On the Upper Bound for the Number of Spanning Trees of a Connected Graph

Rao Li
Dept. of Mathematical Sciences
University of South Carolina Aiken
Aiken, SC 29801
Email: raol@usca.edu

Abstract

Das [4], Feng et al. [5], and Li et al. [13] obtained upper bounds for the number of spanning trees of a connected graph. Using some ideas in [4], [5], and [13] and other established results, we obtain new upper bounds for the number of spanning trees of a connected graph.

Keywords : *Spanning Tree, Laplacian Eigenvalue.*

Received November 10, 2012. *Accepted October* 19, 2013.

1. Introduction

We consider only finite, undirected, connected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let G be a graph of order n. We use G^c to denote the complement of G. We assume that the vertices in G are ordered such that $d_1 \geq d_2 \geq ... \geq d_n$, where d_i, $1 \leq i \leq n$, is the degree of vertex v_i in G. For vertex v_i, we use $N(v_i)$ to denote its neighbors. Grone and Merris in [8] introduced the definition of $d_i^*(G)$. They defined $d_i^*(G)$ as $|\{ v \in V(G) : d(v) \geq i \}|$. For two distinct vertices u and v in G, we define $c_{u,v}$ as $|N(u) \cap N(v)|$; if $d(u) = d_1$ and $d(v) = d_2$, we further define

$$a_{u,v} := \begin{cases} \frac{d_2+2+\sqrt{(d_2+2)^2-8d_2+4c_{u,v}}}{2} & if\ uv \in E(G), \\ \frac{d_2+1+\sqrt{(d_2+1)^2-4c_{u,v}}}{2} & if\ uv \notin E(G). \end{cases}$$

and $a := \max\{ a_{u,v} : \text{where } d(u) = d_1, d(v) = d_2, \text{ and } u \neq v \}$. Notice that $a_{u,v}$ follows from [3]. We use t to denote the number of spanning

JCMCC **95** (2015), pp. 257-269

trees of a labeled graph G. $G \vee H$ is defined as the join of the vertex-disjoint graphs of G and H. The Laplacian of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of G and $A(G)$ is the adjacency matrix of G. The eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G) = 0$ of $L(G)$ are called the Laplacian eigenvalues of the graph G. We use S_n to denote the star graph $K_{1,n-1}$. We also use $K_n - e$ to denote the graph obtained by removing one edge e from K_n. Finally, we use $S_n + e$ to denote the graph obtained by adding one edge e to $K_{1,n-1}$.

Using the Matrix Tree Theorem, i.e., $t\,n = \lambda_1 \lambda_2 ... \lambda_{n-1}$, several authors obtained upper bounds for the number of spanning trees of a graph. In particular, Das in [4] proved the following theorem.

Theorem A Let G be a connected graph with n vertices and e edges. Then

$$t \leq \left(\frac{2e - d_1 - 1}{n-2}\right)^{n-2}.$$

Equality holds if and only if G is K_n or S_n.

Feng et al. in [5] proved the following theorem.

Theorem B Let G be a connected graph with n vertices and e edges. Then

$$t \leq \left(\frac{d_1 + 1}{n}\right)\left(\frac{2e - d_1 - 1}{n-2}\right)^{n-2}.$$

Equality holds if and only if G is K_n or S_n.

Li et al. in [13] proved the following theorem.

Theorem C Let G be a non–complete connected graph with $n \geq 4$ vertices and e edges. Then

$$t \leq d_n \left(\frac{2e - d_1 - 1 - d_n}{n-3}\right)^{n-3}.$$

Equality holds if and only if G is S_n, $K_n - e$, or $K_1 \vee (K_1 \cup K_{n-2})$.

Notice that Theorem C is a corrected version of Theorem 3.1 in [13]. Using some ideas in [4], [5], and [13], we in this note will present a new upper bound for the number of spanning trees of a connected graph of order $n \geq 5$.

Theorem 1 Let G be a non-complete connected graph with $n \geq 5$ vertices and e edges. If $G \neq S_n$, then

$$t \leq (n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4},$$

with equality if and only if G is $K_n - e$, $K_2 \vee K_{n-2}^c$, $K_2 \vee S_{n-2}^c$, $K_1 \vee S_{n-1}^c$, $S_n + e$, or $K_1 \vee (K_1 \cup S_{n-2})$.

2. Proofs of Theorem 1

In order to prove Theorem 1, we need several established results. Theorem 2 below is Lemma 13.1.3 on Page 280 in [7].

Theorem 2 Let G be a graph of order n. Then $\lambda_i(G) = n - \lambda_{n-i}(G^c)$ for each i with $1 \leq i \leq n-1$. In particular, $\lambda_1(G) \leq n$.

The following Theorem 3 was proven by Fiedler in [6].

Theorem 3 Let G be a non-complete graph of order n. Then $\lambda_{n-1}(G) \leq \kappa(G) \leq \kappa'(G) \leq d_n$, where $\kappa(G)$ and $\kappa'(G)$ are the connectivity and edge-connectivity of G, respectively.

Kirkland in [10] and Li and Fan in [11] characterized the graphs with $\lambda_{n-1}(G) = \kappa(G)$. Their results are stated in Theorem 4 below.

Theorem 4 Let G be a graph of order n with $1 \leq \kappa(G) \leq n-2$. Then $\lambda_{n-1}(G) = \kappa(G) = k$ if and only if there exists a vertex subset $S \subset V(G)$ with $|S| = k$, such that $G = G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, $m \geq 2$, and $\kappa(G[S]) \geq 2k - n$ if $\lfloor \frac{n}{2} \rfloor < k \leq n-2$, where G_1, G_2, ..., G_m are the components of $G[V(G) - S]$.

Theorem 5 below was proven by Grone and Merris in [8].

Theorem 5 Let G be a graph containing at least one edge. Then $\lambda_1(G) \geq d_1(G) + 1$. Moreover, if G is connected on $n > 1$ vertices, the equality holds if and only if $d_1(G) = n - 1$.

The statement in Theorem 6 was conjectured by Grone and Merris in [8]. Bai in [1] proved that the conjecture is true.

Theorem 6 Let G be a graph of order n. Then $\sum_{i=1}^{k} \lambda_i(G) \leq \sum_{i=1}^{k} d_i^*(G)$ for each k with $1 \leq k \leq n$.

Both Theorem 7 and Theorem 8 below were proven by Das in [4].

Theorem 7 Let G be a simple connected graph of order n. Then $\lambda_2(G) = \lambda_3(G) = ... = \lambda_{n-1}(G)$ if and only if G is K_n or S_n or K_{d_1,d_1}.

Theorem 8 Let G be a simple connected graph of order n. Then $\lambda_1(G) = \lambda_2(G) = ... = \lambda_{n-2}(G)$ if and only if G is K_n or $K_n - e$.

Theorem 9 below follows from results obtained by Das in [3].

Theorem 9 Let G be a simple connected graph of order $n > 2$. If G is not S_n, then $\lambda_2 \geq a$.

Now we will prove Theorem 1.

Proof of Theorem 1. Let G be a non-complete graph connected graph with $n \geq 5$ vertices and e edges. Suppose that $G \neq S_n$. We assume that x and y are two distinct vertices in G such that $a_{x,y} = a$ with $d(x) = d_1$ and $d(y) = d_2$. Notice first that $0 \leq c_{x,y} \leq d_2 - 1$ if $xy \in E$, and $0 \leq c_{x,y} \leq d_2$ if $xy \notin E$. By the definition of a, we can easily verify that $d_2 \leq a \leq d_2 + 1$.

From Theorem 6, we have that $\lambda_1 + \lambda_2 \leq d_1^* + d_2^* = n + d_2^*$. Theorem 5 then implies that $\lambda_2 \leq n + d_2^* - d_1 - 1$.

From Theorem 2, Theorem 9, and the AM–GM inequality, we have that

$$t = \frac{1}{n}\prod_{i=1}^{n-1}\lambda_i = \frac{\lambda_1\lambda_2\lambda_{n-1}}{n}\prod_{i=3}^{n-2}\lambda_i$$

$$\leq (n + d_2^* - d_1 - 1)\lambda_{n-1}\left(\frac{\sum_{i=3}^{n-2}\lambda_i}{n-4}\right)^{n-4}$$

$$\leq (n + d_2^* - d_1 - 1)\lambda_{n-1}\left(\frac{2e - \lambda_1 - \lambda_2 - \lambda_{n-1}}{n-4}\right)^{n-4}$$

$$\leq (n + d_2^* - d_1 - 1)\lambda_{n-1}\left(\frac{2e - d_1 - 1 - a - \lambda_{n-1}}{n-4}\right)^{n-4}.$$

Now consider the function

$$f(z) = z\left(\frac{2e - d_1 - 1 - a - z}{n-4}\right)^{n-4},$$

where $0 < z \le d_n$. It is easy to see that

$$f'(z) = \frac{(2e - d_1 - 1 - a - z)^{n-5}[2e - d_1 - 1 - a - (n-3)z]}{(n-4)^{n-4}}.$$

Notice first that $2e - d_1 - 1 - a - z \ge d_1 + d_2 + d_3 + d_4 + \cdots + d_{n-1} + d_n - d_1 - 1 - d_2 - 1 - d_n \ge 0$.

If $d_3 \ge 2$, then $2e - d_1 - 1 - a - (n-3)z \ge d_1 + d_2 + d_3 + d_4 + \cdots + d_{n-1} + d_n - d_1 - 1 - d_2 - 1 - (n-3)d_n \ge d_3 - 2 \ge 0$.

If $d_3 = 1$, then $d_3 = d_4 = \cdots = d_n = 1$. Obviously, $d_2 \ne 1$. Otherwise by the assumption that G is connected we have that $v_i v_j \notin E$, where $2 \le i \ne j \le n$, which implies that G is S_n, a contradiction. Now we assume that $d_2 \ge 2$. Again since G is connected, we have that $v_i v_j \notin E$, where $3 \le i \ne j \le n$. Hence, for each i with $3 \le i \le n$, v_i is adjacent to exactly one of v_1 and v_2. Since $d(v_1) = d_1 \ge d(v_2) = d_2 = 2 > d_3 = d_4 = \cdots = d_n = 1$, we have that $\{x, y\} = \{v_1, v_2\}$. Since G is connected, $xy \in E$. Clearly, $c_{x,y} = 0$. Hence, by the definition of a, we have that $a = d_2$. So

$$\begin{aligned} &2e - d_1 - 1 - a - (n-3)z \\ &\ge d_1 + d_2 + d_3 + d_4 + \cdots + d_{n-1} + d_n - d_1 - 1 - d_2 - (n-3)d_n \\ &\ge d_3 - 1 \ge 0. \end{aligned}$$

Thus $f(z)$ is increasing when $0 < z \le d_n$. So

$$\begin{aligned} t &\le (n + d_2^* - d_1 - 1)f(\lambda_{n-1}) \le (n + d_2^* - d_1 - 1)f(d_n) \\ &= (n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4}. \end{aligned}$$

Suppose that

$$t = (n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4}.$$

In review of the proof above, we have that

$$\lambda_1 = n = d_1 + 1, \lambda_2 = n + d_2^* - d_1 - 1 = d_2^* = a, \lambda_3 = \cdots = \lambda_{n-2}, \lambda_{n-1} = d_n.$$

Since $\lambda_1 \ge \lambda_2$, $\lambda_2 \ge \lambda_3$, and $\lambda_{n-2} \ge \lambda_{n-1}$, we have the following eight cases.

Case 1. $\lambda_1 = \lambda_2 = \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

This implies $d_1 + 1 = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = d_n$. This is contrary to $d_1 \geq d_n$. Hence this case is impossible.

Case 2. $\lambda_1 = \lambda_2 = \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

By Theorem 8 and the assumption that $G \neq K_n$, we have that G is $K_n - e$ with $n \geq 5$.

Case 3. $\lambda_1 = \lambda_2 > \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

From $d_2 + 1 \geq a = \lambda_2 = \lambda_1 = d_1 + 1$, we have that $(a-1) = d_1 = d_2 = (n-1)$. Thus $d(x) = d(y) = (n-1)$. Hence $N(x) = V - \{x\}$ and $N(y) = V - \{y\}$. In particular, $xy \in E$.

Since $G \neq K_n$ and $\lambda_{n-1} = d_n$, Theorem 4 implies that G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n$ and $m \geq 2$. In fact, from the proof of Theorem 1 in [11], we can see that S is a cut-set of G and $G_1, G_2, \cdots, G_m$ are the components of $G[V(G) - S]$. Since $d(x) = d(y) = (n-1)$, both x and y are in S.

We claim that $|S| = 2$. Suppose, to the contrary, that $|S| \geq 3$. Since $N(x) = V - \{x\}$ and $N(y) = V - \{y\}$, G^c has at least four components. Notice that the Laplacian eigenvalues of G are

$$(n, n, \lambda_3, \cdots, \lambda_{n-1}, 0).$$

Thus the Laplacian eigenvalues of G^c are

$$((n - \lambda_{n-1}), (n - \lambda_{n-2}), \cdots, (n - \lambda_3), 0, 0, 0)$$

and G^c has three components, a contradiction. Hence $|S| = 2$.

Now we have that $\lambda_3 = \cdots = \lambda_{n-1} = d_n = |S| = 2$. Therefore $G^c - \{x, y\}$ has Laplacian eigenvalues

$$((n-2), (n-2), \cdots, (n-2), 0).$$

By Theorem 8, we have that $G^c - \{x,y\}$ is K_{n-2} or $K_{n-2} - e$. Since $G^c - \{x,y\}$ and $K_{n-2} - e$ have different sets of Laplacian eigenvalues, $G^c - \{x,y\} \neq K_{n-2} - e$. Thus $G^c - \{x,y\} = K_{n-2}$. Hence G is $K_2 \vee K_{n-2}^c$ with $n \geq 5$.

Case 4. $\lambda_1 = \lambda_2 > \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

Using arguments similar to those in Case 3, we have that $(a-1) = d_1 = d(x) = d_2 = d(y) = (n-1)$, $N(x) = V - \{x\}$, $N(y) = V - \{y\}$, and $xy \in E$. Moreover, $S = \{x, y\}$ is a cutset of G and G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n = \lambda_{n-1} = 2$ and $m \geq 2$.

Therefore $G^c - \{x, y\}$ has Laplacian eigenvalues

$$((n-\lambda_{n-1}), (n-\lambda_{n-2}), \cdots, (n-\lambda_3), 0).$$

By Theorem 7, we have that $G^c - \{x, y\}$ is K_{n-2}, S_{n-2}, or $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. Since $(n-\lambda_{n-1}) > (n-\lambda_{n-2})$, $G^c - \{x, y\}$ cannot be K_{n-2}. If $G^c - \{x, y\}$ is $K_{\frac{n-2}{2}, \frac{n-2}{2}}$, the minimum degree of G is $\frac{n}{2}$, which is not equal to $d_n = 2$ when $n \geq 5$. Thus $G^c - \{x, y\}$ cannot be $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. Therefore $G^c - \{x, y\} = S_{n-2}$. Hence G is $K_2 \vee S^c_{n-2}$ with $n \geq 5$.

Case 5. $\lambda_1 > \lambda_2 = \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

By Theorem 7 and the assumptions that $G \neq K_n$ and $G \neq S_n$, we have that G is K_{d_1, d_1}. Since $d_1 = (n-1)$, G must be K_2, contradicting the assumption that $n \geq 5$. Hence this case is impossible.

Case 6. $\lambda_1 > \lambda_2 = \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

From $d_1 = (n-1)$, we have that $d(x) = (n-1)$ and $N(x) = V - \{x\}$. Since G is not K_n and $\lambda_{n-1} = d_n$, Theorem 4 implies that G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n$ and $m \geq 2$. Clearly, x is in S.

We claim that $|S| = 1$. Suppose, to the contrary, that $|S| \geq 2$. Since $N(x) = V - \{x\}$, G^c has at least three components. Notice that the Laplacian eigenvalues of G are

$$(n, \lambda_2, \lambda_3, \cdots, \lambda_{n-1}, 0).$$

Thus the Laplacian eigenvalues of G^c are

$$((n-\lambda_{n-1}), (n-\lambda_{n-2}), \cdots, (n-\lambda_2), 0, 0)$$

and therefore G^c has two components, a contradiction. Hence $|S| = 1$.

Now we have that $\lambda_{n-1} = d_n = |S| = 1$. Therefore $G^c - \{x\}$ has Laplacian eigenvalues

$$((n-1), (n-\lambda_{n-2}), \cdots, (n-\lambda_2), 0).$$

By Theorem 7, we have that $G^c - \{x\}$ is K_{n-1}, S_{n-1}, or $K_{\frac{n-1}{2},\frac{n-1}{2}}$. Since $(n-1) = (n-\lambda_{n-1}) > (n-\lambda_{n-2})$, $G^c - \{x\}$ cannot be K_{n-1}. If $G^c - \{x\}$ is $K_{\frac{n-1}{2},\frac{n-1}{2}}$, the minimum degree of G is $\frac{n-1}{2}$, which is not equal to $d_n = 1$ when $n \geq 5$. Thus $G^c - \{x\}$ cannot be $K_{\frac{n-1}{2},\frac{n-1}{2}}$. Therefore $G^c - \{x\} = S_{n-1}$. Hence G is $K_1 \vee S_{n-1}^c$ with $n \geq 5$.

Case 7. $\lambda_1 > \lambda_2 > \lambda_3$, and $\lambda_{n-2} = \lambda_{n-1}$.

Again, since $d_1 = (n-1)$, we have that $d(x) = (n-1)$ and $N(x) = V - \{x\}$. Since G is not K_n and $\lambda_{n-1} = d_n$, Theorem 4 implies that G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$, where $|S| = d_n$ and $m \geq 2$. Clearly, x is in S.

We claim again that $|S| = 1$. Suppose, to the contrary, that $|S| \geq 2$. Since $N(x) = V - \{x\}$, G^c has at least three components. Since the Laplacian eigenvalues of G are

$$(n, \lambda_2, \lambda_3, \cdots, \lambda_{n-1}, 0),$$

the Laplacian eigenvalues of G^c are

$$((n-d_n), (n-d_n), \cdots, (n-d_n), (n-\lambda_2), 0, 0).$$

Since $n-\lambda_2 = \lambda_1 - \lambda_2 > 0$, G^c has two components, a contradiction. Hence $|S| = 1$.

Now we have that $\lambda_{n-1} = d_n = |S| = 1$. Therefore G^c has Laplacian eigenvalues

$$((n-1), (n-1), \cdots, (n-1), (n-\lambda_2), 0, 0),$$

and $G^c - \{x\}$ has Laplacian eigenvalues

$$((n-1), (n-1), \cdots, (n-1), (n-\lambda_2), 0).$$

By Theorem 8, we have that $G^c - \{x\}$ is K_{n-1} or $K_{n-1} - e$. Since $(n-1) = (n-d_n) > (n-\lambda_2)$, $G^c - \{x\}$ cannot be K_{n-1}. Therefore $G^c - \{x\} = K_{n-1} - e$. Hence G is $S_n + e$ with $n \geq 5$.

Case 8. $\lambda_1 > \lambda_2 > \lambda_3$, and $\lambda_{n-2} > \lambda_{n-1}$.

Using arguments similar to those in Case 7, we have that $d(x) = (n-1)$, $N(x) = V - \{x\}$, G is the same as the graph $G[S] \vee (G_1 \cup G_2 \cup \cdots \cup G_m)$ with $|S| = d_n = \lambda_{n-1} = 1$ and $m \geq 2$, and x is in S.

Since $d_n = 1$, v_n is only adjacent to x in G and v_n is adjacent to each vertex of $V - \{x\}$ in G^c. Since the Laplacian eigenvalues of G are

$$(n, \lambda_2, \cdots, \lambda_{n-2}, \lambda_{n-1}, 0),$$

the Laplacian eigenvalues of G^c are

$$((n-1), (n-\lambda_{n-2}), \cdots, (n-\lambda_3), (n-\lambda_2), 0, 0).$$

This implies that the Laplacian eigenvalues of $G^c - \{x\}$ are

$$((n-1), (n-\lambda_{n-2}), \cdots, (n-\lambda_3), (n-\lambda_2), 0).$$

Therefore the Laplacian eigenvalues of $(G^c - \{x\})^c$ are

$$((n-1)-(n-\lambda_2), (n-1)-(n-\lambda_3), \cdots, (n-1)-(n-\lambda_{n-2}), 0, 0),$$

which further implies that the Laplacian eigenvalues of $(G^c - \{x\})^c - \{v_n\}$, i.e., $G - \{x, v_n\}$, are

$$((\lambda_2 - 1), (\lambda_3 - 1), \cdots, (\lambda_{n-2} - 1), 0).$$

Since $(\lambda_3 - 1) = \cdots = (\lambda_{n-2} - 1)$, we, by Theorem 7, have that $(G^c - \{x\})^c - \{v_n\} = G - \{x, v_n\}$ is K_{n-2}, S_{n-2}, or $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. Since $\lambda_2 - 1 > \lambda_3 - 1$, $G - \{x, v_n\}$ cannot be K_{n-2}. If $G - \{x, v_n\}$ is $K_{\frac{n-2}{2}, \frac{n-2}{2}}$, then $\lambda_2 = d_2^* = (n-1) \neq \frac{n}{2} + 1 = a$ when $n \geq 5$. Thus $G - \{x, v_n\}$ cannot be $K_{\frac{n-2}{2}, \frac{n-2}{2}}$, Therefore $G - \{x, v_n\} = S_{n-2}$. Hence G is $K_1 \vee (K_1 \cup S_{n-2})$ with $n \geq 5$.

Next we show that if G is $K_n - e$, $K_2 \vee K_{n-2}^c$, $K_2 \vee S_{n-2}^c$, $K_1 \vee S_{n-1}^c$, $S_n + e$, or $K_1 \vee (K_1 \cup S_{n-2})$, then

$$t = (n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4}.$$

If G is $K_n - e$ with $n \geq 5$, then the Laplacian eigenvalues of G^c are

$$(2, 0, \cdots, 0, 0, 0).$$

Thus the Laplacian eigenvalues of G are

$$(n, n, \cdots, n, (n-2), 0).$$

Hence the number of spanning trees of G is $t = \prod_{i=1}^{n-1} \lambda_i / n = (n-2)n^{n-3}$. Now

$$(n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4} = (n-2)n^{n-3},$$

which is the number of spanning trees of $K_n - e$ with $n \geq 5$.

If G is $K_2 \vee K_{n-2}^c$ with $n \geq 5$, then the Laplacian eigenvalues of G^c are

$$((n-2),(n-2),\cdots,(n-2),0,0,0).$$

Thus the Laplacian eigenvalues of G are

$$(n,n,2,2,\cdots,2,0).$$

Hence the number of spanning trees of G is $t = \prod_{i=1}^{n-1} \lambda_i/n = n2^{n-3}$. Now

$$(n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4} = n2^{n-3},$$

which is the number of spanning trees of $K_2 \vee K_{n-2}^c$ with $n \geq 5$.

If G is $K_2 \vee S_{n-2}^c$ with $n \geq 5$, then the Laplacian eigenvalues of G^c are

$$((n-2),1,\cdots,1,0,0,0).$$

Thus the Laplacian eigenvalues of G are

$$(n,n,(n-1),\cdots,(n-1),2,0).$$

Hence the number of spanning trees of G is $t = \prod_{i=1}^{n-1} \lambda_i/n = 2n(n-1)^{n-4}$. Now

$$(n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4} = 2n(n-1)^{n-4},$$

which is the number of spanning trees of $K_2 \vee S_{n-2}^c$ with $n \geq 5$.

If G is $K_1 \vee S_{n-1}^c$ with $n \geq 5$, then the Laplacian eigenvalues of G^c are

$$((n-1),1,\cdots,1,0,0).$$

Thus the Laplacian eigenvalues of G are

$$(n,(n-1),\cdots,(n-1),1,0).$$

Hence the number of spanning trees of G is $t = \prod_{i=1}^{n-1} \lambda_i/n = (n-1)^{n-3}$. Now

$$(n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4} = (n-1)^{n-3},$$

which is the number of spanning trees of $K_1 \vee S^c_{n-1}$ with $n \geq 5$.

If G is $S_n + e$ with $n \geq 5$, then

$$(n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4} = 3,$$

which is the number of spanning trees of $S_n + e$ with $n \geq 5$.

If G is $K_1 \vee (K_1 \cup S_{n-2})$ with $n \geq 5$, from the above proof of Case 8, we have that the Laplacian eigenvalues of $G - \{ x, v_n \}$ are

$$((\lambda_2 - 1), (\lambda_3 - 1), \cdots, (\lambda_{n-2} - 1), 0).$$

Since $G - \{ x, v_n \}$ is S_{n-2}, $\lambda_2 - 1 = n - 2$ and $\lambda_3 - 1 = \cdots = \lambda_{n-2} - 1 = 1$. Thus $\lambda_1 = n$, $\lambda_2 = n - 1$, $\lambda_3 = \cdots = \lambda_{n-2} = 2$, and $\lambda_{n-1} = 1$. Hence the number of spanning trees of G is $t = \prod_{i=1}^{n-1} \lambda_i / n = (n-1)2^{n-4}$. Now

$$(n + d_2^* - d_1 - 1)d_n \left(\frac{2e - d_1 - 1 - a - d_n}{n-4}\right)^{n-4} = (n-1)2^{n-4},$$

which is the number of spanning trees of $K_1 \vee (K_1 \cup S_{n-2})$ with $n \geq 5$.

This completes the proof of Theorem 1.

Obviously, if G is $K_2 \vee K^c_{n-2}$, $K_2 \vee S^c_{n-2}$, $S_n + e$, or $K_1 \vee (K_1 \cup S_{n-2})$ with $n \geq 5$, then the upper bound in Theorem 1 is less than the upper bounds in Theorems A, B, or C.

Recall that Li and Pan in [12] proved that $\lambda_2(G) \geq d_2(G)$ for a connected graph of order $n \geq 3$ and Guo in [9] proved that $\lambda_3(G) \geq d_3(G) - 1$ for a connected graph of order $n \geq 4$. Using those results, we can present another upper bound for the number of spanning trees in a connected graph of order $n \geq 6$.

Theorem 10 Let G be a non-complete connected graph with $n \geq 6$ vertices and e edges. Then

$$t \leq (n+d_2^*-d_1-1)(n+d_2^*+d_3^*-d_1-d_2-1)d_n \left(\frac{2e - d_1 - d_2 - d_3 - d_n}{n-5}\right)^{n-5}.$$

The proof of Theorem 10 is similar to the proof of Theorem 1. It can be fulfilled as long as we realize that $\lambda_3 = \lambda_1 + \lambda_2 + \lambda_3 - \lambda_1 - \lambda_2 \leq d_1^* + d_2^* + d_3^* - d_1 - 1 - d_2 = n + d_2^* + d_3^* - d_1 - d_2 - 1$. The complete proof

Theorem 10 is skipped here. It is interesting to decide if the upper bound in Theorem 10 is attainable for some graphs.

3. Acknowledgments

The author would like to thank the referee for his or her suggestions and comments which improve this paper.

References

[1] H. Bai, The Grone–Merris conjecture, *Trans. Amer. Math. Soc.* 363 (2011) 4463 - 4474.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York (1976).

[3] K. Ch. Das, The largest two Laplacian eigenvalues of a graph, *Linear and Multilinear Algebra* 52 (2004) 441 - 460.

[4] K. Ch. Das, A sharp bound for the number of spanning trees of a graph, *Graphs Combin.* 23 (2007) 625 - 632.

[5] L. Feng, G. Yu, Z. Jiang, and L. Ren, Sharp upper bounds for the number of spanning trees of a graph, *Appl. Anal. Discrete Math.* 2 (2008) 255 - 259.

[6] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.* 23 (1973) 298 - 305.

[7] C. Godsil and G. Royle, Algebraic Graph Theory, Springer–Verlag New York (2001).

[8] R. Grone and R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.* 7 (1994) 221 - 229.

[9] J. Guo, On the third largest Laplacian eigenvalue of a graph, *Linear and Multilinear Algebra* 55 (2007) 93 - 102.

[10] S. Kirkland, A bound on algebraic connectivity of a graph in terms of the number of the cutpoints, *Linear and Multilinear Algebra* 47 (2000) 93 - 103.

[11] J. Li and Y. Fan, Note on the algebraic connectivity of a graph, *J. Univ. Sci. Technol. China* 32 (2002) 1 - 6.

[12] J. Li and Y. Pan, A note on the second largest eigenvalue of the two Laplacian matrix of a graph, *Linear and Multilinear Algebra* 48 (2000) 117 - 121.

[13] J. Li, W. Shiu, and A. Chang, The number of spanning trees of a graph, *Applied Mathematics Letters* 23 (2010) 286 – 290.