v_{10}\rangle \cup \langle v_2, u_3, u_4, v_7, v_4, u_7, v_8, v_9, u_8\rangle$,
$H_3 = \langle u_1, u_3, v_4, u_5, v_2, v_5, u_6, v_7, u_8\rangle \cup \langle v_1, v_3, u_4, u_7, v_6, v_9, u_{10}, u_9, v_8\rangle$.

52. $C_2^{6,10}$: $J_{16} \mapsto \{6,10\ ;\ 16\ ;\ 16\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, v_3, v_2\rangle \cup \langle v_8, v_7, v_4, v_5, u_6, u_7, v_6, u_9, u_8, v_9, v_{10}\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, v_6, v_7, u_4, u_7, u_8\rangle \cup \langle v_2, u_3, v_4, u_5, u_6, v_9, v_8, u_9, v_{10}\rangle$,
$H_3 = \langle u_1, u_3, u_4, v_5, v_2, u_5, v_6, v_9, u_{10}, u_9, u_6, v_7, u_8\rangle \cup \langle v_1, v_3, v_4, u_7, v_8\rangle$.

53. $C_2^{10,6}$: $J_{16} \mapsto \{10,6\ ;\ 16\ ;\ 16\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, v_5, v_6, v_7, v_4, v_3, v_2\rangle \cup \langle v_8, v_9, u_8, u_7, u_6, u_9, v_{10}\rangle$,
$H_2 = \langle u_1, v_3, u_2, v_5, v_4, u_5, u_6, v_7, u_8\rangle \cup \langle v_2, u_3, u_4, u_7, v_8, u_9, v_6, v_9, v_{10}\rangle$,
$H_3 = \langle u_1, u_3, v_4, u_7, v_6, u_5, v_2, v_5, u_6, v_9, u_{10}, u_9, u_8\rangle \cup \langle v_1, v_3, u_4, v_7, v_8\rangle$.

54. $R_2^{2,6}$: $J_8 \mapsto \{2,6]\ ;\ 8]\ ;\ 8]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, v_4, v_5, v_6, u_5)$,
$H_2 = \langle u_1, u_3, u_4, v_3, u_2, v_5, u_6, u_5, v_2\rangle$,
$H_3 = \langle u_1, v_3, v_4, u_5, u_4, v_5, v_2, u_3, v_1\rangle$.

55. $R_2^{2,6,4}: J_{12} \mapsto \{2,6,4] \; ; \; 12] \; ; \; 12]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, v_4, v_5, u_4, u_5) \cup (u_7, v_8, v_7, v_6)$,
$H_2 = \langle u_1, u_3, u_4, u_7, u_8, v_7, u_6, u_5, v_4, v_3, u_2, v_5, v_2\rangle$,
$H_3 = \langle u_1, v_3, u_4, v_7, v_4, u_7, u_6, v_5, v_6, u_5, v_2, u_3, v_1\rangle$.

56. $R_2^{4,8}: J_{12} \mapsto \{4,8] \; ; \; 12] \; ; \; 12]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, v_3, v_2\rangle \cup (u_4, v_7, v_8, u_7, v_4, u_5, v_6, v_5)$,
$H_2 = \langle u_1, v_3, v_4, v_5, u_2, u_5, u_6, v_7, u_8, u_7, u_4, u_3, v_2\rangle$,
$H_3 = \langle u_1, u_3, v_4, v_7, v_6, u_7, u_6, v_5, v_2, u_5, u_4, v_3, v_1\rangle$.

57. $R_2^{8,4}: J_{12} \mapsto \{8,4] \; ; \; 12] \; ; \; 12]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_4, u_5, u_2, v_5, v_4, v_3, v_2\rangle \cup (u_7, v_8, v_7, v_6)$,
$H_2 = \langle u_1, v_3, u_2, u_3, v_4, u_5, u_6, u_7, u_8, v_7, u_4, v_5, v_2\rangle$,
$H_3 = \langle u_1, u_3, v_2, u_5, v_6, v_5, u_6, v_7, v_4, u_7, u_4, v_3, v_1\rangle$.

58. $R_2^{4,6,6}: J_{16} \mapsto \{4,6,6] \; ; \; 16] \; ; \; 16]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, v_3, v_2\rangle \cup (u_4, u_5, u_6, v_5, v_4, u_7) \cup (v_6, v_7, v_8, v_9, v_{10}, u_9)$,
$H_2 = \langle u_1, v_3, u_4, u_3, v_4, v_7, u_8, u_9, u_{10}, v_9, u_6, u_7, v_6, v_5, u_2, u_5, v_2\rangle$,
$H_3 = \langle u_1, u_3, v_2, v_5, u_4, v_7, u_6, u_9, v_8, u_7, u_8, v_9, v_6, u_5, v_4, v_3, v_1\rangle$.

59. $R_2^{6,10}: J_{16} \mapsto \{6,10] \; ; \; 16] \; ; \; 16]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, v_3, v_2\rangle \cup (v_4, v_5, v_6, u_7, u_6, u_9, v_{10}, v_9, v_8, v_7)$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_4, u_7, u_8, u_9, u_{10}, v_9, v_6, v_7, u_6, u_5, v_4, u_3, v_2\rangle$,
$H_3 = \langle u_1, u_3, u_4, v_7, u_8, v_9, u_6, v_5, v_2, u_5, v_6, u_9, v_8, u_7, v_4, v_3, v_1\rangle$.

60. $R_2^{12,4}: J_{16} \mapsto \{12,4] \; ; \; 16] \; ; \; 16]\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, u_3, u_2, u_5, u_4, u_7, u_6, v_7, v_6, v_5, v_4, v_3, v_2\rangle \cup (v_8, v_9, v_{10}, u_9)$,
$H_2 = \langle u_1, v_3, u_2, v_5, u_6, u_9, u_{10}, v_9, u_8, u_7, v_6, u_5, v_4, v_7, u_4, u_3, v_2\rangle$,
$H_3 = \langle u_1, u_3, v_4, u_7, v_8, v_7, u_8, u_9, v_6, v_9, u_6, u_5, v_2, v_5, u_4, v_3, v_1\rangle$.

61. $LR_2^{2,4,4,2}: J_{12} \mapsto \{2,4,4,2 \; ; \; 8,4 \; ; \; 6,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, u_4, u_5) \cup (v_4, v_5, u_6, u_7) \cup \langle v_6, v_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, v_4, v_7, u_4, v_3, u_2, v_5, v_2\rangle \cup \langle u_6, u_5, v_6, u_7, v_8\rangle$,
$H_3 = \langle u_1, v_3, v_4, u_5, v_2, u_3, v_1\rangle \cup \langle u_6, v_7, u_8, u_7, u_4, v_5, v_6\rangle$.

62. $LR_2^{2,4,4,4,2}: J_{16} \mapsto \{2,4,4,4,2 \; ; \; 10,6 \; ; \; 6,10\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, u_4, u_5) \cup (u_6, u_7, u_8, u_9) \cup (v_4, v_5, v_6, v_7) \cup \langle v_8, v_9, v_{10}\rangle$,
$H_2 = \langle u_1, u_3, v_4, u_7, v_8, v_7, u_4, v_3, u_2, v_5, v_2\rangle \cup \langle u_8, v_9, u_6, u_5, v_6, u_9, v_{10}\rangle$,
$H_3 = \langle u_1, v_3, v_4, u_5, v_2, u_3, v_1\rangle \cup \langle u_8, v_7, u_6, v_5, u_4, u_7, v_6, v_9, u_{10}, u_9, v_8\rangle$.

63. $LR_2^{2,4,6}: J_{12} \mapsto \{2,4,6 \; ; \; 8,4 \; ; \; 6,6\} = \{H_1; H_2; H_3\}$, *where*
$H_1 = \langle v_1, v_3, v_2\rangle \cup (u_2, u_3, u_4, u_5) \cup \langle v_6, v_7, u_6, v_5, v_4, u_7, v_8\rangle$,
$H_2 = \langle u_1, u_3, v_4, v_3, u_2, v_5, v_6, u_5, v_2\rangle \cup \langle u_6, u_7, u_4, v_7, v_8\rangle$,
$H_3 = \langle u_1, v_3, u_4, v_5, v_2, u_3, v_1\rangle \cup \langle u_6, u_5, v_4, v_7, u_8, u_7, v_6\rangle$.

64. $LR_2^{2,4,4,6}$: $J_{16} \mapsto \{2,4,4,6\ ;\ 8,8\ ;\ 2,14\} = \{H_1;H_2;H_3\}$, *where*
$H_1 = \langle v_1,v_3,v_2\rangle \cup (u_2,u_3,u_4,v_5) \cup (u_5,u_6,u_7,v_4) \cup \langle v_8,v_7,v_6,u_9,u_8, v_9,v_{10}\rangle$,
$H_2 = \langle u_1,v_3,u_2,u_5,u_4,v_7,v_4,u_3,v_2\rangle \cup \langle u_8,u_7,v_6,v_5,u_6,v_9,v_8,u_9,v_{10}\rangle$,
$H_3 = \langle u_1,u_3,v_1\rangle \cup \langle u_8,v_7,u_6,u_9,u_{10},v_9,v_6,u_5,v_2,v_5,v_4,v_3,u_4,u_7,v_8\rangle$.

65. $LR_2^{2,4,4,8,6}$: $J_{24} \mapsto \{2,4,4,8,6\ ;\ 8,16\ ;\ 2,22\} = \{H_1;H_2;H_3\}$, *where*
$H_1 = \langle v_1,v_3,v_2\rangle \cup (u_2,u_3,u_4,v_5) \cup (u_5,u_6,u_7,v_4) \cup (u_8,v_9,v_8,u_{11},u_{10}, u_9,v_6,v_7) \cup \langle v_{12},v_{11},v_{10},u_{13},u_{12},v_{13},v_{14}\rangle$,
$H_2 = \langle u_1,v_3,u_2,u_5,u_4,v_7,v_4,u_3,v_2\rangle \cup \langle u_{12},u_{11},v_{10},v_9,u_6,v_5,v_6,u_7, v_8,u_9,u_8,v_{11},u_{10},v_{13},v_{12},u_{13},v_{14}\rangle$,
$H_3 = \langle u_1,u_3,v_1\rangle \cup \langle u_{12},v_{11},v_8,v_7,u_6,u_9,v_{10},v_{13},u_{14},u_{13},u_{10},v_9,v_6, u_5,v_2,v_5,v_4,v_3,u_4,u_7,u_8,u_{11},v_{12}\rangle$.

4 2-Factorizations of $\langle E_j, E_{j+1}, E_{j+2}\rangle_{n,n} \otimes \overline{K}_2$

In this section we show that $\langle E_j, E_{j+1}, E_{j+2}\rangle_{n,n} \otimes \overline{K}_2$ has a decomposition into almost any bipartite 2-factor of $K_{2n,2n}$. We first prove the following before proving our main result.

Lemma 4.1. *The following holds:*
(i) $J_k \to \{[k],[k],[k]\}$, *when* $k \equiv 0 (\text{mod } 4) \geq 8$.
(ii) $J_{4+k} \to \{[4,k],[4,k],[4,k]\}$, *when* $k \equiv 0 (\text{mod } 4) \geq 4$.

Proof. (i) $J_k \to \{[k],[k],[k]\}$
If $k \equiv 4 (\text{mod } 8), k \geq 20$, the construction is given by $L^{12} \oplus (\frac{k-20}{8})P \oplus R^8$. If $k \equiv 0 (\text{mod } 8) \geq 16$, the construction is given by $L^8 \oplus (\frac{k-16}{8})P \oplus R^8$. The construction for the remaining cases $J_8 \to \{[8],[8],[8]\}$ and $J_{12} \to \{[12],[12],[12]\}$ follows from Lemma 3.3(1,8).
(ii) $J_{4+k} \to \{[4,k],[4,k],[4,k]\}$
If $k \equiv 0 (\text{mod } 4) \geq 20$ and $k \not\equiv 0 (\text{mod } 8)$, the construction is given by $L^{4,4} \oplus (\frac{k-12}{8})P \oplus R^8$. If $k \equiv 0 (\text{mod } 8) \geq 16$, the construction is given by $L^{4,8} \oplus (\frac{k-16}{8})P \oplus R^8$. The remaining cases $J_8 \to \{[4,4],[4,4],[4,4]\}$ and $J_{12} \to \{[4,8],[4,8],[4,8]\}$ are given in Lemma 3.3(2 & 3). □

Theorem 4.2. *Suppose that m is an even integer and F is a bipartite 2-factor of order $2m$, with the provision that if $m \equiv 2$ (mod 4) and F is not a collection of 4-cycles, then $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$ has a 2-factorization into $\{H_1, H_2, H_3\}$, where $H_i \cong F$, $1 \leq i \leq 3$.*

Proof. Without loss of generality we may assume that the given 2-factor F can be decomposed into 2-regular subgraphs $F_1, F_2, \ldots, F_t$ such that each F_i is isomorphic to either $[4,k]$, $k \in \{4,8,12,16,\ldots,\}$ or $[k]$, $k \in \{8,12,16,20,\ldots,\}$. If $m \equiv 0 (\text{mod } 4)$ and $F \cong [4,4,\ldots,4]$, then F can be decomposed into copies of $[4,4]$. If some of the components of F are C_4, then F has a decomposition in which each F_i is isomorphic to either $[4,k]$ or $[k]$. If F has no C_4, then F has a decomposition in which each F_i is isomorphic to $[k]$. By Lemma 4.1, we have $J_k \to \{[k],[k],[k]\}$ and $J_{4+k} \to \{[4,k],[4,k],[4,k]\}$. Then by Lemmas 3.1 and 3.2, we get the required 2-factorization $\{H_1, H_2, H_3\}$ of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$. $\square$

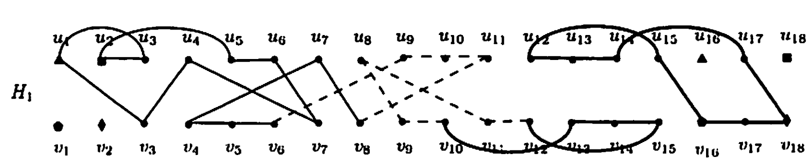
Lemma 4.3. *For $m \geq 4$, if $H_1 \cong H_3$ is a bipartite 2-regular graph of order $2m$ and H_2 is a cycle of length $2m$, then $J_{2m} \to \{H_1, H_2, H_3\}$ with the following possible exceptions: (i) H_1 is a C_4-factor (ii) at least two components of H_1 are C_4s and all other components are of order greater than and divisible by 4.*

Proof. If $H_1 \cong H_2$, then the proof follows by Lemma 4.1(i). So assume that $H_1 \ncong H_2$. We give a construction for $J_{2m} \to \{H_1, H_2, H_3\}$ based on the structure of H_1. Let p, q, r, s and t be positive integers. Then the order of the components of H_1 will be a combination of the following:

(i) $k_1, k_2, \ldots, k_p$, where $k_i \equiv 0 (\bmod\ 4) \geq 8,\ 1 \leq i \leq p$.
(ii) $k'_1, k'_2, \ldots, k'_q$, where $k'_i = 6,\ 1 \leq i \leq q$.
(iii) $k''_1, k''_2, \ldots, k''_r$, where $k''_i \equiv 2 (\bmod\ 8) \geq 10,\ 1 \leq i \leq r$.
(iv) $k'''_1, k'''_2, \ldots, k'''_s$, where $k'''_i \equiv 6 (\bmod\ 8)\ k'''_i \geq 14,\ 1 \leq i \leq s$.
(v) $k^{iv}_1, k^{iv}_2, \ldots, k^{iv}_t$, where $k^{iv}_i = 4,\ 1 \leq i \leq t$.

By the hypothesis, $H_1 \ncong [k^{iv}_1, k^{iv}_2, \ldots, k^{iv}_t]$ i.e., H_1 does not contain only 4-cycles. The number of possible types of H_1 is $\binom{4}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5} = 30$. Without loss of generality, assume that $k_1 \leq k_2 \leq \cdots \leq k_p$, $k''_1 \leq k''_2 \leq \cdots \leq k''_r$, $k'''_1 \leq k'''_2 \leq \cdots \leq k'''_s$. Now we construct the required cycle decomposition in all the 30 types as follows:

Type 1: $H_1 \cong [k_1, k_2, \ldots, k_p]$ where $p \geq 2$ and $k_i \equiv 0 (mod\ 4)\ \geq 8$.
Case 1: $k_p = 8$.
By our assumption $k_1 = k_2 = \cdots = k_p = 8$. If $p = 2$, then by Lemma 3.3(9), we have $J_{16} \to \{H_1, H_2, H_3\} \cong \{[8,8],[16],[8,8]\}$, i.e., $H_1 \cong H_3 \cong [8,8]$, $H_2 \cong [16]$. If $p \geq 3$, then the construction $\{L_1^{8,4} \oplus (p-3)C_1^{4,4} \oplus R_1^{4,8}\}$, gives our requirement to get $J_{k_1+k_2+\cdots+k_p} \to \{H_1, H_2, H_3\}$, where $H_1 \cong H_3 \cong [k_1, k_2, \ldots, k_p] \cong [8,8,\ldots,8]$, $H_2 \cong [k_1 + k_2 + \cdots + k_p] \cong [8+8+\cdots+8]$. For example $J_{32} \to \{[8,8,8,8],[32],[8,8,8,8]\}$, by the construction $\{L_1^{8,4} \oplus C_1^{4,4} \oplus R_1^{4,8}\}$, see Figure 4.1.



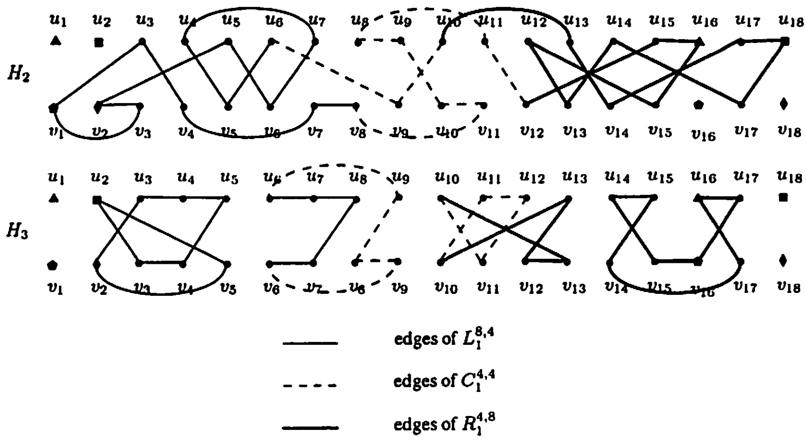


Figure 4.1. The graph $J_{32} = J_{12} \oplus J_8 \oplus J_{12}$

Case 2: $k_i = 8$ for some i and $k_{i+1} \geq 12,\ 1 \leq i \leq p-1$.
The construction for the case is $\{L_1^{8,4} \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus (\frac{k_3-z_3}{8})P \oplus M_3 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus R\}$, where

$$M_i = \begin{cases} C_1^{4,4}, & k_i \equiv 0 (\text{mod } 8) \\ C_1^{8,4}, & k_i \equiv 4 (\text{mod } 8), 2 \leq i \leq p-1, \end{cases}$$

$$z_i = \begin{cases} 8, & k_i \equiv 0 (\text{mod } 8) \\ 12, & k_i \equiv 4 (\text{mod } 8), 2 \leq i \leq p-1, \end{cases}$$

$$z_p = \begin{cases} 12, & k_p \equiv 4 (\text{mod } 8) \\ 16, & k_p \equiv 0 (\text{mod } 8), \end{cases}$$

$$\text{and } R = \begin{cases} R^{12}, & k_p \equiv 0 (\text{mod } 8) \\ R^8, & k_p \equiv 4 (\text{mod } 8). \end{cases}$$

Case 3: $k_1 \geq 12$. The construction for this case is
$\{L^8 \oplus (\frac{k_1-z_1}{8})P \oplus M_1 \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus R\}$,

$$\text{where } M_1 = \begin{cases} C_1^{4,4}, & k_1 \equiv 4 (\text{mod } 8) \\ C_1^{8,4}, & k_1 \equiv 0 (\text{mod } 8), \end{cases}$$

$$\text{and } z_1 = \begin{cases} 12, & k_1 \equiv 4 (\text{mod } 8) \\ 16, & k_1 \equiv 0 (\text{mod } 8). \end{cases}$$

The other terms $M_i, z_i,\ 2 \leq i \leq p$ and R are as in Case 2.

For future use, we denote

$$LS^{(1)} = \begin{cases} L_1^{8,4} \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus (\frac{k_3-z_3}{8})P \oplus M_3 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \\ \oplus M_{p-1}, \qquad k_1 = 8 \\ L^8 \oplus (\frac{k_1-z_1}{8})P \oplus M_1 \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \\ \oplus M_{p-1}, \qquad k_1 \geq 12. \end{cases}$$

Type 2: $H_1 \cong [k'_1, k'_2, \ldots, k'_q]$.
Since $k'_i = 6$, $1 \leq i \leq q$ and the order of H_1 is congruent to 0(mod 4) (by the definition of J_{2m}), q must be even. If $q = 2$, then by Lemma 3.3(4), we have $J_{12} \to \{[6,6],[12],[6,6]\}$. If $q \geq 4$, the construction is $\{L_1^{6,2} \oplus (\frac{q-4}{2})C_1^{4,6,2} \oplus R_1^{4,6,6}\}$.
Type 3: $H_1 \cong [k''_1, k''_2, \ldots, k''_r]$.
Since $k''_i \equiv 2(\text{mod } 8) \geq 10$, $1 \leq i \leq r$ and the order of H_1 is congruent to 0(mod 4), r must be even.
Case 1: $k''_r = 10$.
If $r = 2$, then by Lemma 3.3(11), we have $J_{20} \to \{[10,10],[20],[10,10]\}$. If $r \geq 4$, the construction is $\{L_1^{10,6} \oplus M''_2 \oplus M''_3 \oplus \cdots \oplus M''_{r-2} \oplus R_1^{6,10}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 2 \leq i \leq r-2. \end{cases}$$

Case 2: $k''_i = 10$ for some i and $k''_{i+1} \geq 18$, $1 \leq i \leq r-1$.
The construction is $\{L_1^{10,6} \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-18}{8})P \oplus R^{12}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 2 \leq i \leq r-1. \end{cases}$$

Case 3: $k''_1 \geq 18$. The construction is
$\{L^{12} \oplus (\frac{k''_1-18}{8})P \oplus M''_1 \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-18}{8})P \oplus R^{12}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 1 \leq i \leq r-1. \end{cases}$$

Type 4: $H_1 \cong [k'''_1, k'''_2, \ldots, k'''_s]$.
We observe that s is even. The construction is $\{L^8 \oplus (\frac{k'''_1-14}{8})P \oplus M'''_1 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 1 \leq i \leq s-1. \end{cases}$$

Type 5: $H_1 \cong [k_1, k_2, \ldots, k_p, k'_1, k'_2, \ldots, k'_q]$.
We observe that q is even.
Case 1: $k_i = 8$ for some i, $1 \leq i \leq p$.
The construction is
$\{L_1^{8,4} \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus M_p \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where z_i and M_i, $2 \leq i \leq p-1$ are as in Case 2 of Type 1.

$$\text{and } M_p = \begin{cases} C_1^{4,4}, & k_p \equiv 0 (\text{mod } 8) \\ C_1^{8,4}, & k_p \equiv 4 (\text{mod } 8) \end{cases}$$

Case 2: $k_1 \geq 12$. The construction is
$L^8 \oplus (\frac{k_1-z_1}{8})P \oplus M_1 \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus M_p \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where z_1 and M_1 are as in Case 3 of Type 1 and the other terms are as in previous case.

For future use, we denote

$$LS^{(5)} = \begin{cases} L_1^{8,4} \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \\ \oplus M_p, \qquad k_1 = 8 \\ L^8 \oplus (\frac{k_1-z_1}{8})P \oplus M_1(\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus \\ M_{p-1} \quad \oplus (\frac{k_p-z_p}{8})P \oplus M_p, \qquad k_1 \geq 12. \end{cases}$$

Type 6: $H_1 \cong [k_1, k_2, \ldots, k_p, k''_1, k''_2, \ldots, k''_r]$.
We observe that r is even.
Case 1: $k''_r = 10$. The construction is
$\{LS^{(5)} \oplus M''_1 \oplus M''_2 \oplus \cdots \oplus M''_{r-2} \oplus R_1^{6,10}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 1 \leq i \leq r-2. \end{cases}$$

Case 2: $k_r \geq 18$. The construction is
$\{LS^{(5)} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-18}{8})P \oplus R^{12}\}$,

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 1 \leq i \leq r-1. \end{cases}$$

Type 7: $H_1 \cong [k_1, k_2, \ldots, k_p, k'''_1, k'''_2, \ldots, k'''_s]$.
We observe that s is even. The construction for this case is
$\{LS^{(5)} \oplus (\frac{k'''_1-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_2-14}{8})P \oplus M'''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$,
where

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 2 \leq i \leq s-1. \end{cases}$$

Type 8: $H_1 \cong [4, k_1, k_2, \ldots, k_p]$.

Case 1: $k_p = 8$.

If $p = 1$, then by Lemma 3.3(6), we have $J_{12} \to \{[4,8],[12],[4,8]\}$. For $p \geq 2$, the construction is $\{L_1^{4,4} \oplus (p-2)C_1^{4,4} \oplus R_1^{4,8}\}$.

Case 2: $k_p \geq 12$. The construction for this case is $\{L_1^{4,4} \oplus (\frac{k_1-z_1}{8})P \oplus M_1 \oplus (\frac{k_2-z_2}{8})P \oplus M_2 \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus R\}$, where

$$M_i = \begin{cases} C_1^{4,4}, & k_i \equiv 0 (\mathrm{mod}\ 8) \\ C_1^{8,4}, & k_i \equiv 4 (\mathrm{mod}\ 8),\ 1 \leq i \leq p-1, \end{cases}$$

$$z_i = \begin{cases} 8, & k_i \equiv 0 (\mathrm{mod}\ 8) \\ 12, & k_i \equiv 4 (\mathrm{mod}\ 8),\ 1 \leq i \leq p-1, \end{cases}$$

$$z_p = \begin{cases} 12, & k_p \equiv 4 (\mathrm{mod}\ 8) \\ 16, & k_p \equiv 0 (\mathrm{mod}\ 8), \end{cases}$$

$$\text{and } R = \begin{cases} R^{12}, & k_p \equiv 0 (\mathrm{mod}\ 8) \\ R^{8}, & k_p \equiv 4 (\mathrm{mod}\ 8). \end{cases}$$

Type 9: $H_1 \cong [k'_1, k'_2, \ldots, k'_q, k''_1, k''_2, \ldots, k''_r]$.

We observe that $q + r$ is even. Hence q and r are of same parity.

Case 1: $r = 1$. The construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k''_1-10}{8})P \oplus R^8\}$.

Case 2: $k''_r = 10$ and both q and $r \geq 3$ are odd. The construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus C_1^{8,4} \oplus M''_2 \oplus M''_3 \oplus \cdots \oplus M''_{r-2} \oplus R_1^{6,10}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even},\ 2 \leq i \leq r-2. \end{cases}$$

Case 3: $k''_r \geq 18$ and both q and $r \geq 3$ are odd. The construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k''_1-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-18}{8})P \oplus R^{12}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even},\ 2 \leq i \leq r-1. \end{cases}$$

Case 4: $k''_r = 10$ and both q and r are even. The construction is $\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus M''_1 \oplus M''_2 \oplus \cdots \oplus M''_{r-2} \oplus R_1^{6,10}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd},\ 1 \leq i \leq r-2. \end{cases}$$

Case 5: $k''_r \geq 18$ and both q and r are even. The construction is $\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-18}{8})P \oplus R^{12}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, \ 1 \leq i \leq r-1. \end{cases}$$

Type 10: $H_1 \cong [k'_1, k'_2, \ldots, k'_q, k'''_1, k'''_2, \ldots, k'''_s]$.
We observe that $q+s$ is even. Hence q and s are of same parity.
Case 1: Both q and s are even. The construction for this case is $\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k'''_1-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_2-14}{8})P \oplus M'''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, \ 2 \leq i \leq s-1. \end{cases}$$

Case 2: Both q and s are odd. If $s = 1$, the construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k'''_1-14}{8})P \oplus R^{12}\}$. If $s \geq 3$, the construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k'''_1-6}{8})P \oplus M'''_1 \oplus (\frac{k'''_2-14}{8})P \oplus M'''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd}, \ 3 \leq i \leq s-1. \end{cases}$$

Type 11: $H_1 \cong [k'_1, k'_2, \ldots, k'_q, k^{iv}_1, k^{iv}_2, \ldots, k^{iv}_t]$.
We observe that q is even. The construction is $\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus (\lfloor \frac{t-1}{2} \rfloor)C_1^{2,4,2} \oplus R\}$, where

$$R = \begin{cases} R_1^{2,6}, & t \text{ is odd} \\ R_1^{2,6,4}, & t \text{ is even}. \end{cases}$$

Type 12: $H_1 \cong [k''_1, k''_2, \ldots, k''_r, k'''_1, k'''_2, \ldots, k'''_s]$.
We observe that $r+s$ is even. Hence r and s are of same parity.
Case 1: $k''_i = 10$ for some i, $1 \leq i \leq r$ and both r and s are even. Then the construction is $\{L_1^{10,6} \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k'''_1-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_2-14}{8})P \oplus M'''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, \ 2 \leq i \leq r, \end{cases}$$

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, \ 2 \leq i \leq s-1. \end{cases}$$

Case 2: $k_i'' = 10$ for some i, $1 \leq i \leq r$ and both r and s are odd. Then the construction is $\{L_1^{10,6} \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where M_i'' is as above and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd}, \ 1 \leq i \leq s-1. \end{cases}$$

Case 3: $k_1'' \geq 18$ for some i, $1 \leq i \leq r$ and both r and s are even. Then the construction is $\{L^{12} \oplus (\frac{k_1''-18}{8})P \oplus M_1'' \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, \ 1 \leq i \leq r \end{cases}$$

and M_i''' is as in Case 1.
Case 4: $k_1'' \geq 18$ for some i, $1 \leq i \leq r$ and both r and s are odd. Then the construction is $\{L^{12} \oplus (\frac{k_1''-18}{8})P \oplus M_1'' \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where M_i'' is as in previous case and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd}, \ 1 \leq i \leq s-1. \end{cases}$$

Type 13: $H_1 \cong [k_1'', k_2'', \ldots, k_r'', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that r is even. In this type, define

$$L = \begin{cases} L^8, & t \text{ is odd} \\ L_1^{4,8}, & t \text{ is even} \end{cases}$$

Case 1: $r = 2$. Then the construction is
$\{L \oplus (\frac{k_1''-10}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus \cdots \oplus (\frac{k_2''-10}{8})P \oplus R^8\}$.
Case 2: $k_r'' = 10$ and $r \geq 4$. Then the construction is
$\{L \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus C_1^{8,4} \oplus M_3'' \oplus \cdots \oplus M_{r-2}'' \oplus R_1^{6,10}\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, \ 3 \leq i \leq r-2. \end{cases}$$

Case 3: $k''_r \geq 18$ and $r \geq 4$. Then the construction is
$\{L^{\oplus}(\frac{k''_1-10}{8})P \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{k''_2-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k''_3-10}{8})P \oplus M''_3 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-18}{8})P \oplus R^{12}\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 3 \leq i \leq r-1. \end{cases}$$

<u>Type 14:</u> $H_1 \cong [k'''_1, k'''_2, \ldots, k'''_s, k^{iv}_1, k^{iv}_2, \ldots, k^{iv}_t]$.
We observe that s is even.
Case 1: $s = 2$. Then the construction is
$\{L^{12} \oplus (\frac{k'''_1-14}{8})P \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{k'''_2-14}{8})P \oplus R\}$, where

$$R = \begin{cases} R^{12}, & t \text{ is odd} \\ R_1^{12,4}, & t \text{ is even.} \end{cases}$$

Case 2: $s \geq 4$. Then the construction is
$\{L^8 \oplus (\frac{k'''_1-14}{8})P \oplus M'''_1 \oplus \cdots \oplus (\frac{k'''_{s-3}-14}{8})P \oplus M'''_{s-3} \oplus (\frac{k'''_{s-2}-14}{8})P \oplus C_1^{8,4} \oplus (\frac{k'''_{s-1}-6}{8})P \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{k'''_s-14}{8})P \oplus R\}$, where R is as above and

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 1 \leq i \leq s-3. \end{cases}$$

<u>Type 15:</u> $H_1 \cong [k_1, k_2, \ldots, k_p, k'_1, k'_2, \ldots, k'_q, k''_1, k''_2, \ldots, k''_r]$.
We observe that q and r are of same parity.
Case 1: Both q and r are even. The construction is
$\{LS^{(5)} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where

$$M''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{4,4}, & i \text{ is even, } 1 \leq i \leq r. \end{cases}$$

Case 2: Both q and r are odd. The construction is
$\{LS^{(1)} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_{r-1}-10}{8})P \oplus M''_{r-1} \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k_p-z_p}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where

$$M''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{4,4}, & i \text{ is even, } 1 \leq i \leq r-1, \end{cases}$$

$$M''_r = \begin{cases} C_1^{6,6}, & k_p \equiv 0 (\text{mod } 8) \\ C_1^{6,10}, & k_p \equiv 4 (\text{mod } 8), \end{cases}$$

$$\text{and } z_p = \begin{cases} 8, & k_p \equiv 0 (\text{mod } 8) \\ 12, & k_p \equiv 4 (\text{mod } 8). \end{cases}$$

Type 16: $H_1 \cong [k_1, k_2, \ldots, k_p, k'_1, k'_2, \ldots, k'_q, k'''_1, k'''_2, \ldots, k'''_s]$.
We observe that q and s are of same parity.
Case 1: Both q and s are even. The construction is
$\{LS^{(5)} \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus (\frac{k'''_1-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_2-14}{8})P \oplus M'''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$.

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 2 \le i \le s-1. \end{cases}$$

Case 2: Both q and s are odd. The construction for $s = 1$ is $\{LS^{(5)} \oplus (\frac{k'''_1-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$ and for $s \ge 3$ is $\{LS^{(5)} \oplus (\frac{k'''_1-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus (\frac{k'''_2-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_3-14}{8})P \oplus M'''_3 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd}, 3 \le i \le s-1. \end{cases}$$

Type 17: $H_1 \cong [k_1, k_2, \ldots, k_p, k'_1, k'_2, \ldots, k'_q, k^{iv}_1, k^{iv}_2, \ldots, k^{iv}_t]$.
We observe that q is even. The construction is
$\{LS^{(5)} \oplus (\frac{q-2}{2})C^{2,6,4} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus R\}$, where

$$R = \begin{cases} R_1^{2,6}, & t \text{ is even} \\ R_1^{2,6,4}, & t \text{ is odd} \end{cases}$$

Type 18: $H_1 \cong [k_1, k_2, \ldots, k_p, k''_1, k''_2, \ldots, k''_r, k'''_1, k'''_2, \ldots, k'''_s]$.
We observe that r and s are of same parity.
Case 1: Both r and s are odd. The construction is
$\{LS^{(5)} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k'''_1-14}{8})P \oplus M'''_1 \oplus (\frac{k'''_2-14}{8})P \oplus M'''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{4,4}, & i \text{ is even}, 1 \le i \le r. \end{cases}$$

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd}, 1 \le i \le s-1. \end{cases}$$

Case 2: Both r and s are even. The construction is
$\{LS^{(5)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where M_i'' is as in Case 1 and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 2 \le i \le s-1. \end{cases}$$

<u>Type 19:</u> $H_1 \cong [k_1, k_2, \ldots, k_p, k_1'', k_2'', \ldots, k_r'', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that r is even. The construction is
$\{LS^{(1)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_{r-1}''-10}{8})P \oplus M_{r-1}'' \oplus (\frac{k_p-z_p}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_r''-10}{8})P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{4,4}, & i \text{ is even}, 1 \le i \le r-2, \end{cases}$$

$$M_{r-1}'' = \begin{cases} C_1^{6,6}, & k_p \equiv 0 (\text{mod } 8) \\ C_1^{6,10}, & k_p \equiv 4 (\text{mod } 8), \end{cases}$$

$$z_p = \begin{cases} 8, & k_p \equiv 0 (\text{mod } 8) \\ 12, & k_p \equiv 4 (\text{mod } 8), \end{cases}$$

$$\text{and } R = \begin{cases} R^8, & t \text{ is odd} \\ R_1^{8,4}, & t \text{ is even.} \end{cases}$$

<u>Type 20:</u> $H_1 \cong [k_1, k_2, \ldots, k_p, k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that s is even.
Case 1: $s = 2$. The construction is
$\{LS^{(5)} \oplus (\frac{k_1'''-6}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_2'''-14}{8})P \oplus R\}$, where

$$R = \begin{cases} R^{12}, & t \text{ is odd} \\ R_1^{12,4}, & t \text{ is even.} \end{cases}$$

Case 2: $s \ge 4$. The construction is
$\{LS^{(5)} \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-3}'''-14}{8})P \oplus M_{s-3}''' \oplus (\frac{k_{s-2}'''-14}{8})P \oplus C_1^{8,4} \oplus (\frac{k_{s-1}'''-6}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where R is as above and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 2 \le i \le s-3. \end{cases}$$

Type 21: $H_1 \cong [k'_1, k'_2, \ldots, k'_q, k''_1, k''_2, \ldots, k''_r, k'''_1, k'''_2, \ldots, k'''_s]$.
Case 1: q, r and s are even. The construction is
$\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k'''_1-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_2-14}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 1 \le i \le r, \end{cases}$$

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 2 \le i \le s-1. \end{cases}$$

Case 2: q is even r and s are odd. The construction is
$\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k''_1-10}{8})P \oplus M''_1 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k'''_1-14}{8})P \oplus M'''_1 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where M''_i is as in Case 1 and

$$M'''_i = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd}, 1 \le i \le s-1. \end{cases}$$

Case 3: q and r are odd and s is even. The construction is
$\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k''_1-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k'''_1-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k'''_2-14}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where M'''_i is as in Case 1 and

$$M''_i = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even}, 2 \le i \le r. \end{cases}$$

Case 4: q and s are odd and r is even. The construction is
$\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k''_1-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k''_2-10}{8})P \oplus M''_2 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{k'''_1-14}{8})P \oplus M'''_1 \oplus \cdots \oplus (\frac{k'''_{s-1}-14}{8})P \oplus M'''_{s-1} \oplus (\frac{k'''_s-14}{8})P \oplus R^8\}$, where M''_i is as in case 3 and M'''_i is as in Case 2.

Type 22: $H_1 \cong [k'_1, k'_2, \ldots, k'_q, k''_1, k''_2, \ldots, k''_r, k^{iv}_1, k^{iv}_2, \ldots, k^{iv}_t]$.
We observe that q and r are of same parity.
Case 1: Both q and r are even. The construction is
$\{L \oplus (\frac{k''_1-10}{8})P \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{k''_2-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k''_3-10}{8})P \oplus M''_3 \oplus \cdots \oplus (\frac{k''_r-10}{8})P \oplus M''_r \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where

$$L = \begin{cases} L^8, & t \text{ is odd} \\ L_1^{4,8}, & t \text{ is even} \end{cases}$$

$$\text{and } M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 3 \le i \le r. \end{cases}$$

Case 2: Both q and r are odd.
If $r = 1$, then the construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1''-10}{8})P \oplus R\}$, where

$$R = \begin{cases} R^8, & t \text{ is even} \\ R_1^{8,4}, & t \text{ is odd.} \end{cases}$$

If $r \ge 3$ and $k_1'' = 10$, then the construction is $\{L_1^{10,6} \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_{r-1}''-10}{8})P \oplus M_{r-1}'' \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_r''-10}{8})P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 2 \le i \le r-1, \end{cases}$$

$$R = \begin{cases} R^8, & t \text{ is odd} \\ R_1^{8,4}, & t \text{ is even.} \end{cases}$$

If $r \ge 3$ and $k_1'' \ge 18$, then the construction is $\{L^{12} \oplus (\frac{k_1''-18}{8})P \oplus M_1'' \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_{r-1}''-10}{8})P \oplus M_{r-1}'' \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_r''-10}{8})P \oplus R\}$, where R is as above and

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 2 \le i \le r-1. \end{cases}$$

Type 23: $H_1 \cong [k_1', k_2', \ldots, k_q', k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that q and s are of same parity.
Case 1: Both q and s are odd.
If $s = 1$, then the construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1'''-14}{8})P \oplus R\}$, where

$$R = \begin{cases} R^{12}, & t \text{ is even} \\ R_1^{12,4}, & t \text{ is odd.} \end{cases}$$

If $s \ge 3$, then the construction is $\{L_1^{6,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1'''-6}{8})P \oplus C_1^{4,4} \oplus (\frac{k_2'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_3'''-14}{8})P \oplus M_3''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 3 \le i \le s-1, \end{cases}$$

$$\text{and } R = \begin{cases} R^8, & t \text{ is even} \\ R_1^{8,4}, & t \text{ is odd.} \end{cases}$$

Case 2: Both q and s are even. The construction is $\{L_1^{6,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus C_1^{4,4} \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1, \end{cases}$$

$$\text{and } R = \begin{cases} R^8, & t \text{ is even} \\ R_1^{8,4}, & t \text{ is odd.} \end{cases}$$

Type 24: $H_1 \cong [k_1'', k_2'', \ldots, k_r'', k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that r and s are of same parity. In this type, define

$$L = \begin{cases} L^8, & t \text{ is odd} \\ L_1^{4,8}, & t \text{ is even} \end{cases}$$

Case 1: Both r and s are odd.
If $r = s = 1$, then the construction is $\{L \oplus (\frac{k_1''-10}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1'''-14}{8})P \oplus R^{12}\}$.

If $r = 1$ and $s \ge 3$, then the construction is $\{L \oplus (\frac{k_1''-10}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1'''-6}{8})P \oplus C_1^{4,4} \oplus (\frac{k_2'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_3'''-14}{8})P \oplus M_3''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 3 \le i \le s-1. \end{cases}$$

If $r \ge 3$, then the construction is $\{L \oplus (\frac{k_1''-10}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_2''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_3''-10}{8})P \oplus M_3'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 3 \le i \le r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 1 \le i \le s-1. \end{cases}$$

Case 2: Both r and s are even. The construction is $\{L \oplus (\frac{k_1''-10}{8})P \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{k_2''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_3''-10}{8})P \oplus M_3'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 3 \le i \le r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1. \end{cases}$$

Type 25: $H_1 \cong [k_1, k_2, \ldots, k_p, k_1', k_2', \ldots, k_q', k_1'', k_2'', \ldots, k_r'', k_1''', k_2''', \ldots, k_s''']$.
Case 1: q, r and s are even. The construction is $\{LS^{(5)} \oplus (\frac{q}{2})C_1^{2,6,4} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 1 \le i \le r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1. \end{cases}$$

Case 2: q is even r and s are odd. The construction is $\{LS^{(5)} \oplus (\frac{q}{2})C_1^{2,6,4} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R^8\}$, where M_i'' is as in Case 1 and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 1 \le i \le s-1. \end{cases}$$

Case 3: q and r are odd and s is even. The construction is $\{LS^{(5)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-2}'''-14}{8})P \oplus M_{s-2}''' \oplus (\frac{k_{s-1}'''-14}{8})P \oplus C_1^{8,4} \oplus (\frac{k_s'''-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where M_i'' is as in Case 1 and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 1 \le i \le s-2. \end{cases}$$

Case 4: q and s are odd and r is even.
If $s = 1$, then the construction is $\{LS^{(5)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$. If $s \ge 3$, then the construction is

$\{LS^{(5)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-2}'''-14}{8})P \oplus M_{s-2}''' \oplus (\frac{k_{s-1}'''-14}{8})P \oplus C_1^{8,4} \oplus (\frac{k_s'''-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus R_1^{2,6}\}$, where M_i'' is as in case 1 and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even}, 2 \leq i \leq s-2. \end{cases}$$

Type 26: $H_1 \cong [k_1, k_2, \ldots, k_p, k_1', k_2', \ldots, k_q', k_1'', k_2'', \ldots, k_r'', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that q and r are of same parity.

Case 1: Both q and r are even. The construction is $\{LS^{(5)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus R\}$, where

$$R = \begin{cases} R_1^{2,6,4}, & t \text{ is odd} \\ R_1^{2,6}, & t \text{ is even} \end{cases}$$

$$\text{and } M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd}, 1 \leq i \leq r. \end{cases}$$

Case 2: Both q and r are odd. The construction is $\{LS^{(5)} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_{r-1}''-10}{8})P \oplus M_{r-1}'' \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_r''-10}{8})P \oplus R\}$, where

$$R = \begin{cases} R^8, & t \text{ is odd} \\ R_1^{8,4}, & t \text{ is even} \end{cases}$$

$$\text{and } M_i'' = \begin{cases} C_1^{4,4} & \text{if } i \text{ is even} \\ C_1^{6,6} & \text{if } i \text{ is odd}, 1 \leq i \leq r-1. \end{cases}$$

Type 27: $H_1 \cong [k_1.k_2, \ldots, k_p, k_1', k_2', \ldots, k_q', k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$.
We observe that q and s are of same parity. In this type, define

$$R = \begin{cases} R_1^{2,6}, & t \text{ is even} \\ R_1^{2,6,4}, & t \text{ is odd} \end{cases}$$

Case 1: Both q and s are odd.
If $s = 1$, then the construction is $\{LS^{(5)} \oplus (\frac{k_1'''-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus R\}$.

If $s \geq 3$, then the construction is $\{LS^{(5)} \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-2}'''-14}{8})P \oplus M_{s-2}''' \oplus (\frac{k_{s-1}'''-14}{8})P \oplus C_1^{8,4} \oplus (\frac{k_s'''-6}{8})P \oplus (\frac{q-1}{2})C_1^{2,6,4} \oplus$

$(\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus R\}$, where

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-2. \end{cases}$$

Case 2: Both q and s are even. The construction is $\{LS^{(5)} \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus C_1^{8,4} \oplus (\frac{q-2}{2})C_1^{2,6,4} \oplus (\lfloor \frac{t}{2} \rfloor)C_1^{2,4,2} \oplus R\}$, where

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1. \end{cases}$$

Type 28: $H_1 \cong [k_1, k_2, \ldots, k_p, k_1'', k_2'', \ldots, k_r'', k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$. We observe that r and s are of same parity. In this type, define

$$R = \begin{cases} R^8, & t \text{ is odd} \\ R_1^{8,4}, & t \text{ is even.} \end{cases}$$

Case 1: Both r and s are even. The construction is $\{LS^{(5)} \oplus (\frac{k_1'''-6}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even, } 2 \le i \le r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1. \end{cases}$$

Case 2: Both r and s are odd.
If $r = s = 1$, then the construction is $\{LS^{(5)} \oplus (\frac{k_1'''-6}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1''-10}{8})P \oplus R\}$.

If $r = 1$ and $s \ge 3$, then the construction is $\{LS^{(5)} \oplus (\frac{k_1'''-6}{8})P \oplus (\lfloor \frac{t+1}{2} \rfloor)C_1^{2,4,2} \oplus (\frac{k_1''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_2'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_3'''-14}{8})P \oplus M_3''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 3 \le i \le s-1. \end{cases}$$

If $r \geq 3$, then the construction is $\{LS^{(5)} \oplus (\frac{k_1'''-6}{8})P \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{k_1''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_2'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_3'''-14}{8})P \oplus M_3''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even, } 2 \leq i \leq r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 3 \leq i \leq s-1. \end{cases}$$

Type 29: $H_1 \cong [k_1', k_2', \ldots, k_q', k_1'', k_2'', \ldots, k_r'', k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$. In this type, define

$$R = \begin{cases} R^8, & t \text{ is even} \\ R_1^{8,4}, & t \text{ is odd} \end{cases}$$

Case 1: q, r and s are even. The construction is $\{L_1^{6,2} \oplus (\lfloor\frac{t}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 1 \leq i \leq r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \leq i \leq s-1. \end{cases}$$

Case 2: q is even r and s are odd. The construction is $\{L_1^{6,2} \oplus (\lfloor\frac{t}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{q-2}{2})C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k_1''-10}{8})P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where M_i'' is as in Case 1 and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 1 \leq i \leq s-1. \end{cases}$$

Case 3: q and r are odd and s is even. The construction is $\{L_1^{6,2} \oplus (\lfloor\frac{t}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k_1''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even, } 2 \leq i \leq r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1. \end{cases}$$

Case 4: q and s are odd and r is even. The construction is $\{L_1^{6,2} \oplus (\lfloor \frac{t}{2} \rfloor) C_1^{2,4,2} \oplus (\frac{q-1}{2}) C_1^{4,6,2} \oplus (\frac{k_1''-10}{8}) P \oplus C_1^{8,4} \oplus (\frac{k_2''-10}{8}) P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8}) P \oplus M_r'' \oplus (\frac{k_1'''-14}{8}) P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8}) P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8}) P \oplus R\}$, where M_i'' is as in Case 3 and M_i''' is as in Case 2.

<u>Type 30:</u> $H_1 \cong [k_1, k_2, \ldots, k_p, k_1', k_2', \ldots, k_q', k_1'', k_2'', \ldots, k_r'', k_1''', k_2''', \ldots, k_s''', k_1^{iv}, k_2^{iv}, \ldots, k_t^{iv}]$. In this type, define

$$R = \begin{cases} R^8, & t \text{ is odd} \\ R_1^{8,4}, & t \text{ is even} \end{cases}$$

Case 1: q, r and s are even. The construction is $\{LS^{(5)} \oplus (\lfloor \frac{t+1}{2} \rfloor) C_1^{2,4,2} \oplus (\frac{q-2}{2}) C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k_1''-10}{8}) P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8}) P \oplus M_r'' \oplus (\frac{k_1'''-14}{8}) P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8}) P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8}) P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8}) P \oplus R\}$, where

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is even} \\ C_1^{6,6}, & i \text{ is odd, } 1 \le i \le r, \end{cases}$$

$$\text{and } M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is odd} \\ C_1^{8,8}, & i \text{ is even, } 2 \le i \le s-1. \end{cases}$$

Case 2: q is even r and s are odd. The construction is $\{LS^{(5)} \oplus (\lfloor \frac{t+1}{2} \rfloor) C_1^{2,4,2} \oplus (\frac{q-2}{2}) C_1^{4,6,2} \oplus C_1^{4,4} \oplus (\frac{k_1''-10}{8}) P \oplus M_1'' \oplus \cdots \oplus (\frac{k_r''-10}{8}) P \oplus M_r'' \oplus (\frac{k_1'''-14}{8}) P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8}) P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8}) P \oplus R\}$, where M_i'' is as in Case 1 and

$$M_i''' = \begin{cases} C_1^{6,6}, & i \text{ is even} \\ C_1^{8,8}, & i \text{ is odd, } 1 \le i \le s-1. \end{cases}$$

Case 3: q and r are odd and s is even. The construction is $\{LS^{(5)} \oplus (\lfloor \frac{t+1}{2} \rfloor) C_1^{2,4,2} \oplus (\frac{q-1}{2}) C_1^{4,6,2} \oplus (\frac{k_1''-10}{8}) P \oplus C_1^{8,4} \oplus (\frac{k_2''-10}{8}) P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8}) P \oplus M_r'' \oplus (\frac{k_1'''-14}{8}) P \oplus C_1^{10,6} \oplus (\frac{k_2'''-14}{8}) P \oplus M_2''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8}) P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8}) P \oplus R\}$, where M_i''' is as in Case 1 and

$$M_i'' = \begin{cases} C_1^{4,4}, & i \text{ is odd} \\ C_1^{6,6}, & i \text{ is even, } 2 \le i \le r. \end{cases}$$

Case 4: q and s are odd and r is even. The construction is $\{LS^{(5)} \oplus (\lfloor\frac{t+1}{2}\rfloor)C_1^{2,4,2} \oplus (\frac{q-1}{2})C_1^{4,6,2} \oplus (\frac{k_1''-10}{8})P \oplus C_1^{8,4} \oplus (\frac{k_2''-10}{8})P \oplus M_2'' \oplus \cdots \oplus (\frac{k_r''-10}{8})P \oplus M_r'' \oplus (\frac{k_1'''-14}{8})P \oplus M_1''' \oplus \cdots \oplus (\frac{k_{s-1}'''-14}{8})P \oplus M_{s-1}''' \oplus (\frac{k_s'''-14}{8})P \oplus R\}$, where M_i'' is as in Case 3 and M_i''' is as in Case 2. □

Lemma 4.4. *For $m \geq 4$, if H_1 is a bipartite 2-regular graph of order $2m$ and $H_2 \cong H_3$ is a cycle of length $2m$, then $J_{2m} \to \{H_1, H_2, H_3\}$ with the following possible exceptions: (i) H_1 is a C_4-factor or (ii) more than one component of H_1 is C_4 and all other components are of order $r \equiv 0 (\mathrm{mod}\ 4) > 4$.*

Proof. Replacing the terms $L_1^{a,b}, C_1^{a,b}, C_1^{a,b,c}, R_1^{a,b}, R_1^{a,b,c}$ in the proof of Lemma 4.3 by $L_2^{a,b}, C_2^{a,b}, C_2^{a,b,c}, R_2^{a,b}, R_2^{a,b,c}$ respectively, we get the required decomposition. □

Theorem 4.5. *There exists a 2-factorization $\{H_1, H_2, H_3\}$ of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$ such that (i) $H_1 \cong H_3$ is a bipartite 2-factor (non isomorphic to C_4-factor) (ii) H_2 is a Hamilton cycle .*

Proof. We get the required 2-factorization of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$, by Lemmas 3.2 and 4.3, except the case when at least two components of H_1 are of order 4 and at least one component is of order greater than and divisible by 4. The idea of the proof for the remaining cases is as follows. First we decompose $J_{2m} \mapsto \{a, b_1, b_2, \ldots, b_k, c\ ;\ c_1, c_2\ ;\ a, b_1, b_2, \ldots, b_k, c\}$, then by contracting the vertices u_1 with u_m, u_2 with u_{m+2}, v_1 with v_m, and v_2 with v_{m+2} in J_{2m}, we get the required 2-factorization of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$. Without loss of generality, we have $H_1 \cong H_3 \cong [4, \ldots, 4, k_1, k_2, \ldots, k_p]$, where $p \neq 0$ and $k_i \equiv 0(\mathrm{mod}\ 4) \geq 8$. Let t be the number of C_4 in H_1. For convenience we write $k_j = 8$, for some j, $0 \leq j \leq p$. We consider the remaining proof in 4 cases.
Case 1: t even, j odd: Then the construction is $\{(\frac{t-2}{2})C_1^{2,4,2} \oplus LR_1^{2,4,4,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus M_p \oplus (\frac{j-1}{2})C_1^{2,8,6}\}$, where

$$M_i = \begin{cases} C_1^{6,6}, & k_i \equiv 4(\mathrm{mod}\ 8) \\ C_1^{10,6}, & k_i \equiv 0(\mathrm{mod}\ 8),\ j+1 \leq i \leq p, \end{cases}$$

$$\text{and } z_i = \begin{cases} 12, & k_i \equiv 4(\mathrm{mod}\ 8) \\ 16, & k_i \equiv 0(\mathrm{mod}\ 8). \end{cases}$$

For example to get $H_1 \cong H_3 \cong [4, 4, 4, 4, 8]$ and $H_2 \cong [24]$ in $\langle E_0, E_1, E_2\rangle_{6,6} \otimes \overline{K}_2$, by the construction $C_1^{2,4,2} \oplus LR_1^{2,4,4,6}$, first we decompose $J_{24} \mapsto \{2, 4, 4, 4, 4, 6\ ;\ 18, 6\ ;\ 2, 4, 4, 4, 4, 6\}$, then we contract the vertices u_1 with u_{12}, u_2 with u_{14}, v_1 with v_{12} and v_2 with v_{14}, see Figure 4.2.

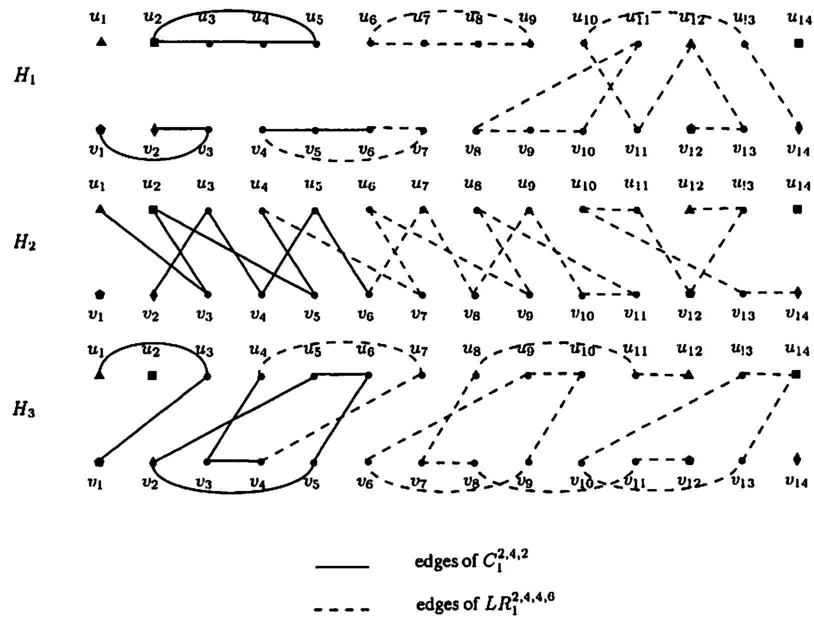


Figure 4.2. The graph $J_{24} = J_8 \oplus J_{16}$

Case 2: t even, $j \neq p$ even: Then the construction is $\{(\frac{t-2}{2})C_1^{2,4,2} \oplus LR_1^{2,4,4,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p - z_p}{8})P \oplus (\frac{j}{2})C_1^{2,8,6}\}$, where

$$M_i = \begin{cases} C_1^{6,6}, & k_i \equiv 4 (\text{mod } 8) \\ C_1^{10,6}, & k_i \equiv 0 (\text{mod } 8),\ j+1 \leq i \leq p-2, \end{cases}$$

$$z_i = \begin{cases} 12, & k_i \equiv 4 (\text{mod } 8) \\ 16, & k_i \equiv 0 (\text{mod } 8), \end{cases}$$

$$M_{p-1} = \begin{cases} C_1^{6,6}, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ C_1^{6,10}, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \\ C_1^{10,6}, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ C_1^{10,2}, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \end{cases}$$

$$\text{and } z_p = \begin{cases} 8, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ 12, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \\ 8, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ 4, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \end{cases}$$

If $j = p$, the construction is $\{(\frac{t-2}{2})C_1^{2,4,2} \oplus LR_1^{2,4,4,8,6} \oplus (\frac{j-2}{2})C_1^{2,8,6}\}$.
Case 3: t odd, $j \neq p$ odd: Then the construction is $\{(\frac{t-1}{2})C_1^{2,4,2} \oplus LR_1^{2,4,8,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus (\frac{j-1}{2})C_1^{2,8,6}\}$, where M_i and z_i are as in Case 2. If $j = p$, the construction is $\{(\frac{t-3}{2})C_1^{2,4,2} \oplus LR_1^{2,4,4,8,2} \oplus (\frac{j-1}{2})C_1^{2,8,6}\}$.
Case 4: t odd, $j \neq 0$ even: The construction is $\{(\frac{t-1}{2})C_1^{2,4,2} \oplus LR_1^{2,4,8,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus M_p \oplus (\frac{j-1}{2})C_1^{2,8,6}\}$, where M_i and z_i are as in Case 1. If $j = 0$ and $z_p \neq 12$, then the construction is $\{(\frac{t-3}{2})C_1^{2,4,2} \oplus LR_1^{2,4,4,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus M_p\}$, where

$$M_i = \begin{cases} C_1^{6,6}, & k_i \equiv 4 \pmod 8 \\ C_1^{10,6}, & k_i \equiv 0 \pmod 8), j+1 \leq i \leq p-1, \end{cases}$$

$$z_i = \begin{cases} 12, & k_i \equiv 4 \pmod 8 \\ 16, & k_i \equiv 0 \pmod 8, \end{cases}$$

$$M_p = \begin{cases} C_1^{10,2}, & k_i \equiv 0 \pmod 8 \\ C_1^{14,2}, & k_i \equiv 4 \pmod 8, \end{cases}$$

$$\text{and } z_p = \begin{cases} 16, & k_i \equiv 0 \pmod 8 \\ 20, & k_i \equiv 4 \pmod 8. \end{cases}$$

If $j = 0$ and $z_p = 12$, then the construction is $\{(\frac{t-1}{2})C_1^{2,4,2} \oplus LR_1^{2,4,10} \oplus (p-1)C_1^{2,10}\}$. □

Theorem 4.6. *There exists a 2-factorization $\{H_1, H_2, H_3\}$ of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$ such that (i) H_1 is a bipartite 2-factor (ii) H_2 and H_3 are Hamilton cycles.*

Proof. By Lemmas 3.2 and 4.4, we get the required 2-factorization of $\langle E_0, E_1, E_2\rangle_{\frac{m}{2},\frac{m}{2}} \otimes \overline{K}_2$, except when (i) H_1 is a C_4-factor (ii) at least two components of H_1 is of order 4 and all other components has order greater than and divisible by 4. Also, we get the required factorization of the missing cases for smaller values of m, from Lemma 3.3. Following is the construction for the missing cases for larger values of m which is similar to the one given in Theorem 4.5. Without loss of generality, let $H_1 \cong [4, \ldots, 4, k_1, k_2, \ldots, k_p]$, where each $k_i \equiv 0 \pmod 4) \geq 8$ and $p \geq 0$. Let t be the number of C_4 in H_1. For convenience, we write $k_j = 8,\ 0 \leq j \leq p$. We consider the remaining proof in 2 cases.

Case 1: $p = 0$. We observe that $t \geq 3$. If t is odd, the construction is $\{LR_2^{2,4,4,2} \oplus (\frac{t-3}{2})C_2^{2,4,2}\}$ and if t is even, the construction is $\{LR_2^{2,4,4,4,2} \oplus (\frac{t-4}{2})C_2^{2,4,2}\}$.
Case 2: $p \geq 1$. Then we consider the following subcases.
Subcase (i):t even, j odd: Then the construction is $\{(\frac{t-2}{2})C_2^{2,4,2} \oplus LR_2^{2,4,4,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus M_p \oplus (\frac{j-1}{2})C_2^{2,8,6}\}$, where

$$M_i = \begin{cases} C_2^{6,6}, & k_i \equiv 4 (\text{mod } 8) \\ C_2^{10,6}, & k_i \equiv 0 (\text{mod } 8), \ j+1 \leq i \leq p, \end{cases}$$

$$\text{and } z_i = \begin{cases} 12, & k_i \equiv 4 (\text{mod } 8) \\ 16, & k_i \equiv 0 (\text{mod } 8). \end{cases}$$

Subcase (ii): Both t and j are odd: Replace $LR_2^{2,4,4,6}$ by $LR_2^{2,4,6}$ in Subcase (i), to get the required construction.
Subcase (iii): Both t and $j \neq p$ are even: The construction is $\{(\frac{t-2}{2})C_2^{2,4,2} \oplus LR_2^{2,4,4,6} \oplus (\frac{k_{j+1}-z_{j+1}}{8})P \oplus M_{j+1} \oplus \cdots \oplus (\frac{k_{p-1}-z_{p-1}}{8})P \oplus M_{p-1} \oplus (\frac{k_p-z_p}{8})P \oplus (\frac{j}{2})C_2^{2,8,6}\}$, where

$$M_i = \begin{cases} C_2^{6,6}, & k_i \equiv 4 (\text{mod } 8) \\ C_2^{10,6}, & k_i \equiv 0 (\text{mod } 8), \ j+1 \leq i \leq p-2, \end{cases}$$

$$z_i = \begin{cases} 12, & k_i \equiv 4 (\text{mod } 8) \\ 16, & k_i \equiv 0 (\text{mod } 8), \end{cases}$$

$$M_{p-1} = \begin{cases} C_2^{6,6}, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ C_2^{6,10}, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \\ C_2^{10,6}, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ C_2^{10,2}, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \end{cases}$$

$$\text{and } z_p = \begin{cases} 8, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ 12, & k_{p-1} \equiv 4 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \\ 8, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 0 (\text{mod } 8) \\ 4, & k_{p-1} \equiv 0 (\text{mod } 8) \text{ and } k_p \equiv 4 (\text{mod } 8) \end{cases}$$

If $j = p$, then the construction is $\{(\frac{t-2}{2})C_2^{2,4,2} \oplus LR_2^{2,4,4,8,6} \oplus (\frac{j-2}{2})C_2^{2,8,6}\}$.
Subcase (iv): t odd, $j \neq p$ even: Replace $LR_2^{2,4,4,6}$ by $LR_2^{2,4,6}$ in Subcase 3, to get the required construction. If $j = p$, then the construction is $\{(\frac{t-3}{2})C_2^{2,4,2} \oplus LR_2^{2,4,4,2} \oplus (\frac{j}{2})C_2^{2,8,6}\}$. □

Theorem 4.7. *Let $n \geq 3$ and $0 \leq j \leq n-1$. The graph $\langle E_j, E_{j+1}, E_{j+2}\rangle_{n,n} \otimes \overline{K}_2$, has a factorization into three 2-factors such that (i) two of them are isomorphic to a given 2-factor (which is non isomorphic to a C_4-factor) of $K_{2n,2n}$ and one is a Hamilton cycle, (ii) one of them is isomorphic to a given 2-factor of $K_{2n,2n}$ and two of them are Hamilton cycles, (iii) all the three are isomorphic to a given 2-factor of $K_{2n,2n}$ with components of order divisible by 4 (non isomorphic to a C_4-factor, when n is odd).*

Proof. Follows from Theorems 4.2, 4.5 and 4.6, by taking $m = 2n$, since $\langle E_0, E_1, E_2\rangle_{n,n} \cong \langle E_j, E_{j+1}, E_{j+2}\rangle_{n,n}$, for any j, $0 \leq j \leq n-1$. □

Remark 4.8. *Existence of 2-factorization of $\langle E_j, E_{j+1}, E_{j+2}\rangle_{n,n} \otimes \overline{K}_2$, such that (i) two of the 2-factors are C_4-factors and one is a Hamilton cycle or (ii) all of the 2-factors are C_4-factors, when n odd, is unknown.*

5 Bipartite Hamilton-Waterloo Problem

As a consequence of our results in the previous sections, we show the existence of Bipartite Hamilton-Waterloo Problem, when F_2 is a refinement of F_1, with few exceptions.

Theorem 5.1. *Suppose that F_1 and F_2 are bipartite 2-factors of order 4n, with F_2 a refinement of F_1 and no component of F_1 is a C_4 or C_6, then $(\alpha, \beta) \in BHWP(2n, 2n; F_1, F_2)$ whenever $\alpha + \beta = n$, except possibly when $\alpha = 1$ and (i) F_2 is a C_4-factor or (ii) F_2 has more than one C_4 with all other components of an order $r \equiv 0 (\mathrm{mod}\ 4) > 4$ or (iii) F_2 has components with an order $r \equiv 2(\mathrm{mod}\ 4)$, when n is even.*

Proof. Case 1. n odd: For convenience we write $K_{2n,2n} = (\langle E_0, E_1, E_2\rangle_{n,n} \oplus \langle E_3, E_4\rangle_{n,n} \oplus \langle E_5, E_6\rangle_{n,n} \oplus \cdots \oplus \langle E_{n-2}, E_{n-1}\rangle_{n,n}) \otimes \overline{K}_2$. By taking each component of F_1 as H_2 in Lemmas 4.3 & 4.4 and by applying Lemmas 3.1 & 3.2, we get a factorization of $\langle E_0, E_1, E_2\rangle_{n,n} \otimes \overline{K}_2$, into three 2-factors H_1', H_2' and H_3' such that $H_1' \cong F_2$, $H_2' \cong F_1$ and H_3' is isomorphic to either F_1 or F_2 as required. Further by Lemma 2.1, $\langle E_j, E_{j+1}\rangle_{n,n} \otimes \overline{K}_2$, has a decomposition into two 2-factors isomorphic to a given 2-factor of $K_{2n,2n}$.
Case 2. n even: For convenience we write $K_{2n,2n} = (\langle E_0, E_1, E_2\rangle_{n,n} \oplus \langle E_3, E_4, E_5\rangle_{n,n} \oplus \langle E_6, E_7\rangle_{n,n} \oplus \langle E_8, E_9\rangle_{n,n} \oplus \cdots \oplus \langle E_{n-2}, E_{n-1}\rangle_{n,n}) \otimes \overline{K}_2$. By Theorem 4.2 and Lemmas 3.1,3.2,4.3 & 4.4, we get the required factorization of the graphs $\langle E_0, E_1, E_2\rangle_{n,n} \otimes \overline{K}_2$ and $\langle E_3, E_4, E_5\rangle_{n,n} \otimes \overline{K}_2$, as in Case 1. Further by Lemma 2.1, $\langle E_j, E_{j+1}\rangle_{n,n} \otimes \overline{K}_2$, has a decomposition into two 2-factors isomorphic to a given 2-factor of $K_{2n,2n}$. □

Theorem 5.2. *If F_1 is a Hamilton cycle and F_2 is any 2-factor of $K_{2n,2n}$, then $(\alpha, \beta) \in BHWP(2n, 2n; F_1, F_2)$, except possibly the case $\alpha = 1$ when* *(i) F_2 is a C_4-factor, where n is odd or (ii) F_2 is a C_4-factor or order of the components of F_2 are congruent to 2(mod 4), when n is even.*

Proof. Follows from Theorems 4.7 and 5.1. □

Lemma 5.3. $(1,1) \notin BHWP(4,4;[4,4],[8])$,

Proof. The result follows immediately as the graph $K_{4,4}$ cannot be decomposed into a Hamilton cycle and a C_4-factor. □

In the sense of non-existence, we also prove the following.

Lemma 5.4. $(2,1) \notin BHWP(6,6;[4,4,4],[12])$.

Proof. We prove the result by contradiction. Assume that $K_{6,6}$ has a 2-factorization $\{F,G,H\}$, such that F and G are C_4-factors and H is a Hamilton cycle. Let $U = \{u_1,u_2,u_3,u_4,u_5,u_6\}$ and $V = \{v_1,v_2,v_3,v_4,v_5,v_6\}$ be the partite sets of $K_{6,6}$. Consider $u_1 \in U$. Let v_1 and v_2 be the neighbors of u_1 in G. Since G is a C_4-factor, there is another vertex, say $u_2 \in U$ such that $N_G(u_2) = \{v_1,v_2\}$. Let $N_F(u_2) = \{v_3,v_4\}$. If $N_F(u_1) = \{v_3,v_4\}$, then H can not be a Hamilton cycle. Therefore, there is some vertex, say $u_3 \in U$, such that $N_F(u_3) = \{v_3,v_4\}$. Since $N_G(u_1) = \{v_1,v_2\}$ and $N_F(u_1) = \{v_5,v_6\}$, we have $N_H(u_1) = \{v_3,v_4\}$. Now $N_G(u_3) = \{v_5,v_6\}$ and there is some vertex, say $u_4 \in U$, such that (u_3,v_5,u_4,v_6) is a cycle in G. All these facts imply that v_3 and v_4 will be adjacent to u_4 in H, contradicting the fact that H is a hamilton cycle. Hence $K_{6,6}$ can not have a such 2-factorization. □

The possible exceptions in Theorems 5.1 and 5.2 and actual exceptions in Lemmas 5.3 and 5.4 leads one to wonder whether it is ever possible to have a decomposition of $K_{2n,2n}$ into a single Hamilton cycle and C_4-factors.

We conjecture that Lemmas 5.3 and 5.4 are the only exceptions.
Conjecture: For every $n \geq 4$, $(1,n-1) \in BHWP(2n,2n;H,F)$, where H is a Hamilton cycle of $K_{2n,2n}$ and F is a C_4-factor of $K_{2n,2n}$.

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