Up-Down Sequences: Inversions, Coinversions, and the Sum of Major Indices

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Abstract

For $n \geq 1$ we call a sequence $s_1, s_2, \ldots, s_n$ an up-down sequence of length n when (i) $s_1 = 1$; (ii) $s_i \in \{1, 2, 3, 4\}$, for $2 \leq i \leq n$; and, (iii) $|s_i - s_{i-1}| = 1$, for $2 \leq i \leq n$. We count the number of inversions and coinversions for all such up-down sequences of length n, as well as the sum of the major indices for all these sequences of length n.

1 Introduction

For $n \geq 1$, we call a sequence $s_1, s_2, \ldots, s_n$ an *up-down sequence* (of length n) if

- (i) $s_1 = 1$;
- (ii) $s_i \in \{1, 2, 3, 4\}$, for $2 \leq i \leq n$; and,
- (iii) $|s_1 - s_{i-1}| = 1$, for $2 \leq i \leq n$.

(This example appears in [5] where it was contributed by Clark Kimberling and Neven Juric.)

If we let a_n count the number of up-down sequences of length n we find that

- $a_1 = 1$, for the sequence 1;
- $a_2 = 1$, for the sequence 1, 2; and,

- $a_3 = 2$, for the sequences $1, 2, 1$ and $1, 2, 3$.

In Table 1 we find the up-down sequences for when $n = 4, 5, 6, 7$. (From this point on, we omit the commas between the consecutive entries of a sequence.) Here the sequences for $n = 6$ are obtained by (i) appending '2' at the end of each sequence for $n = 5$; and, (ii) appending '34' at the end of each sequence for $n = 4$. The situation is similar for n even, $n \geq 4$. When $n = 7$, the sequences are now obtained by (i) appending '3' at the end of each sequence for $n = 6$; and, (ii) appending '21' at the end of each sequence for $n = 5$. This situation is similar for n odd, $n \geq 3$. In general we find that

$$a_n = a_{n-1} + a_{n-2},\ n \geq 3,\ a_1 = 1,\ a_2 = 1,$$

so $a_n = F_n$, the nth Fibonacci number.

[The Fibonacci numbers are defined recursively by $F_0 = 0, F_1 = 1$, and for $n \geq 2, F_n = F_{n-1} + F_{n-2}$. Further, for $n \geq 0, F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{5}}{2}$, the golden ratio, and $\beta = \frac{1-\sqrt{5}}{2}$. This form is referred to as the Binet form for F_n and one finds that $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$, and $\alpha - \beta = \sqrt{5}$.]

$n=4$	$n=5$	$n=6$	$n=7$
1212	12123	121232	1212323
1232	12323	123232	1232323
1234	12343	123432	1234323
	12121	121212	1212123
	12321	123212	1232123
		121234	1212343
		123234	1232343
		123434	1234343
			1212321
			1232321
			1234321
			1212121
			1232121

Table 1

2 Counting the Numbers of 1's, 2's, 3's, and 4's

For $n \geq 1$, we let

- w_n = the number of 1's that occur among the F_n sequences;
- t_n = the number of 2's that occur among the F_n sequences;
- h_n = the number of 3's that occur among the F_n sequences; and,
- f_n = the number of 4's that occur among the F_n sequences.

In Table 2 we have the values of w_n, t_n, h_n, and f_n, for $1 \leq n \leq 10$.

n	w_n	t_n	h_n	f_n
1	1	0	0	0
2	1	1	0	0
3	3	2	1	0
4	4	5	2	1
5	9	9	6	1
6	13	19	11	5
7	27	33	25	6
8	40	65	44	19
9	80	111	90	25
10	120	210	155	65

Table 2

In [3] the following general results are derived. [Here L_n denotes the nth Lucas number, where $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Further, the Binet form for the Lucas numbers is $L_n = \alpha^n + \beta^n$, for $n \geq 0$.]

$$w_n = \frac{1}{8}L_n + \frac{27}{40}F_n - \frac{1}{20}nL_n + \frac{1}{4}nF_n + (-1)^n\left(-\frac{1}{8}L_n + \frac{1}{8}F_n\right), \quad n \geq 1 \tag{1}$$

$$t_n = \frac{1}{8}L_n - \frac{7}{40}F_n + \frac{1}{20}nL_n + \frac{1}{4}nF_n + (-1)^n\left(-\frac{1}{8}L_n + \frac{3}{8}F_n\right), \quad n \geq 1 \tag{2}$$

$$h_n = -\frac{1}{8}L_n - \frac{17}{40}F_n + \frac{1}{20}nL_n + \frac{1}{4}nF_n + (-1)^n\left(\frac{1}{8}L_n - \frac{3}{8}F_n\right), \quad n \geq 1 \tag{3}$$

$$f_n = -\frac{1}{8}L_n - \frac{3}{40}F_n - \frac{1}{20}nL_n + \frac{1}{4}nF_n + (-1)^n\left(\frac{1}{8}L_n - \frac{1}{8}F_n\right), \quad n \geq 1 \tag{4}$$

Further, one finds the following relationships among w_n, t_n, h_n, and f_n.

$$t_n = f_{n+2} + \left(\frac{1}{2}\right)\left(1 + (-1)^{n-1}\right) F_{n-1},\ n \geq 1 \tag{5}$$

$$t_n = h_n + \left(\frac{1}{2}\right)\left(1 + (-1)^{n}\right) F_{n} + \left(\frac{1}{2}\right)\left(1 + (-1)^{n-1}\right) F_{n-1},\ n \geq 1 \tag{6}$$

$$w_n = t_{n-2} + \left(\frac{1}{2}\right)\left(1 + (-1)^{n}\right) F_{n} + \left(\frac{1}{2}\right)\left(1 + (-1)^{n-1}\right) L_{n-1},\ n \geq 3 \tag{7}$$

$$w_n = h_{n-2} + \left(\frac{1}{2}\right)\left(1 + (-1)^{n}\right) L_{n-1} + \left(\frac{1}{2}\right)\left(1 + (-1)^{n+1}\right) F_{n+1},\ n \geq 3 \tag{8}$$

$$w_n = f_n + \left(\frac{1}{2}\right)\left(1 + (-1)^{n}\right) F_{n} + \frac{1}{2}\left(1 + (-1)^{n+1}\right) F_{n+1},\ n \geq 3 \tag{9}$$

$$f_{n+2} = h_n + \left(\frac{1}{2}\right)\left(1 + (-1)^{n}\right) F_{n},\ n \geq 1 \tag{10}$$

Finally, if we let $e_{n,i}$ count the number of up-down sequences of length n that end in i, for $1 \leq i \leq 4$, we find that

$$e_{n,1} = e_{n,3} = 0,\ e_{n,2} = F_{n-1},\ e_{n,4} = F_{n-2},\ n \text{ even}, \tag{11}$$

$$e_{1,1} = 1, \tag{12}$$

$$e_{n,2} = e_{n,4} = 0, n \text{ odd}, \tag{13}$$

$$e_{n,1} = F_{n-2}, n \text{ odd}, n \geq 3, \tag{14}$$

$$e_{n,3} = F_{n-1}, n \text{ odd}, n \geq 1. \tag{15}$$

3 Inversions

Following the development in Definition 6.11 on p. 216 of [4], if $x_1, x_2, \ldots, x_n$ is a sequence of length n (whose elements are taken from a totally ordered set), an *inversion* occurs for all i, j where $1 \leq i < j \leq n$, but $x_i > x_j$.

For example, consider the up-down sequence 12321 of length 5. Here the 2 in position 2 and the 1 in position 5 provide an inversion for the sequence. Likewise, the 3 in position 3 and the 2 in position 4 is another example of an inversion. The 3 in position 3 and the 1 in position 5 provide a third inversion, while the 2 in position 4 and the 1 in position 5 provide the fourth and final inversion. In total, there are four inversions for the up-down sequence 12321.

In this section our objective is to count, for a given $n \geq 1$, the total number of inversions that occur among the F_n up-down sequences of length n. We designate this number by inv_n and find, for example, that

$$\begin{array}{lllll} inv_1 = 0 & inv_2 = 0 & inv_3 = 1 & inv_4 = 2 & inv_5 = 10 \\ inv_6 = 20 & inv_7 = 63 & inv_8 = 119 & inv_9 = 309 & inv_{10} = 562 \end{array}$$

To motivate the recurrence relation for inv_n, we examine the following special cases—one for when n is even and the other for when n is odd.

Case 1: $inv_6 = inv_4 + f_4 + inv_5 + (f_5 + h_5)$
(i) Here the summand f_4 accounts for the inversions that arise for the 3 in 34 that follows each of the f_4 4's that occurs in the F_4 sequences of length 4.
(ii) The summand $(f_5 + h_5)$ accounts for the inversions that result from the 2 that now follows each of the f_5 4's and h_5 3's that occur in the F_5 sequences of length 5.

Case 2: $inv_7 = inv_6 + f_6 + inv_5 + (f_5 + h_5) + (f_5 + h_5 + t_5) + F_5$
(i) Here the summand f_6 accounts for the inversions that arise for the 3 that follows each of the f_6 4's that occur in the F_6 sequences of length 6.
(ii) The summand $(f_5 + h_5)$ accounts for the inversions that arise for the 2 in 21 that follows each of the f_5 4's and h_5 3's that occur in the F_5 sequences of length 5.
(iii) The summand $(f_5 + h_5 + t_5)$ accounts for the inversions that arise for the 1 in 21 that follows each of the f_5 4's, h_5 3's, and t_5 2's that occur in the F_5 sequences of length 5.
(iv) Finally, the summand F_5 accounts for each inversion that arises when 21 is appended at the end of each of the F_5 sequences of length 5.

Consequently, to consider both the even and odd cases simultaneously, we arrive at the recurrence relation

$$\begin{aligned} inv_n = inv_{n-1} + inv_{n-2} &+ \left(\frac{1}{2}\right)(1 + (-1)^n)(f_{n-1} + h_{n-1} + f_{n-2}) \\ &+ \left(\frac{1}{2}\right)(1 + (-1)^{n-1})(f_{n-1} + 2f_{n-2} + 2h_{n-2} + t_{n-2} + F_{n-2}), \\ & n > 2,\ inv_1 = 0,\ inv_2 = 0. \end{aligned}$$

Using the results from Section 2 this recurrence relation takes the form

$$
\begin{aligned}
inv_n &= inv_{n-1} + inv_{n-2} \\
&+ \left(\frac{1}{2}\right)[1+(-1)^n]\Big[-\frac{1}{8}L_{n-1} - \frac{3}{40}F_{n-1} - \frac{1}{20}(n-1)L_{n-1} \\
&+ \frac{1}{4}(n-1)F_{n-1} + (-1)^{n-1}\left(\frac{1}{8}L_{n-1} - \frac{1}{8}F_{n-1}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^n]\Big[-\frac{1}{8}L_{n-1} - \frac{17}{40}F_{n-1} + \frac{1}{20}(n-1)L_{n-1} \\
&+ \frac{1}{4}(n-1)F_{n-1} + (-1)^{n-1}\left(\frac{1}{8}L_{n-1} - \frac{3}{8}F_{n-1}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^n]\Big[-\frac{1}{8}L_{n-2} - \frac{3}{40}F_{n-2} - \frac{1}{20}(n-2)L_{n-2} \\
&+ \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(\frac{1}{8}L_{n-2} - \frac{1}{8}F_{n-2}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\Big[-\frac{1}{8}L_{n-1} - \frac{3}{40}F_{n-1} - \frac{1}{20}(n-1)L_{n-1} \\
&+ \frac{1}{4}(n-1)F_{n-1} + (-1)^{n-1}\left(\frac{1}{8}L_{n-1} - \frac{1}{8}F_{n-1}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\,(2)\Big[-\frac{1}{8}L_{n-2} - \frac{3}{40}F_{n-2} - \frac{1}{20}(n-2)L_{n-2} \\
&+ \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(\frac{1}{8}L_{n-2} - \frac{1}{8}F_{n-2}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\,(2)\Big[-\frac{1}{8}L_{n-2} - \frac{17}{40}F_{n-2} + \frac{1}{20}(n-2)L_{n-2} \\
&+ \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(\frac{1}{8}L_{n-2} - \frac{3}{8}F_{n-2}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\Big[\frac{1}{8}L_{n-2} - \frac{7}{40}F_{n-2} + \frac{1}{20}(n-2)L_{n-2} \\
&+ \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(-\frac{1}{8}L_{n-2} - \frac{3}{8}F_{n-2}\right)\Big] \\
&+ \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\,F_{n-2}.
\end{aligned}
$$

Using the Binet forms for the Fibonacci and Lucas numbers, now the recurrence relation becomes

$$\begin{aligned}
inv_n &= inv_{n-1} + inv_{n-2} \\
&+ \left(-\frac{29}{100}\sqrt{5}\right)\alpha^n + \left(\frac{29}{100}\sqrt{5}\right)\beta^n + \frac{7}{40}(-1+\sqrt{5})n\alpha^n + \frac{7}{40}(-1-\sqrt{5})n\beta^n \\
&+ \left(\frac{1}{2} - \frac{17}{100}\sqrt{5}\right)(-\alpha)^n + \left(\frac{1}{2} + \frac{17}{100}\sqrt{5}\right)(-\beta)^n \\
&+ \left(\frac{9}{40} - \frac{1}{8}\sqrt{5}\right)n(-\alpha)^n + \left(\frac{9}{40} + \frac{1}{8}\sqrt{5}\right)n(-\beta)^n .
\end{aligned}$$

Using the techniques presented in Chapter 7 of [1] and Chapter 10 of [2], we learn that the homogeneous part of the solution has the form

$$c_1\alpha^n + c_2\beta^n,$$

while the particular part has the form

$$A_1n\alpha^n + B_1n\beta^n + A_2n^2\alpha^n + B_2n^2\beta^n + A_3(-\alpha)^n + B_3(-\beta)^n + A_4n(-\alpha)^n + B_4n(-\beta)^n.$$

To determine A_1, A_2 we substitute $inv_n = A_1n\alpha^n + A_2n^2\alpha^n$ into the recurrence relation

$$inv_n = inv_{n-1} + inv_{n-2} + \left(-\frac{29}{100}\sqrt{5}\right)\alpha^n + \frac{7}{40}\left(-1+\sqrt{5}\right)n\alpha^n.$$

Upon dividing through by α^{n-2} this then leads to

$$\begin{aligned}
A_1n\alpha^2 + A_2n^2\alpha^2 &= A_1(n-1)\alpha + A_2(n^2-2n+1)\alpha + A_1(n-2) + A_2(n^2-4n+4) \\
&\quad + \left(-\frac{29}{100}\sqrt{5}\right)\alpha^2 + \frac{7}{40}(-1+\sqrt{5})n\alpha^2.
\end{aligned}$$

Comparing coefficients for $n^2, n,$ and $n^0(=1)$, we have

$$\begin{aligned}
n^2 : A_2\alpha^2 &= A_2\alpha + A_2, \text{ which reduces to } 0 = 0 \text{ because } \alpha^2 = \alpha + 1; \\
n : A_1\alpha^2 &= A_1\alpha - 2A_2\alpha + A_1 - 4A_2 + \frac{7}{40}\left(-1+\sqrt{5}\right)\alpha^2; \\
n^0(=1) : 0 &= -A_1\alpha + A_2\alpha - 2A_1 + 4A_2 - \frac{29\sqrt{5}}{100}\alpha^2.
\end{aligned}$$

Solving the last two equations simultaneously we arrive at

$$A_1 = -\tfrac{9}{50} - \tfrac{3\sqrt{5}}{40},\ A_2 = \tfrac{7\sqrt{5}}{200}.$$

Then a similar calculation yields

$$B_1 = -\tfrac{9}{50} + \tfrac{3\sqrt{5}}{40},\ B_2 = -\tfrac{7\sqrt{5}}{200}.$$

Turning now to A_3 and A_4, we substitute $inv_n = A_3(-\alpha)^n + A_4 n(-\alpha)^n$ into the recurrence relation

$$inv_n = inv_{n-1} + inv_{n-2} + \left(\frac{1}{2} - \frac{17\sqrt{5}}{100}\right)(-\alpha)^n + \left(\frac{9}{40} - \frac{\sqrt{5}}{8}\right) n(-\alpha)^n.$$

We divide the result by $(-\alpha)^{n-2}$ to find that

$$\begin{aligned} A_3(-\alpha)^2 + A_4 n(-\alpha)^2 &= A_3(-\alpha) + A_4(n-1)(-\alpha) + A_3 + A_4(n-2) \\ &\quad + \left(\frac{1}{2} - \frac{17}{100}\sqrt{5}\right)(-\alpha)^2 + \left(\frac{9}{40} - \frac{1}{8}\sqrt{5}\right) n(-\alpha)^2. \end{aligned}$$

Comparing the coefficients for n and $n^0 (= 1)$ we find that

$$\begin{aligned} A_4(-\alpha)^2 &= A_4(-\alpha) + A_4 + \left(\frac{9}{40} - \frac{1}{8}\sqrt{5}\right)(-\alpha)^2 \text{ and} \\ A_3(-\alpha)^2 &= A_3(-\alpha) + A_4\alpha + A_3 - 2A_4 + \left(\frac{1}{2} - \frac{17}{100}\sqrt{5}\right)(-\alpha)^2. \end{aligned}$$

From these equations it follows that

$$A_3 = -\frac{1}{4} + \frac{63}{400}\sqrt{5},\ A_4 = -\frac{1}{10} + \frac{1}{40}\sqrt{5},$$

and a similar calculation provides

$$B_3 = -\frac{1}{4} - \frac{63}{400}\sqrt{5},\ B_4 = -\frac{1}{10} - \frac{1}{40}\sqrt{5}.$$

Consequently,

$$\begin{aligned} inv_n &= c_1\alpha^n + c_2\beta^n + \left(-\frac{9}{50} - \frac{3}{40}\sqrt{5}\right) n\alpha^n + \left(-\frac{9}{50} + \frac{3}{40}\sqrt{5}\right) n\beta^n \\ &\quad + \frac{7}{200}\sqrt{5}n^2\alpha^n - \frac{7}{200}\sqrt{5}n^2\beta^n \\ &\quad + \left(-\frac{1}{4} + \frac{63}{400}\sqrt{5}\right)(-\alpha)^n + \left(-\frac{1}{4} - \frac{63}{400}\sqrt{5}\right)(-\beta)^n \\ &\quad + \left(-\frac{1}{10} + \frac{1}{40}\sqrt{5}\right) n(-\alpha)^n + \left(-\frac{1}{10} - \frac{1}{40}\sqrt{5}\right) n(-\beta)^n. \end{aligned}$$

Then from the initial conditions—namely $inv_1 = 0,\ inv_2 = 0$, we find that

$$c_1 = \frac{1}{4} + \frac{277}{2000}\sqrt{5},\ c_2 = \frac{1}{4} - \frac{277}{2000}\sqrt{5}.$$

Consequently, for $n \geq 1$,

$$\begin{aligned} inv_n &= \left(\frac{1}{4} + \frac{277}{2000}\sqrt{5}\right)\alpha^n + \left(\frac{1}{4} - \frac{277}{2000}\sqrt{5}\right)\beta^n \\ &+ \left(-\frac{9}{50} - \frac{3}{40}\sqrt{5}\right)n\alpha^n + \left(-\frac{9}{50} + \frac{3}{40}\sqrt{5}\right)n\beta^n \\ &+ \frac{7}{200}\sqrt{5}n^2\alpha^n - \frac{7}{200}\sqrt{5}n^2\beta^n \\ &+ \left(-\frac{1}{4} + \frac{63}{400}\sqrt{5}\right)(-\alpha)^n + \left(-\frac{1}{4} - \frac{63}{400}\sqrt{5}\right)(-\beta)^n \\ &+ \left(-\frac{1}{10} + \frac{1}{40}\sqrt{5}\right)n(-\alpha)^n + \left(-\frac{1}{10} - \frac{1}{40}\sqrt{5}\right)n(-\beta)^n \\ &= \left(\frac{7}{40}n^2 - \frac{3}{8}n + \frac{277}{400}\right)F_n + \left(-\frac{9}{50}n + \frac{1}{4}\right)L_n \\ &+ (-1)^n\left(\frac{1}{8}n + \frac{63}{80}\right)F_n + (-1)^n\left(-\frac{1}{10}n - \frac{1}{4}\right)L_n. \end{aligned}$$

4 Coinversions

Referring this time to Exercise 6.84 on p. 239 of [4], if $x_1, x_2, \ldots, x_n$ is a sequence of length n (whose elements are taken from a totally ordered set), a *coinversion* occurs for all i, j where $1 \leq i < j \leq n$ and $x_i < x_j$.

If we consider the up-down sequence 12321 of length 5, the 1 in the first position and the 2 in the second position provide a coinversion for the sequence. Likewise, the 1 in the first position, together with the 3 in the third position or the 2 in the fourth position, provide two more coinversions. Finally, the 2 in the second position and the 3 in the third position provide a fourth (and final) coinversion.

In this section we shall determine, for a given $n \geq 1$, a formula that counts the total number of coinversions that occur among the F_n up-down sequences of length n. We shall designate this number by $coinv_n$ and we find, for example, that

$coinv_1 = 0$	$coinv_2 = 1$	$coinv_3 = 4$	$coinv_4 = 13$	$coinv_5 = 29$
$coinv_6 = 74$	$coinv_7 = 144$	$coinv_8 = 328$	$coinv_9 = 604$	$coinv_{10} = 1287.$

To motivate the recurrence relation for $coinv_n$, we consider the following special cases—one for when n is even and the other for when n is odd.

Case 1: $coinv_6 = coinv_5 + w_5 + coinv_4 + 2(w_4 + t_4) + h_4 + F_4$
(i) Here the summand w_5 arises from when the sixth (last) entry in the

sequence is 2, which when appended to each of the F_5 sequences, provides w_5 coinversions.
(ii) When 34 is appended to each of the F_4 sequences, two coinversions are created for each occurrence of 1 or 2 in the F_4 sequences. These are accounted for by the summand $2(w_4 + t_4)$.
(iii) In addition to the coinversions in (ii), the final 4 (in position 6) also provides a coinversion for each 3 that occurs in the F_4 sequences.
(iv) Finally, when 34 is appended to each of the F_4 sequences, this also provides a new coinversion —namely, 34 itself.

Case 2: $coinv_7 = coinv_6 + (w_6 + t_6) + coinv_5 + w_5$
(i) When 3 is appended to each of the F_6 sequences, the summand $(w_6 + t_6)$ counts the coinversions for each occurrence of 1 or 2 in the F_6 sequences.
(ii) The summand w_5 accounts for the coinversions that arise when we append 21 to each of the F_5 sequences. Each occurrence of 1 in these sequences provides a coinversion with the 2 in 21.

The observations for these two cases now lead us to the following recurrence relation:

$$\begin{aligned} coinv_n &= coinv_{n-1} + coinv_{n-2} \\ &+ \left(\frac{1}{2}\right)[1 + (-1)^n]\,[w_{n-1} + 2(w_{n-2} + t_{n-2}) + h_{n-2} + F_{n-2}] \\ &+ \left(\frac{1}{2}\right)\left[1 + (-1)^{n-1}\right][w_{n-1} + t_{n-1} + w_{n-2}], \\ &n > 2,\ coinv_1 = 0,\ coinv_2 = 1. \end{aligned}$$

From the results in Section 2 this recurrence relation becomes

$$
\begin{aligned}
coinv_n = {} & coinv_{n-1} + coinv_{n-2} \\
& + \left(\frac{1}{2}\right)[1+(-1)^n]\left[\frac{1}{8}L_{n-1} + \frac{27}{40}F_{n-1} - \frac{1}{20}(n-1)L_{n-1}\right. \\
& \left. + \frac{1}{4}(n-1)F_{n-1} + (-1)^{n-1}\left(-\frac{1}{8}L_{n-1} + \frac{1}{8}F_{n-1}\right)\right] \\
& + \left(\frac{1}{2}\right)[1+(-1)^n](2)\left[\frac{1}{8}L_{n-2} + \frac{27}{40}F_{n-2} - \frac{1}{20}(n-2)L_{n-2}\right. \\
& \left. + \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(-\frac{1}{8}L_{n-2} + \frac{1}{8}F_{n-2}\right)\right] \\
& + \left(\frac{1}{2}\right)[1+(-1)^n](2)\left[\frac{1}{8}L_{n-2} - \frac{7}{40}F_{n-2} + \frac{1}{20}(n-2)L_{n-2}\right. \\
& \left. + \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(-\frac{1}{8}L_{n-2} + \frac{3}{8}F_{n-2}\right)\right] \\
& + \left(\frac{1}{2}\right)[1+(-1)^n]\left[-\frac{1}{8}L_{n-2} - \frac{17}{40}F_{n-2} + \frac{1}{20}(n-2)L_{n-2}\right. \\
& \left. + \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(\frac{1}{8}L_{n-2} - \frac{3}{8}F_{n-2}\right)\right] \\
& + \left(\frac{1}{2}\right)[1+(-1)^n]F_{n-2} \\
& + \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\left[\frac{1}{8}L_{n-1} + \frac{27}{40}F_{n-1} - \frac{1}{20}(n-1)L_{n-1}\right. \\
& \left. + \frac{1}{4}(n-1)F_{n-1} + (-1)^{n-1}\left(-\frac{1}{8}L_{n-1} + \frac{1}{8}F_{n-1}\right)\right] \\
& + \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\left[\frac{1}{8}L_{n-1} - \frac{7}{40}F_{n-1} + \frac{1}{20}(n-1)L_{n-1}\right. \\
& \left. + \frac{1}{4}(n-1)F_{n-1} + (-1)^{n-1}\left(-\frac{1}{8}L_{n-1} + \frac{3}{8}F_{n-1}\right)\right] \\
& + \left(\frac{1}{2}\right)[1+(-1)^{n-1}]\left[\frac{1}{8}L_{n-2} + \frac{27}{40}F_{n-2} - \frac{1}{20}(n-2)L_{n-2}\right. \\
& \left. + \frac{1}{4}(n-2)F_{n-2} + (-1)^{n-2}\left(-\frac{1}{8}L_{n-2} + \frac{1}{8}F_{n-2}\right)\right].
\end{aligned}
$$

Using the Binet forms for the Fibonacci and Lucas numbers once again, this recurrence relation now becomes

$$\begin{aligned}coinv_n =&coinv_{n-1} + coinv_{n-2}\\&+\left(\frac{3}{8}-\frac{13}{200}\sqrt{5}\right)\alpha^n+\left(\frac{3}{8}+\frac{13}{200}\sqrt{5}\right)\beta^n\\&+\left(-\frac{7}{40}+\frac{7}{40}\sqrt{5}\right)n\alpha^n+\left(-\frac{7}{40}-\frac{7}{40}\sqrt{5}\right)n\beta^n\\&+\left(-\frac{3}{8}+\frac{29}{200}\sqrt{5}\right)(-\alpha)^n+\left(-\frac{3}{8}-\frac{29}{200}\sqrt{5}\right)(-\beta)^n\\&+\left(-\frac{9}{40}+\frac{1}{8}\sqrt{5}\right)n(-\alpha)^n+\left(-\frac{9}{40}-\frac{1}{8}\sqrt{5}\right)n(-\beta)^n,\end{aligned}$$

for which the homogeneous part of the solution has the form

$$c_1^*\alpha^n + c_2^*\beta^n,$$

while the particular part has the form

$$A_1^*n\alpha^n+B_1^*n\beta^n+A_2^*n^2\alpha^n+B_2^*n^2\beta^n+A_3^*(-\alpha)^n+B_3^*(-\beta)^n+A_4^*n(-\alpha)^n+B_4^*n(-\beta)^n.$$

Computing as we did in Section 3, we substitute $coinv_n = A_1^*n\alpha^n + A_2^*n^2\alpha^n$ into the recurrence relation

$$coinv_n = coinv_{n-1}+coinv_{n-2}+\left(\frac{3}{8}-\frac{13}{200}\sqrt{5}\right)\alpha^n+\left(-\frac{7}{40}+\frac{7}{40}\sqrt{5}\right)n\alpha^n.$$

After dividing through by α^{n-2} and comparing the coefficients for n and $n^0(=1)$, we find that

$$A_1^* = \left(\frac{3}{25}+\frac{3}{40}\sqrt{5}\right) \text{ and } A_2^* = \frac{7}{200}\sqrt{5},$$

and a similar calculation yields

$$B_1^* = \left(\frac{3}{25}-\frac{3}{40}\sqrt{5}\right) \text{ and } B_2^* = -\frac{7}{200}\sqrt{5}.$$

To determine A_3^* and A_4^*, we substitute $coinv_n = A_3^*(-\alpha)^n + A_4^*n(-\alpha)^n$ into the recurrence relation

$$coinv_n = coinv_{n-1}+coinv_{n-2}+\left(-\frac{3}{8}+\frac{29}{200}\sqrt{5}\right)(-\alpha)^n+\left(-\frac{9}{40}+\frac{1}{8}\sqrt{5}\right)n(-\alpha)^n.$$

From here we learn that

$$A_3^* = \left(\frac{1}{4} - \frac{53}{400}\sqrt{5}\right) \text{ and } A_4^* = \left(\frac{1}{10} - \frac{1}{40}\sqrt{5}\right),$$

and a similar calculation yields

$$B_3^* = \left(\frac{1}{4} + \frac{53}{400}\sqrt{5}\right) \text{ and } B_4^* = \left(\frac{1}{10} + \frac{1}{40}\sqrt{5}\right).$$

Then the initial conditions—namely $coinv_1 = 0$ and $coinv_2 = 1$, lead us to

$$c_1^* = \left(-\frac{1}{4} - \frac{343}{2000}\sqrt{5}\right) \text{ and } c_2^* = \left(-\frac{1}{4} + \frac{343}{2000}\sqrt{5}\right).$$

Consequently, for $n \geq 1$,

$$\begin{aligned} coinv_n &= \left(-\frac{1}{4} - \frac{343}{2000}\sqrt{5}\right)\alpha^n + \left(-\frac{1}{4} + \frac{343}{2000}\sqrt{5}\right)\beta^n \\ &+ \left(\frac{3}{25} + \frac{3}{40}\sqrt{5}\right)n\alpha^n + \left(\frac{3}{25} - \frac{3}{40}\sqrt{5}\right)n\beta^n + \frac{7}{200}\sqrt{5}n^2\left(\alpha^n - \beta^n\right) \\ &+ \left(\frac{1}{4} - \frac{53}{400}\sqrt{5}\right)(-\alpha)^n + \left(\frac{1}{4} + \frac{53}{400}\sqrt{5}\right)(-\beta)^n \\ &+ \left(\frac{1}{10} - \frac{1}{40}\sqrt{5}\right)n(-\alpha)^n + \left(\frac{1}{10} + \frac{1}{40}\sqrt{5}\right)n(-\beta)^n \\ &= \left(\frac{1}{400}\right)\left[70n^2 + 150n - 343\right]F_n + \left(\frac{1}{100}\right)\left[12n - 25\right]L_n \\ &+ (-1)^{n+1}\left(\frac{1}{80}\right)\left[10n + 53\right]F_n + (-1)^n\left(\frac{1}{20}\right)\left[2n + 5\right]L_n. \end{aligned}$$

5 The Sum of Major Indices

Following the development in Definition 6.27 on p. 221 of [4], this final time we consider a sequence $x_1, x_2, \ldots, x_n$ of length n (with elements taken from a totally ordered set). If $x_i > x_{i+1}$, we say that a *descent* occurs at position i, for $1 \leq i \leq n-1$. The *descent set* of the sequence is then the set of all $1 \leq i \leq n-1$ where $x_i > x_{i+1}$. The *major index* of the sequence is the sum of the elements in its descent set.

For example, the up-down sequence 123434 (of length 6) has descent set $\{4\}$ and major index 4. The up-down sequence 1234321 (of length 7) has descent set $\{4, 5, 6\}$, so its major index is 15.

If we consider the five up-down sequences of length 5, we find the results in Table 3. Consequently, the sum of the major indices for the up-down

Sequence	Descent Set	Major Index
12123	{2}	2
12323	{3}	3
12343	{4}	4
12121	{2,4}	6
12321	{3,4}	7

Table 3

sequences of length 5 is 22.

In this final section we shall solve a recurrence relation for the sum of the major indices of the F_n up-down sequences of length n. This result will be denoted by sum_n and we find, for example, that

$$\begin{array}{lllll} sum_1 = 0 & sum_2 = 0 & sum_3 = 2 & sum_4 = 5 & sum_5 = 22 \\ sum_6 = 46 & sum_7 = 131 & sum_8 = 251 & sum_9 = 606 & sum_{10} = 1110. \end{array}$$

As we did in the previous two sections, we'll motivate the recurrence relation for sum_n by considering the following two cases.

Case 1: $sum_6 = sum_5 + (6-1)F_4 + sum_4 + (6-2)F_2$
(i) Here the summand $(6-1)F_4$ accounts for the descent that arises at position $5(=6-1)$, when we append 2 to each of the F_4 up-down sequences of length 5 that end in 3.
(ii) The summand $(6-2)F_2$ accounts for the descent that arises at position $4(=6-2)$, when 34 is appended to the one $(=F_2)$ up-down sequence of length 4 that ends in 4.

Case 2: $sum_7 = sum_6 + (7-1)F_4 + sum_5 + (7-2)F_4 + (7-1)F_5$
(i) Here the summand $(7-1)F_4$ counts the F_4 descents that arise at position $6(=7-1)$ when 3 is appended to each of the F_4 up-down sequences of length 6 that end in 4.
(ii) The summand $(7-2)F_4$ accounts for the descents that arise at position $5(=7-2)$ when 21 is appended to each of the F_4 up-down sequences of length 5 that end in 3.
(iii) The summand $(7-1)F_5$ accounts for each additional descent that arises at position $6(=7-1)$ when 21 is appended to each of the F_5 up-down sequences of length 5.

In general, for n even, with $n \geq 4$,

$$sum_n = sum_{n-1} + sum_{n-2} + (n-1)F_{n-2} + (n-2)F_{n-4},$$

while, for n odd, with $n \geq 3$,

$$\begin{aligned} sum_n &= sum_{n-1} + sum_{n-2} + (n-1)F_{n-3} + (n-1)F_{n-2} + (n-2)F_{n-3} \\ &= sum_{n-1} + sum_{n-2} + (n-1)F_{n-1} + (n-2)F_{n-3}. \end{aligned}$$

Consequently, for $n \geq 3$,

$$\begin{aligned} sum_n = {} & sum_{n-1} + sum_{n-2} \\ & + \left(\frac{1}{2}\right) \left[1 + (-1)^{n-1}\right] \left[(n-1)F_{n-1} + (n-2)F_{n-3}\right] \\ & + \left(\frac{1}{2}\right) \left[1 + (-1)^{n}\right] \left[(n-1)F_{n-2} + (n-2)F_{n-4}\right], \end{aligned}$$

with $sum_1 = sum_2 = 0$.

Using the Binet form for the Fibonacci numbers, this recurrence relation becomes

$$\begin{aligned} sum_n = {} & sum_{n-1} + sum_{n-2} \\ & + \left(-\frac{1}{4} + \frac{1}{4}\sqrt{5}\right) n\alpha^n + \left(-\frac{1}{4} - \frac{1}{4}\sqrt{5}\right) n\beta^n \\ & + \left(\frac{1}{2} - \frac{2}{5}\sqrt{5}\right) \alpha^n + \left(\frac{1}{2} + \frac{2}{5}\sqrt{5}\right) \beta^n \\ & + \left(-\frac{7}{4} + \frac{3}{4}\sqrt{5}\right) n(-\alpha)^n + \left(-\frac{7}{4} - \frac{3}{4}\sqrt{5}\right) n(-\beta)^n \\ & + \left(3 - \frac{13}{10}\sqrt{5}\right) (-\alpha)^n + \left(3 + \frac{13}{10}\sqrt{5}\right) (-\beta)^n. \end{aligned}$$

Consequently, the solution has the form

$$\begin{aligned} sum_n = {} & c_1^{**}\alpha^n + c_2^{**}\beta^n \\ & + (A_1^{**} + A_2^{**}n)n\alpha^n + (B_1^{**} + B_2^{**}n)n\beta^n \\ & + (A_3^{**} + A_4^{**}n)(-\alpha)^n + (B_3^{**} + B_4^{**}n)(-\beta)^n. \end{aligned}$$

Continuing with the methods we used in Sections 3 and 4, we now find that

$$A_1^{**} = -\frac{1}{20}\sqrt{5} \text{ and } A_2^{**} = \frac{1}{20}\sqrt{5}$$

while

$$B_1^{**} = \frac{1}{20}\sqrt{5} \text{ and } B_2^{**} = -\frac{1}{20}\sqrt{5}.$$

Further similar calculations tell us that

$$A_3^{**} = \left(\frac{1}{4} - \frac{3}{40}\sqrt{5}\right) \text{ and } A_4^{**} = \left(\frac{1}{2} - \frac{1}{4}\sqrt{5}\right)$$

with

$$B_3^{**} = \left(\frac{1}{4} + \frac{3}{40}\sqrt{5}\right) \text{ and } B_4^{**} = \left(\frac{1}{2} + \frac{1}{4}\sqrt{5}\right).$$

Therefore,

$$\begin{aligned} sum_n &= c_1^{**}\alpha^n + c_2^{**}\beta^n \\ &+ \left(-\frac{1}{20}\sqrt{5}\right) n\alpha^n + \left(\frac{1}{20}\sqrt{5}\right) n^2\alpha^n \\ &+ \left(\frac{1}{20}\sqrt{5}\right) n\beta^n + \left(-\frac{1}{20}\sqrt{5}\right) n^2\beta^n \\ &+ \left(\frac{1}{2} - \frac{1}{4}\sqrt{5}\right) n(-\alpha)^n + \left(\frac{1}{4} - \frac{3}{40}\sqrt{5}\right) (-\alpha)^n \\ &+ \left(\frac{1}{2} + \frac{1}{4}\sqrt{5}\right) n(-\beta)^n + \left(\frac{1}{4} + \frac{3}{40}\sqrt{5}\right) (-\beta)^n, \end{aligned}$$

and from the initial conditions it follows that

$$c_1^{**} = \left(-\frac{1}{4} - \frac{1}{8}\sqrt{5}\right) \text{ and } c_2^{**} = \left(-\frac{1}{4} + \frac{1}{8}\sqrt{5}\right).$$

Consequently, for $n \geq 1$,

$$\begin{aligned} sum_n &= \left(-\frac{1}{4} - \frac{1}{8}\sqrt{5}\right) \alpha^n + \left(-\frac{1}{4} + \frac{1}{8}\sqrt{5}\right) \beta^n \\ &+ \left(-\frac{1}{20}\sqrt{5}\right) n\alpha^n + \left(\frac{1}{20}\sqrt{5}\right) n^2\alpha^n \\ &+ \left(\frac{1}{20}\sqrt{5}\right) n\beta^n + \left(-\frac{1}{20}\sqrt{5}\right) n^2\beta^n \\ &+ \left(\frac{1}{2} - \frac{1}{4}\sqrt{5}\right) n(-\alpha)^n + \left(\frac{1}{4} - \frac{3}{40}\sqrt{5}\right) (-\alpha)^n \\ &+ \left(\frac{1}{2} + \frac{1}{4}\sqrt{5}\right) n(-\beta)^n + \left(\frac{1}{4} + \frac{3}{40}\sqrt{5}\right) (-\beta)^n \\ &= \left(\frac{1}{8}\right) \left[-5 - 2n + 2n^2 - 3(-1)^n - 10(-1)^n n\right] F_n \\ &+ \left(\frac{1}{4}\right) \left[-1 + (-1)^n + 2(-1)^n n\right] L_n. \end{aligned}$$

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