On quadrilaterals and 4-path in claw-free graphs *

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Abstract

Let G be a claw-free graph of order $4k$, where k is a positive integer. In this paper, it is proved that if the degree sum $d(u)+d(v)$ is at least $4k-2$ for every pair of nonadjacent vertices $u, v \in V(G)$, then G has a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, unless G is isomorphic to $K_{4k_1+2} \bigcup K_{4k_2+2}$ or $K_{4k_1+1} \bigcup K_{4k_2+3}$, where $k_1 \geq 0, k_2 \geq 0, k_1 + k_2 = k-1$. We further showed that the requirement about claw-free is indispensable and the degree condition is sharp.

Keywords: claw-free, vertex-disjoint, quadrilaterals, 4-path
MSC(2010): 05C38, 05C70

1 Introduction

In this paper, all graphs are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [1]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\delta(G)$ to denote the vertex set, edge set and minimum degree in G respectively. Besides, $\sigma_2(G)=\min\{d(x) + d(y) \mid x, y \in V(G), x \neq y, xy \notin E(G)\}$ is the minimum degree sum of nonadjacent vertices. The order of G is $|G|$ and its size is $e(G)=|E|$. A set of graphs is said to be vertex-disjoint if no two of them have any common vertex. An

*This work was supported by the TianYuan Special Funds of the National Natural Science Foundation of China(Grant No. 11226292) and the Fundamental Research Funds for the Central Universities(Grant No. K5051370010, K5051306014).

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independent set of the graph G is a set of vertices with no edge between them. For $x \in V(G)$, $N_G(x)=\{y \in V(G) \mid xy \in E(G)\}$. If H is a subgraph of G, then $N_H(x)=N_G(x)\bigcap V(H)$ and $d(x,H)=|N_H(x)|$. Suppose X and Y be two vertex-disjoint subgraphs of G or two disjoint subsets of $V(G)$. We define $G[X]$ to be the subgraph of G induced by X and $e(X,Y)$ to be the number of edges between X and Y. A k-cycle is a cycle of order k, and a m-path is a path of order m. Particularly, a quadrilateral is a cycle of order 4 and a triangle is a cycle of order 3. For a k-cycle $C=x_1x_2...x_kx_1$, $x_ix_{i+1}(1 \le i \le k$ is an integer) is an edge in C, where the subscript is reduced modulo k when it is larger than k.

For two vertex-disjoint graphs G and H, $G \cup H$ is the union of G and H without adding any edge between G and H. A claw is a complete bipartite graph $K_{1,3}$. A graph is said claw-free if it does not contain an induced subgraph isomorphic to a claw. For two graphs G and H, we use $G \simeq H$ to denote that G is isomorphic to H.

A long-standing conjecture on quadrilaterals comes from Erdős [2], which has been proved by Wang [6] recently.

Theorem 1.1 *(Erdős [2], Wang [6]) For a graph G of order $n = 4k$, where k is a positive integer, if the minimum degree $\delta(G) \ge 2k$, then G contains k vertex-disjoint quadrilaterals.*

There are many results related to this theorem(see [3, 4, 7, 8, 9, 10]). Among all the results, Yan and Liu [7] showed the following theorem in 2003.

Theorem 1.2 *(Yan and Liu [7]) Let G be a graph with $|G|=4k$, where $k > 0$. If $\sigma_2(G) \ge 4k-1$, then G contains $k-1$ vertex-disjoint quadrilaterals and a 4-path such that all of them are vertex-disjoint.*

We improve the above result by reducing the degree condition by 1 for claw-free graphs. Before giving our result, we define two types of exceptional graphs: $M(k_1,k_2)=K_{4k_1+2} \bigcup K_{4k_2+2}$ and $N(k_1,k_2)=K_{4k_1+1} \bigcup K_{4k_2+3}$, where $k_1 \ge 0, k_2 \ge 0$. Our main result is as follows.

Theorem 1.3 *Let k be a positive integer. If G is a claw-free graph with $|G|=4k$ and $\sigma_2(G) \ge 4k-2$, then G has a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, unless $G \simeq M(k_1,k_2)$ or $G \simeq N(k_1,k_2)$, where $k_1 \ge 0, k_2 \ge 0, k_1+k_2 = k-1$.*

To illustrate that the condition of G being claw-free is indispensable, we consider a graph isomorphic to the complete bipartite graph $K_{2k+1,2k-1}$. Obviously, $|K_{2k+1,2k-1}| = 4k$, $\sigma_2(K_{2k+1,2k-1}) = 4k-2$ and $K_{2k+1,2k-1}$

contains a claw. Since any quadrilateral and 4-path must contain two vertices of each partite set, $K_{2k+1,2k-1}$ does not contain a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint. Since $K_{2k+1,2k-1}$ contains an independent set with at least three vertices, it follows that $K_{2k+1,2k-1}$ does not belong to the two types of graph $M(k_1,k_2)$ or $N(k_1,k_2)$.

Furthermore, the degree condition of our result is sharp, just considering the graph $(K_{4k_1+2}-e)\bigcup K_{4k_2+2}$, denoted by $S(k)$, where e is an edge in K_{4k_1+2}, $k_1+k_2=k-1, k_1>k_2\geq 0$. Obviously, $|S(k)|=4k$, $\sigma_2(S(k))=4k-3$ and $S(k)$ is claw-free. Furthermore, $S(k)$ does not contain a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint and $S(k)$ does not belong to the two types of graph $M(k_1,k_2)$ or $N(k_1,k_2)$.

In the next section, we give some useful lemmas. Then we prove our main result in Section 3.

2 Technical lemmas

In this section, G is a graph of order $n\geq 3$.

Lemma 2.1 *(see [5]) Let C be a quadrilateral and let x and y be two distinct vertices of G not on C. Suppose $d(x,C)+d(y,C)\geq 5$, then $G[V(C)\bigcup\{x,y\}]$ contains a quadrilateral C' and an edge e such that C' and e are vertex-disjoint and e is incident with exactly one of x and y.*

Lemma 2.2 *(see [5]) Let C be a quadrilateral and let P_1 and P_2 be two paths of order 2 in G. Suppose C, P_1 and P_2 are vertex-disjoint and $e(C,P_1\bigcup P_2)\geq 9$. Then $G[C\bigcup P_1\bigcup P_2]$ contains a quadrilateral C' and a path P of order 4 such that C' and P are vertex-disjoint.*

Lemma 2.3 *(see [5]) Let P_1,P_2 be two vertex-disjoint paths of order 4 in G. If $e(P_1,P_2)\geq 6$, then $G[V(P_1)\bigcup V(P_2)]$ contains a quadrilateral.*

Lemma 2.4 *(see [5]) Let C be a quadrilateral and P_1 and P_2 be two paths of order 4 in G. Suppose C, P_1 and P_2 are vertex-disjoint and $e(C,P_1\bigcup P_2)\geq 17$, then $G[V(C\bigcup P_1\bigcup P_2)]$ contains two vertex-disjoint quadrilaterals.*

Lemma 2.5 *(see [4]) Let P_1 and P_2 be two vertex-disjoint paths of order 4 in G. If $G[V(P_1\bigcup P_2)]$ doesn't contain a quadrilateral, then $V(P_1\bigcup P_2)$ can be partitioned into four pairs of nonadjacent vertices.*

Proof. Denote $P_1 = x_1x_2x_3x_4$ and $P_2 = y_1y_2y_3y_4$. Since P_1 and P_2 do not contain a quadrilateral, we have $x_1x_4 \notin E(G)$ and $y_1y_4 \notin E(G)$. If $e(\{x_2, x_3\}, \{y_1, y_4\}) \geq 1$, without loss of generality, say $x_2y_1 \in E(G)$, then $x_3y_2 \notin E(G)$ and $x_2y_3 \notin E(G)$. So $\{x_1, x_4\}$, $\{y_1, y_4\}$, $\{x_2, y_3\}$ and $\{x_3, y_2\}$ are four pairs of nonadjacent vertices. We can get the similar result if $e(\{y_2, y_3\}, \{x_1, x_4\}) \geq 1$. However, if $e(\{x_2, x_3\}, \{y_1, y_4\}) = e(\{y_2, y_3\}, \{x_1, x_4\}) = 0$, then $\{x_2, y_1\}$, $\{x_3, y_4\}$, $\{y_2, x_1\}$ and $\{y_3, x_4\}$ are four pairs of nonadjacent vertices. In each case, $V(P_1 \bigcup P_2)$ can be partitioned into four pairs of nonadjacent vertices. ∎

Lemma 2.6 *Let* $|D|$=4. *If* $G[D]$ *does not contain a path of order* 4, *then* $e(D) \leq 3$.

Proof. Assume to the contrary that $e(D) \geq 4$. Denote the four vertices of D to be x, y, z, w. Since $e(D) \geq 4$, without loss of generality(simply denoted by w.l.o.g. hereafter), assume $xy \in E(G[D])$. Suppose $zw \in E(G[D])$. Since $e(D) \geq 4$, it follows that $e(xy, zw) \geq 2$. Thus $G[D]$ contains a path of order 4, a contradiction. Hence $zw \notin E(G[D])$. Since $e(D) \geq 4$, we have $e(xy, zw) \geq 3$. W.l.o.g., suppose $\{xz, xw, yz\} \subseteq E(G[D])$. Then $G[D]$ contains a 4-path $zyxw$, a contradiction. ∎

The following lemma is obvious.

Lemma 2.7 *If* ab, cd *are two vertex-disjoint edges in* G, $e(ab, cd) \geq 3$, *then* $G[a, b, c, d]$ *contains a quadrilateral.*

Lemma 2.8 *Let* ab, cd *be two vertex-disjoint edges and* $Q = x_1x_2x_3x_4x_1$ *be a quadrilateral in* G *such that all of them are vertex-disjoint. Furthermore, suppose* $G[Q \bigcup ab \bigcup cd]$ *does not contain a quadrilateral and a* 4*-path such that they are vertex-disjoint and* $e(ab, x_jx_{j+1}) = 2$, $e(cd, x_jx_{j+1}) = 2$ *for all* $j \in \{1, 2, 3, 4\}$. *The following two statements hold:*

(*i*) *If* $\{ax_1, ax_2\} \subseteq E(G)$, *then* $\{ax_3, ax_4\} \subseteq E(G)$.

(*ii*) *If* $\{ax_1, bx_1\} \subseteq E(G)$, *then* $\{ax_3, bx_3\} \subseteq E(G)$.

Proof. (i) Suppose $\{ax_1, ax_2\} \subseteq E(G)$. Since $e(ab, x_1x_2) = 2$, it follows that $e(b, x_1x_2) = 0$ and therefore $bx_2 \notin E(G)$. If $bx_3 \in E(G)$, then abx_3x_2a is a quadrilateral. Since $e(x_1x_4, cd) = 2$, $G[x_1, x_4, c, d]$ contains a 4-path. Therefore, $G[Q \bigcup \{a, b, c, d\}]$ contains a quadrilateral and a 4-path such that they are vertex-disjoint, a contradiction. Hence $bx_3 \notin E(G)$. Since $e(ab, x_2x_3) = 2$, it follows that $ax_3 \in E(G)$. With the same proof, $ax_4 \in E(G)$.

(ii) Suppose $\{ax_1, bx_1\} \subseteq E(G)$. Since $e(ab, x_1x_2) = 2$ and $e(ab, x_1x_4) = 2$, it follows that $e(ab, x_2) = 0$ and $e(ab, x_4) = 0$. Since $e(ab, x_2x_3) = 2$, we have $e(ab, x_3) = 2$. Thus, $\{ax_3, bx_3\} \subseteq E(G)$. ∎

Lemma 2.9 *Let G be a claw-free graph with $|G|=4k$ and $\sigma_2(G) \geq 4k-2$, where $k > 1$ is an integer. Suppose G contains $k-1$ quadrilaterals $Q_1, Q_2, \ldots, Q_{k-1}$ and two vertex-disjoint edges e_1, e_2 such that all of them are vertex-disjoint. If there exists $Q_i = x_1x_2x_3x_4x_1$ such that $e(e_1, x_jx_{j+1}) = e(e_2, x_jx_{j+1}) = 2$ for all $j \in \{1,2,3,4\}$, then G has a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint.*

Proof. Suppose on the contrary that G does not contain a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint. Denote $e_1 = xy$, $e_2 = zw$, $D = G[e_1 \bigcup e_2]$ and $H = \bigcup_{l=1}^{k-1} Q_l$. By symmetry, assume $e(x, x_1x_2) \geq e(y, x_1x_2)$ and $e(z, x_1x_2) \geq e(w, x_1x_2)$. Obviously, $e(D) = 2$ and there is no edge between e_1 and e_2. We have $G[e_l \bigcup \{x_j, x_{j+1}\}]$ does not contain a quadrilateral for every $l \in \{1,2\}$ and $j \in \{1,2,3,4\}$, otherwise, $G[D \bigcup Q_i]$ contains a quadrilateral and a 4-path such that they are vertex-disjoint, a contradiction.

Suppose $e(x, x_1x_2) = 2$. By Lemma 2.8, $e(x, Q_i) = 4$ and $e(y, Q_i) = 0$. Thus, $x_1x_3 \in E$ and $x_2x_4 \in E$ for otherwise G contains a claw. If $e(z, x_1x_2) = 2$, then $e(z, Q_i) = 4$ and $e(w, Q_i) = 0$ from Lemma 2.8. This implies that $e(\{y, w\}, H - Q_i) \geq 4k - 2 - 2 = 4(k-2) + 4$. Hence, there is $Q_l \subseteq H - Q_i$ such that $e(\{y, w\}, Q_l) \geq 5$. By Lemma 2.1, $G[Q_l \bigcup \{y, w\}]$ contains a quadrilateral Q_l' and an edge e such that Q_l' and e are vertex-disjoint and e is incident with exact one of y and w, w.o.l.g., let $e = ya$, where $a \in V(Q_l)$. Then $G[D \bigcup Q_l \bigcup Q_i]$ contains two quadrilaterals Q_l', $zx_1x_2x_3z$ and a 4-path $ayxx_4$ such that all of them are vertex-disjoint, a contradiction. Thus, $e(z, x_1x_2) = e(w, x_1x_2) = 1$. Since $G[e_2 \bigcup \{x_1, x_2\}]$ does not contain a quadrilateral, w.l.o.g., assume $zx_1 \in E$ and $wx_1 \in E$. This implies $e(x_1, zw) = e(x_3, zw) = 2$ and $e(x_2x_4, zw) = 0$ from Lemma 2.8. Then $G[D \bigcup Q_i]$ contains a quadrilateral zwx_3x_1z and a 4-path yxx_2x_4 such that they are vertex-disjoint, a contradiction.

Thus, $e(x, x_1x_2) = e(y, x_1x_2) = 1$. By the symmetry of e_1 and e_2, $e(z, x_1x_2) = e(w, x_1x_2) = 1$. Since $G[e_1 \bigcup \{x_1, x_2\}]$ does not contain a quadrilateral, w.l.o.g., let $x_1x \in E$ and $x_1y \in E$. By Lemma 2.8, $e(x_3, e_1) = e(x_1, e_1) = 2$. Since $G[e_2 \bigcup \{x_1, x_2\}]$ does not contain a quadrilateral, it follows that either $e(x_1, e_2) = 2$ or $e(x_2, e_2) = 2$. If $e(x_2, e_2) = 2$, then $e(x_4, e_2) = 2$ from Lemma 2.8, which implies that $G[D \bigcup Q_i]$ contains two vertex-disjoint quadrilaterals xx_3yx_1x and zx_2wx_4z, a contradiction. Hence, $e(x_1, e_2) = 2$. By Lemma 2.8, $e(x_3, e_2) = 2$. Since $e(e_1, Q_i) = e(e_2, Q_i) = 4$, it follows that $e(x_2, xy) = e(x_4, xy) = 0$ and $e(x_2, zw) = e(x_4, zw) = 0$. Hence $G[\{x_1\} \bigcup \{y, x_2, w\}]$ is a claw, a contradiction. ∎

3 Proof of Theorem 1.3

Let G be a claw-free graph with $|G|$=4k and $\sigma_2(G) \geq 4k-2$. Suppose on the contrary that neither G contains a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, nor $G \simeq M(k_1,k_2)$ or $G \simeq N(k_1,k_2)$ for any $k_1 \geq 0, k_2 \geq 0, k_1+k_2 = k-1$. Let G be a maximal counterexample, that is, for any $xy \notin E(G)$, either $G+xy$ contains a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, or $G+xy \simeq M(k_1,k_2)$ or $G+xy \simeq N(k_1,k_2)$ for some $k_1 \geq 0, k_2 \geq 0, k_1+k_2 = k-1$.

Claim 1. G contains $k-1$ vertex-disjoint quadrilaterals.
Proof. If $G+xy \simeq M(k_1,k_2)$ or $G+xy \simeq N(k_1,k_2)$ hold for any $xy \notin E(G)$, obviously G contains $k-1$ vertex-disjoint quadrilaterals. Therefore, we only need to consider the case that $G+xy$ contains a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint for any $xy \notin E(G)$. So G either contains $k-2$ quadrilaterals and two 4-paths such that all of them are vertex-disjoint or contains $k-1$ vertex-disjoint quadrilaterals. Assume on the contrary that G doesn't contain $k-1$ vertex-disjoint quadrilaterals. Then G contains $k-2$ quadrilaterals and two 4-paths such that all of them are vertex-disjoint. Let $Q_1, Q_2 \ldots Q_{k-2}$ be the $k-2$ quadrilaterals and P_1, P_2 be the two 4-paths such that all of them are vertex-disjoint. Denote $H = \bigcup_{i=1}^{k-2} Q_i$, $P_1 = x_1x_2x_3x_4$ and $P_2 = y_1y_2y_3y_4$. Since P_1 and P_2 do not contain a quadrilateral, we have $e(P_1) \leq 4$ and $e(P_2) \leq 4$. Furthermore, $e(P_1,P_2) \leq 5$ from Lemma 2.3. So $\sum_{x \in V(P_1 \bigcup P_2)} d(x, P_1 \bigcup P_2) \leq 8+8+10 = 26$. Since $G[V(P_1 \bigcup P_2)]$ doesn't contain a quadrilateral, by Lemma 2.5, it follows that $V(P_1 \bigcup P_2)$ can be divided into four pairs of nonadjacent vertices. So

$$\sum_{x \in V(P_1 \bigcup P_2)} d(x,H) \geq 4(4k-2) - 26 = 16(k-2) - 2.$$

By Lemma 2.4, $\sum_{x \in V(P_1 \bigcup P_2)} d(x, Q_i) \leq 16$ for all Q_i in H. Therefore

$$16(k-2) - 2 \leq \sum_{x \in V(P_1 \bigcup P_2)} d(x,H) \leq 16(k-2).$$

So we have $3 \leq e(P_1) \leq 4$, $e(P_2) = 4$, $e(P_1,P_2) = 5$ or $e(P_1) = 4$, $3 \leq e(P_2) \leq 4$, $e(P_1,P_2) = 5$ or $e(P_1) = 4$, $e(P_2) = 4$, $4 \leq e(P_1,P_2) \leq 5$.

Assume $e(P_1) = 4$, $e(P_2) = 4$ and $4 \leq e(P_1,P_2) \leq 5$. Without loss of generality, say $\{x_1x_3, y_1y_3\} \subseteq E(G)$. Since $G[V(P_1 \bigcup P_2)]$ does not contain a quadrilateral, we have $e(x_1x_2x_3, y_1y_2y_3) \leq 1$, $e(x_4, y_1y_2y_3) \leq 1$ and $e(y_4, x_1x_2x_3) \leq 1$. Since $e(P_1,P_2) \geq 4$, it follows that $x_4y_4 \in E(G)$,

$e(x_4, y_1y_2y_3) = 1$ and $e(y_4, x_1x_2x_3) = 1$. Obviously, $y_4x_1 \notin E(G)$, $y_4x_2 \notin E(G)$ and so $y_4x_3 \in E(G)$. With the same proof, $x_4y_3 \in E(G)$. Now $x_3x_4y_3y_4x_3$ is a quadrilateral, a contradiction.

Therefore, by symmetry, we have $3 \leq e(P_1) \leq 4$, $e(P_2) = 4$ and $e(P_1, P_2) = 5$. Without loss of generality, say $y_1y_3 \in E(G)$. Since $e(P_1, P_2) = 5$, there exists a vertex $x_i \in V(P_1)$ such that $d(x_i, P_2) = 2$. By symmetry, say $i = 2$ or $i = 1$. If $i = 2$, then $\{x_2y_3, x_2y_4\} \subseteq E(G)$ and $d(x_1, P_2) = d(x_3, P_2) = 0$. Since $d(x_4, P_2) \leq 2$, we have $e(P_1, P_2) \leq 4$, a contradiction. If $i = 1$, we can get a contradiction by a similar argument. Therefore, G contains $k-1$ vertex-disjoint quadrilaterals. ∎

By Claim 1, G contains $k-1$ vertex-disjoint quadrilaterals, denoted by $Q_1, Q_2, \cdots, Q_{k-1}$. Denote $H = \bigcup_{i=1}^{k-1} Q_i$, $D = G - H$. Obviously, D does not contain a 4-path. Now we choose $k-1$ quadrilaterals $Q_1, Q_2, \cdots Q_{k-1}$ such that the number of vertex-disjoint edges of D is maximum.

Claim 2. $e(D) \geq 2$.
Proof. By contradiction, suppose $e(D) \leq 1$. Then we can choose two vertices $z, w \in V(D)$ such that $d(z, D) + d(w, D) = 0$. So $d(z, H) + d(w, H) \geq 4k - 2 - 0 = 4(k-1) + 2$. Therefore, there exists $Q_i \subseteq H$ such that $d(z, Q_i) + d(w, Q_i) \geq 5$. By Lemma 2.1, $G[Q_i \bigcup \{z, w\}]$ contains a quadrilateral Q'_i and an edge e' such that they are vertex-disjoint. Replacing Q_i with Q'_i, we get $H' = (H - Q_i) \bigcup Q'_i$, $D' = G - H'$. Now D' contains more vertex-disjoint edges than D, contradicting our choice of Q_1, $Q_2, \cdots Q_{k-1}$. ∎

In the following, let $V(D) = \{x, y, z, w\}$ and xy be an edge in D.

Since D does not contain a 4-path, we have $e(D) \leq 3$ from Lemma 2.6. Hence,

$$2 \leq e(D) \leq 3. \tag{1}$$

Now we divide the proof into the following two cases: $e(D) = 2$ or $e(D) = 3$.

Case 1. $e(D) = 2$.

In this case, we first prove that D contains two vertex-disjoint edges.

Claim 3. $zw \in E(D)$ and therefore $E(D) = \{xy, zw\}$.
Proof. Suppose $zw \notin E(D)$. Since $e(D) = 2$, w.o.l.g., say $xz \in E(D)$, we have $d(z, D) + d(w, D) = 1$. So $d(z, H) + d(w, H) \geq 4k - 2 - 1 = 4(k-1) + 1$. Therefore, there exists $Q_i \subseteq H$ such that $d(z, Q_i) + d(w, Q_i) \geq 5$. By Lemma 2.1, $G[Q_i \bigcup \{z, w\}]$ contains a quadrilateral Q'_i and an edge e' such that they are vertex-disjoint. Replacing Q_i with Q'_i, we get $H' = (H - Q_i) \bigcup Q'_i$, $D' = G - H'$. D' contains two vertex-disjoint edges xy and e', contradicting our choice of $Q_1, Q_2, \cdots Q_{k-1}$. ∎

For simplicity, denote $e_1 = xy, e_2 = zw$ hereafter. By Claim 3, $E(D) =$

$\{xy, zw\}$. Thus $d(z, D)+d(w, D)+d(x, D)+d(y, D) = 4$. Since $xz \notin E(D)$ and $yw \notin E(D)$, we have

$$e(D, H) \geq 2(4k-2) - 4 = 8(k-1). \quad (2)$$

Claim 4. $e(D, Q_i) = 8$ for each Q_i, where $i = 1, 2 \cdots, k-1$.
Proof. If there is a Q_i in H such that $e(D, Q_i) \geq 9$, by Lemma 2.2, $G[V(Q_i) \bigcup D]$ contains a quadrilateral and a 4-path such that they are vertex-disjoint, a contradiction. So $e(D, Q_i) \leq 8$ for each Q_i and therefore

$$e(D, H) = \sum_{i=1}^{k-1} e(D, Q_i) \leq 8(k-1).$$

By (2), $e(D, H) = 8(k-1)$. Thus, $e(D, Q_i) = 8$ for each $Q_i \subseteq H$. ∎

By Claim 4, for each $Q_i \subseteq H, e(e_1, Q_i) + e(e_2, Q_i) = 8$.

Claim 5. For each $Q_i \subseteq H$, either $e(e_1, Q_i) = 8$ and $e(e_2, Q_i) = 0$ or $e(e_2, Q_i) = 8$ and $e(e_1, Q_i) = 0$.

Proof. Suppose there exits $Q_i \subseteq H$ such that $e(e_1, Q_i) = e(e_2, Q_i) = 4$. Denote Q_i to be $x_1x_2x_3x_4x_1$. Assume that $e(e_1, x_1x_2) \geq 3$. By Lemma 2.7, $G[e_1 \bigcup\{x_1, x_2\}]$ contains a quadrilateral. Thus, $G[e_2 \bigcup\{x_3, x_4\}]$ does not contain a 4-path, which implies $e(e_2, x_3x_4) = 0$. Since $e(e_2, x_1x_2) = e(e_2, Q_i) = 4$, $G[e_2 \bigcup\{x_1, x_2\}]$ contains a quadrilateral from Lemma 2.7. Hence $e(e_1, x_3x_4) = 0$ and therefore $e(e_1, x_1x_2) = 4$. Now we get a claw $G[\{x_2\} \bigcup\{y, w, x_3\}]$, a contradiction. Hence $e(e_1, x_1x_2) \leq 2$. By symmetry, $e(e_1, x_jx_{j+1}) \leq 2$ and $e(e_2, x_jx_{j+1}) \leq 2$ for every $j \in \{1, 2, 3, 4\}$. Since $e(e_1, Q_i) = e(e_2, Q_i) = 4$, it follows that $e(e_1, x_jx_{j+1}) = 2$ and $e(e_2, x_jx_{j+1}) = 2$ for all $j \in \{1, 2, 3, 4\}$. By Lemma 2.9, G contains $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, a contradiction.

Hence, either $e(e_1, Q_i) \geq 5$ or $e(e_2, Q_i) \geq 5$ for every $i \in \{1, 2, \ldots, k-1\}$. By symmetry, say $e(e_1, Q_i) \geq 5$. Denote Q_i to be $x_1x_2x_3x_4x_1$. Then either $e(e_1, x_1x_2) \geq 3$ or $e(e_1, x_3x_4) \geq 3$ holds. Without loss of generality, say $e(e_1, x_1x_2) \geq 3$. By Lemma 2.7, $G[e_1 \bigcup\{x_1, x_2\}]$ contains a quadrilateral. This implies $e(e_2, x_3x_4) = 0$ for otherwise $G[V(Q_i) \bigcup D]$ contains a quadrilateral and a 4-path such that they are vertex-disjoint. Since $e(e_1, x_1x_2) \leq 4$ and $e(e_1, Q_i) \geq 5$, it follows that $e(e_1, x_3x_4) \geq 1$. Hence $G[e_1 \bigcup\{x_3, x_4\}]$ contains a 4-path. By Lemma 2.7, $e(e_2, x_1x_2) \leq 2$. Thus, $e(e_2, Q_i) \leq 2$ and $e(e_1, Q_i) \geq 6$. If $e(e_1, x_3x_4) \geq 3$, then similarly as before, $e(e_2, x_1x_2) = 0$. Hence, $e(e_2, Q_i) = 0$ and $e(e_1, Q_i) = 8$, we complete our proof. Now assume $e(e_1, x_3x_4) \leq 2$. Since $e(e_1, Q_i) + e(e_2, Q_i) = 8$, then

$e(e_1, x_1x_2) = 4$, $e(e_1, x_3x_4) = 2$ and $e(e_2, x_1x_2) = e(e_2, Q_i) = 2$. W.l.o.g., say $zx_1 \in E$. Then $xx_3 \notin E$ and $yx_3 \notin E$ for otherwise $G[e_1 \bigcup \{x_2, x_3\}]$ contains a quadrilateral, which is vertex-disjoint with the 4-path wzx_1x_4. Since $e(e_1, x_3x_4) = 2$, we have $e(e_1, x_4) = 2$. Hence $yx_4 \in E$ and therefore $G[D \bigcup Q_i]$ contains a quadrilateral $x_2x_3x_4yx_2$ and a 4-path xx_1zw such that they are vertex-disjoint, a contradiction. ∎

Now we will complete the proof of Case 1 by proving that G is isomorphic to $M(k_1, k_2)$, where $k_1 \geq 0, k_2 \geq 0, k_1 + k_2 = k - 1$. By Claim 5, w.l.o.g., there exist $k_1 \geq 0$, $k_2 \geq 0, k_1 + k_2 = k - 1$, such that $e(e_1, Q_i) = e(e_2, Q_j) = 8$, $e(e_1, Q_j) = e(e_2, Q_i) = 0$, where $i \in \{1, 2, \cdots, k_1\}$, $j \in \{k_1 + 1, k_1 + 2, \cdots, k_1 + k_2\}$. Denote $H_1 = G[(\bigcup_{m=1}^{k_1} Q_m) \bigcup e_1]$, $H_2 = G[(\bigcup_{l=k_1+1}^{k-1} Q_l) \bigcup e_2]$.

We firstly show that $e(H_1, H_2) = 0$. If $k_1 = 0$ or $k_2 = 0$, then $e(H_1, H_2) = 0$ clearly. Now assume $k_1 \geq 1$ and $k_2 \geq 1$. Obviously, $e(e_1, H_2) = 0$ and $e(e_2, H_1) = 0$. Suppose there is an edge e between $H_1 - x - y$ and $H_2 - z - w$, w.l.o.g, say $e = p_1q_1$, where $p_1 \in V(Q_i)$, $q_1 \in V(Q_j)$, $i \in \{1, 2, \cdots, k_1\}$, $j \in \{k_1 + 1, k_1 + 2, \cdots, k_1 + k_2\}$. Denote $Q_i = p_1p_2p_3p_4p_1$, $Q_j = q_1q_2q_3q_4q_1$. Then $G[D \bigcup Q_i \bigcup Q_j]$ contains two quadrilaterals xyp_3p_4x, zwq_3q_4z and a 4-path $p_2p_1q_1q_2$ such that all of them are vertex-disjoint. Therefore, G has a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, a contradiction. Thus $e(H_1, H_2) = 0$ holds.

Now choose two vertices $v \in V(H_1)$ and $u \in V(H_2)$ arbitrarily. Since $e(H_1, H_2) = 0$, we have $d(v, G) = d(v, H_1) \leq \mid H_1 \mid -1 = 4k_1 + 1$ and $d(u, G) = d(u, H_2) \leq \mid H_2 \mid -1 = 4k_2 + 1$. Therefore, $d(v, G) + d(u, G) \leq 4k - 2$. On the other hand, Since $vu \notin E(G)$, it follows that $d(v, G) + d(u, G) \geq 4k - 2$. Therefore, $d(v, H_1) = \mid H_1 \mid -1$ and $d(u, H_2) = \mid H_2 \mid -1$. Since we choose $v \in V(H_1)$ and $u \in V(H_2)$ arbitrarily, H_1 is isomorphic to K_{4k_1+2} and H_2 is isomorphic to K_{4k_2+2}. So G is isomorphic to $M(k_1, k_2)$, a contradiction, which completes the proof in Case 1.

Case 2. $e(D) = 3$.

As we denoted before, $D = G - H$, $V(D) = \{x, y, z, w\}$ and xy is an edge in D. If $zw \in E(G)$, then $e(xy, zw) = 1$. Obviously, $G[D]$ contains a 4-path, a contradiction. Therefore, $zw \notin E(G)$ and we have $e(xy, \{z, w\}) = 2$. If $\{xz, xw\} \subseteq E(G)$, then $G[D]$ is a claw, contradicting the condition G is claw-free. Therefore, we have $\{xz, yz\} \subseteq E(G)$ by symmetry. Then $G[D]$ contains a triangle $xyzx$ and a vertex w, where $e(xyz, w) = 0$. Note that D has exactly one vertex-disjoint edge in Case 2.

Obviously, $d(w, D) + d(z, D) = 2$. Since $wz \notin E(G)$, we have

$$d(w, H) + d(z, H) \geq 4k - 2 - 2 = 4(k - 1). \qquad (3)$$

If there exists Q_i such that $d(w,Q_i)+d(z,Q_i)\geq 5$, then by Lemma 2.1, $G[Q_i\bigcup\{w,z\}]$ contains a quadrilateral Q_i' and an edge e such that they are vertex-disjoint. Replacing Q_i with Q_i', we get $H'=(H-Q_i)\bigcup Q_i'$, $D'=G-H'$. Now D' contains two vertex-disjoint edges xy and e, contradicting our choice of Q_1, $Q_2,\cdots Q_{k-1}$. Therefore $d(w,Q_i)+d(z,Q_i)\leq 4$ for each $Q_i\subseteq H$. Then

$$d(w,H)+d(z,H)=\sum_{i=1}^{k-1}(d(w,Q_i)+d(z,Q_i))\leq 4(k-1). \quad (4)$$

By (3) and (4), we have $e(\{w,z\},Q_i)=4$ for any $Q_i\subseteq H$. With the same proof, $e(\{w,x\},Q_i)=4$ and $e(\{w,y\},Q_i)=4$ for each $Q_i\subseteq H$. Therefore for each $Q_i\subseteq H$

$$e(\{w,x\},Q_i)=e(\{w,y\},Q_i)=e(\{w,z\},Q_i)=4. \quad (5)$$

Claim 6. For each $Q_i\subseteq H$, either $d(w,Q_i)=4$ or $d(w,Q_i)=0$.
Proof. Suppose on the contrary that $1\leq d(w,Q_i)\leq 3$ for some $Q_i\subseteq H$. Denote Q_i to be $x_1x_2x_3x_4x_1$. We firstly claim there exist two vertices p, $q\in V(D-w)$ and a vertex $h\in V(Q_i)$ such that $ph\in E$ and $qh\in E$. If $d(w,Q_i)\leq 2$, then $e(D-w,Q_i)\geq 6$ by (5) and the claim holds obviously. Now assume $d(w,Q_i)=3$, w.l.o.g., say $\{x_2,x_3,x_4\}\subseteq N_{Q_i}(w)$. Then $e(D-w,x_1x_3)=0$ for otherwise $G[D\bigcup Q_i]$ contains a quadrilateral and a 4-path such that they are vertex-disjoint. By (5), $d(x,Q_i)=d(y,Q_i)=d(z,Q_i)=1$. Therefore, $e(D-w,x_2x_4)=e(D-w,Q_i)=3$, which implies that there exist two vertices p, q in $V(D-w)$ and a vertex $h\in\{x_2,x_4\}$ such that $ph\in E$ and $qh\in E$.

Since $ph\in E$ and $qh\in E$, it follows that $G[D+h-w]$ contains a quadrilateral, denoted by Q_i'. If $d(w,Q_i-h)>0$, then $G[Q_i+w-h]$ either is a claw or contains a 4-path, which is vertex-disjoint with the quadrilateral Q_i', a contradiction. Hence $wh\in E$ and $d(w,Q_i)=1$. By (5), $d(p,Q_i)=d(q,Q_i)=3$. Thus $e(pq,Q_i-h)\geq 6-2=4$, which implies there exists a vertex $b\in V(Q_i-h)$ such that $pb\in E$ and $qb\in E$. Therefore, $G[D+b-w]$ contains a quadrilateral, denoted by Q_i''. Since $d(w,Q_i-b)=1>0$, we have $G[Q_i+w-b]$ either is a claw or contains a 4-path, which is vertex-disjoint with the quadrilateral Q_i'', a contradiction. ∎

Now we will show that G is isomorphic to $N(k_1,k_2)$ and complete our proof. By Claim 6, w.l.o.g., there exist two integers $k_1\geq 0, k_2\geq 0$, $k_1+k_2=k-1$ such that $d(w,Q_t)=4$ and $d(w,Q_j)=0$, where $t\in\{1,2,\cdots,k_1\}$, $j\in\{k_1+1,k_1+2,\cdots,k_1+k_2\}$. Let $H_1=G[(\bigcup_{m=1}^{k_1}Q_m)\bigcup\{w\}]$, $H_2=$

$G[(\bigcup_{j=k_1+1}^{k-1} Q_j) \bigcup \{x, y, z\}]$.

We firstly show that $e(H_1, H_2) = 0$. If $k_1 = 0$ or $k_2 = 0$, then $e(H_1, H_2) = 0$ obviously. Now assume $k_1 \geq 1$ and $k_2 \geq 1$. By (5), $d(x, Q_j) = d(y, Q_j) = d(z, Q_j) = 4$ and $d(x, Q_t) = d(y, Q_t) = d(z, Q_t) = 0$, where $t \in \{1, 2, \cdots, k_1\}$, $j \in \{k_1+1, k_1+2, \cdots, k_1+k_2\}$. Thus $e(w, H_2) = 0$ and $e(\{x, y, z\}, H_1) = 0$. Suppose there exists an edge e between $H_1 - w$ and $H_2 - \{x, y, z\}$, w.o.l.g., say $e = p_1q_1$, where $p_1 \in V(Q_t)$, $q_1 \in V(Q_j)$, $t \in \{1, 2, \cdots, k_1\}$, $j \in \{k_1+1, k_1+2, \cdots, k_1+k_2\}$. Denote $Q_t = p_1p_2p_3p_4p_1$ and $Q_j = q_1q_2q_3q_4q_1$. Then $G[D \bigcup Q_t \bigcup Q_j]$ contains two quadrilaterals $wp_2p_3p_4w$, $xq_2q_3q_4x$ and a 4-path p_1q_1yz such that all of them are vertex-disjoint. Therefore, G contains a spanning subgraph consisting of $k-1$ quadrilaterals and a 4-path such that all of them are vertex-disjoint, a contradiction. Thus $e(H_1, H_2) = 0$ holds.

Now choose two vertices $v \in V(H_1)$ and $u \in V(H_2)$ arbitrarily. Since $e(H_1, H_2) = 0$, we have $d(v, G) = d(v, H_1) \leq |H_1| - 1 = 4k_1$ and $d(u, G) = d(u, H_2) \leq |H_2| - 1 = 4k_2 + 2$. Thus $d(v, G) + d(u, G) \leq 4k - 2$. On the other hand, Since $vu \notin E(G)$, it follows that $d(v, G) + d(u, G) \geq 4k - 2$. Therefore, $d(v, H_1) = |H_1| - 1$ and $d(u, H_2) = |H_2| - 1$. Since we choose $v \in V(H_1)$ and $u \in V(H_2)$ arbitrarily, H_1 is isomorphic to K_{4k_1+1} and H_2 is isomorphic to K_{4k_2+3}. Note that $e(H_1, H_2) = 0$, then G is isomorphic to $N(k_1, k_2)$, a contradiction. This completes the whole proof.

Acknowledgements

The authors wish to thank the referees for their careful reading and suggestion for the revision of this paper.

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