The existence of 3 pairwise additive $B(v, 2, 1)$ for any $v \geq 6$

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Abstract. The existence of additive BIB designs and 2 pairwise additive BIB designs has been discussed with direct and recursive constructions in [6, 9]. In this paper, by reorganizing some methods of constructing pairwise additive BIB designs from other combinatorial structures, it is shown that 3 pairwise additive $B(v, 2, 1)$ can be constructed for any integer $v \geq 6$.

Keywords: incidence matrix; balanced incomplete block (BIB) design; pairwise additive BIB design; nested BIB design; self-orthogonal latin square.

1 Introduction

A *balanced incomplete block* (BIB) *design* is a system $(V, \mathcal{B})$ with v points and b blocks each containing k different points, each point appearing in r different blocks and any two different points appearing in exactly λ blocks [8]. This is denoted by BIBD(v, b, r, k, λ) or $B(v, k, \lambda)$. Let $\boldsymbol{N} = (n_{ij})$ be the $v \times b$ incidence matrix of a BIB design, where $n_{ij} = 1$ or 0 for all i $(= 1, ..., v)$ and j $(= 1, ..., b)$, according as the ith point occurs in the jth block or otherwise. Hence the incidence matrix $\boldsymbol{N}$ satisfies the conditions: (i) $\sum_{j=1}^{b} n_{ij} = r$ for all i, (ii) $\sum_{i=1}^{v} n_{ij} = k$ for all j, (iii) $\sum_{j=1}^{b} n_{ij}n_{i'j} = \lambda$ for all i, i' $(i \neq i') = 1, ..., v$.

Let $s = v/k$, where s need not be an integer unlike other parameters. A set of ℓ BIBD(v, b, r, k, λ) is called *ℓ pairwise additive BIB designs* if ℓ $(2 \leq \ell \leq s)$ corresponding incidence matrices $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_\ell$ of the BIB design satisfy the condition:

(A) $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2}$ is the incidence matrix of a BIBD$(v^* = v = sk,\ b^* = b,\ r^* = 2r,\ k^* = 2k,\ \lambda^* = 2r(2k-1)/(sk-1))$ for any distinct $i_1, i_2 \in \{1, 2, ..., \ell\}$.

This is especially called *additive BIB designs* [5, 9] if $\ell = s$.

Additive BIB designs and pairwise additive BIB designs have been introduced by Matsubara et al. [5] and Sawa et al. [9]. In pairwise additive B(v, k, λ), it is known [9] that $2\lambda/(k-1)$ is a positive integer, which implies $k = 2$ or 3 when $\lambda = 1$. In pairwise additive B(v, 2, 1), it is finally shown [6] that there are 2 pairwise additive B(v, 2, 1) for any integer $v \geq 4$. A problem of constructing 3 pairwise additive B(v, 2, 1) is more difficult than that of 2 pairwise additive B(v, 2, 1), because incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$ of 3 pairwise additive B(v, 2, 1) possess the property that all of $\boldsymbol{N}_1+\boldsymbol{N}_2, \boldsymbol{N}_2+\boldsymbol{N}_3, \boldsymbol{N}_3+\boldsymbol{N}_1$ are incidence matrices of BIB designs respectively. Furthermore, it is shown that $\boldsymbol{N}_1+\boldsymbol{N}_2+\boldsymbol{N}_3$ is an incidence matrix of a BIB design. This construction is combinatorially a challenging problem.

The purpose of this paper is devoted to show the existence of 3 pairwise additive B(v, 2, 1) for any integer $v \geq 6$, by reorganizing some methods which are somehow similar to [9] of constructing some class of ℓ pairwise additive B(v, 2, 1) through self-orthogonal latin squares and nested BIB designs for some $\ell \leq v/2$, though the terms SOLS and nestedness are not used in [9].

2 Construction by self-orthogonal latin squares

In this section, some constructions of pairwise additive BIB designs through self-orthogonal latin squares are presented.

Let $A = (a_{ij})$ and $B = (b_{ij})$ are two latin squares of order v. The latin squares A and B are said to be *orthogonal* if all ordered pairs (a_{ij}, b_{ij}) are distinct. Latin squares $A_1, ..., A_\ell$ are called *mutually orthogonal latin squares of order* v, denoted by MOLS(v), if they are orthogonal in each pair. A *self-orthogonal latin square of order* v, denoted by SOLS(v), is a latin square that is orthogonal to its transpose. A set $\{L_1, ..., L_\ell\}$ of self-orthogonal latin squares of order v is called ℓ SOLS(v), if $\{L_1, L_1^T, \ldots, L_\ell, L_\ell^T\}$ is a set of 2ℓ MOLS(v). A latin square $A = (a_{ij})$ is said to be *idempotent* if $a_{ii} = i$ where $1 \leq i \leq v$. It is here known [4] that any latin squares in ℓ SOLS(v) can be made idempotent by renaming the symbol.

Lemma 2.1 [1] There are 2 SOLS(v) for all $v \geq 7$ except possibly for $v \in \{10, 12, 14, 18, 21, 22, 24, 30, 34\}$.

In [9], additive B(2^n, 2, 1) are essentially constructed through $2^{n-1}-1$ SOLS(v) for any integer n. Similarly, $\ell+1$ pairwise additive B(v, 2, 1) will be obtained through ℓ SOLS(v) as follows.

Theorem 2.2 The existence of ℓ SOLS(v) implies the existence of $\ell+1$ pairwise additive B(v, 2, 1) for a positive integer ℓ.

Proof. Let a set of 2ℓ idempotent MOLS(v) be $\{L_h, L_h^T | 1 \le h \le \ell\}$, where $L_h = (a_{ij}^{(2h-1)})$ and $L_h^T = (a_{ji}^{(2h-1)}) = (a_{ij}^{(2h)})$, derived from the ℓ SOLS(v). Let $\ell + 1$ incidence matrices of B($v, 2, 1$) be

$$\begin{aligned} \boldsymbol{N}_0 &: \{i, j\} \\ \boldsymbol{N}_h &: \{a_{ij}^{(2h-1)}, a_{ij}^{(2h)}\} \end{aligned}$$

where $1 \le i < j \le v$. Then, for $1 \le h < h' \le \ell$, the $v(v-1)/2$ ordered pairs of $\{a_{ij}^{(2h-1)}, a_{ij}^{(2h'-1)}\}$ and $\{a_{ij}^{(2h)}, a_{ij}^{(2h')}\}$ are distinct from each other, because of the property of orthogonality of idempotent latin squares L_h and $L_{h'}$. Similarly, the $v(v-1)/2$ ordered pairs of $\{a_{ij}^{(2h-1)}, a_{ij}^{(2h')}\}$ and $\{a_{ij}^{(2h)}, a_{ij}^{(2h'-1)}\}$ are distinct from each other, because of the property of orthogonality of idempotent latin squares L_h and $L_{h'}^T$. Hence it follows that $\boldsymbol{N}_h + \boldsymbol{N}_{h'}$ forms a B($v, 4, 6$). Finally, for $1 \le h \le \ell$, $\boldsymbol{N}_0 + \boldsymbol{N}_h$ can also form a B($v, 4, 6$), because of the property of idempotent latin squares. Hence the $\ell + 1$ incidence matrices $\boldsymbol{N}_0, \boldsymbol{N}_h$, for $1 \le h \le \ell$, can yield the required designs. □

Thus, Lemma 2.1 and Theorem 2.2 with $\ell = 2$ can produce the following.

Theorem 2.3 There are 3 pairwise additive B($v, 2, 1$) for all $v \ge 7$ except possibly for $v \in \{10, 12, 14, 18, 21, 22, 24, 30, 34\}$.

3 Construction by nested BIB designs

In this section, some constructions of pairwise additive BIB designs through nested BIB designs with a perpendicular array are presented.

Preece [7] introduced the concept of a *nested BIB design* (NBIBD), denoted by NB($v; b_1, b_2; k_1, k_2$). An NB($v; b_1, b_2; k_1, k_2$) is a triple $(V, \mathcal{B}_1, \mathcal{B}_2)$ with v points, $|V| = v$, and two systems of blocks, $|\mathcal{B}_i| = b_i$, $i = 1, 2$, such that (i) the first system is nested within the second, i.e., each block in $\mathcal{B}_2$ is partitioned into u subblocks of size k_1 and the resulting subblocks form $\mathcal{B}_1$, here, $b_1 = ub_2$ and $k_2 = uk_1$, (ii) $(V, \mathcal{B}_i)$ is a BIB design with v points and b_i blocks of k_i points each.

A *perpendicular array* (PA), denoted by PA$_d(g, s)$, is a matrix with g rows and $d\binom{s}{2}$ columns with entries from an s-set S such that (i) each column has g distinct entries, and (ii) each set of 2 rows contains each set of 2 distinct entries of S as a column precisely d times [2]. When $d = 1$, it is simply written as PA(g, s).

Lemma 3.1 [9] There exists a PA(p, p) for any odd prime power p.

In [9], a construction of additive $B(v = sk, k, [s(s-1)/2]\lambda)$ by use of $PA(s, s)$ and resolvable $B(v = sk, k, \lambda)$ is given. Similarly, pairwise additive $B(v = sk, k, \lambda)$ can be obtained through nested BIB designs as follows.

Theorem 3.2 Let $s'(\leq s)$ be an odd prime power and an $NB(v = sk_1; b_1 = s'b_2, b_2; k_1, k_2 = s'k_1)$ exist. Then s' pairwise additive $BIBD(v = sk_1, b = (s'-1)b_1/2, r = (s'-1)b_1/(2s), k = k_1, \lambda = (k_1-1)(s'-1)b_1/[2s(sk_1-1)])$ exist.

Proof. Since s' is an odd prime power, a $PA(s', s')$ of size $s' \times [s'(s'-1)/2]$ exists on account of Lemma 3.1. Let the symbols in this $PA(s', s')$ be from the set $\{1, \ldots, s'\}$. For convenience of representation, let (m, n)-entry of $PA(s', s')$ be $p(m, n)$ $(1 \leq m \leq s', 1 \leq n \leq s'(s'-1)/2)$. For two systems of blocks, $|\mathcal{B}_i| = b_i$, $i = 1, 2$, in the NBIBD, blocks are $B^{(h)} \in \mathcal{B}_2$ $(1 \leq h \leq b_2)$, and subblocks are $B_f^{(h)} \in \mathcal{B}_1$ $(1 \leq h \leq b_2, 1 \leq f \leq s')$. Here $\bigcup_{1 \leq f \leq s'} B_f^{(h)} = B^{(h)}$. Now let

$$\mathcal{B}_j^* = \left(\boldsymbol{B}_j^{(1)} : \cdots : \boldsymbol{B}_j^{(b_2)}\right)$$

where $\boldsymbol{B}_j^{(h)}$ are the juxtaposition of subblocks in $B^{(h)}$ with indices being entries of $PA(s', s')$, i.e.,

$$\boldsymbol{B}_j^{(h)} = \left(B_{p(j,1)}^{(h)} : \cdots : B_{p(j,s'(s'-1)/2)}^{(h)}\right), 1 \leq h \leq b_2, 1 \leq j \leq s'.$$

Then it follows that $(V, \mathcal{B}_j^*)$ is s' pairwise additive $BIBD(v = sk_1, b = (s'-1)b_1/2, r = (s'-1)b_1/(2s), k = k_1, \lambda = (k_1-1)(s'-1)b_1/[2s(sk_1-1)])$. □

The following example illustrates Theorem 3.2 with $NB(12; 66, 22; 2, 6)$.

Example 3.3 Developing the following blocks on Z_{11} gives an $NB(12; 66, 22; 2, 6)$ over $Z_{11} \cup \{\infty\}$:

$$\{\infty, 4|1, 3|5, 9\}, \{0, 6|7, 8|2, 10\} \mod 11$$

and take the following $PA(3, 3)$:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}$$

Now, by associating the symbols 1, 2 and 3 in this array with the first, second and third subblocks respectively in each of base blocks of the above $NB(12; 66, 22; 2, 6)$, 3 pairwise additive $B(12, 66, 11, 2, 1)$ can be obtained by taking the following incidence matrices:

$$\begin{array}{lcll} N_1 & : & \{\infty,4\},\{1,3\},\{5,9\},\{0,6\},\{7,8\},\{2,10\} & \mod 11 \\ N_2 & : & \{5,9\},\{\infty,4\},\{1,3\},\{2,10\},\{0,6\},\{7,8\} & \mod 11 \\ N_3 & : & \{1,3\},\{5,9\},\{\infty,4\},\{7,8\},\{2,10\},\{0,6\} & \mod 11 \end{array}$$

The existence of some nested BIB designs with specific parameters is known below.

Theorem 3.4 [3] The necessary and sufficient condition for the existence of NB$(v; b_1 = v(v-1)/2, b_2 = v(v-1)/6; k_1 = 2, k_2 = 6)$ is that $v \equiv 0, 1 \pmod 3$ and $v \geq 6$.

Thus, Theorems 3.2 and 3.4 can show the following.

Theorem 3.5 There are 3 pairwise additive B$(v, 2, 1)$ for all $v \equiv 0, 1 \pmod 3$ and $v \geq 6$.

4 Main result

In this section, the existence of 3 pairwise additive BIB designs will be shown for all $v \geq 6$.

The methods discussed here do not cover a case of $v = 14$. Hence it is individually constructed.

Lemma 4.1 There are 3 pairwise additive B(14,2,1).

Proof. A development of the following initial blocks on Z_{13} can yield three incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$ of the required BIB design:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,\infty\},\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\} \mod 13 \\
\boldsymbol{N}_2 &: \{2,3\},\{2,\infty\},\{4,6\},\{6,9\},\{8,12\},\{10,2\},\{12,5\} \mod 13 \\
\boldsymbol{N}_3 &: \{4,5\},\{4,6\},\{10,\infty\},\{12,2\},\{3,7\},\{7,12\},\{11,4\} \mod 13
\end{aligned}$$

In fact, it can be confirmed that $\boldsymbol{N}_1 + \boldsymbol{N}_2$, $\boldsymbol{N}_2 + \boldsymbol{N}_3$, $\boldsymbol{N}_3 + \boldsymbol{N}_1$ (and also $\boldsymbol{N}_1 + \boldsymbol{N}_2 + \boldsymbol{N}_3$) are incidence matrices of BIB designs, respectively. □

Finally the main result of this paper is established.

Theorem 4.2 For any $v \geq 6$, there are 3 pairwise additive B$(v, 2, 1)$.

Proof. When $v \neq 6, 10, 12, 14, 18, 21, 22, 24, 30, 34$, Theorem 2.3 shows the existence of 3 pairwise additive B$(v, 2, 1)$. When $v = 6, 10, 12, 18, 21, 22, 24, 30, 34$, Theorem 3.5 shows the existence of 3 pairwise additive B$(v, 2, 1)$. When $v = 14$, the 3 pairwise additive B$(v, 2, 1)$ are given in Lemma 4.1. Hence the proof is completed. □

Acknowledgement. The authors wish to thank the referee for the constructive comments and suggestions to improve the readability of the paper.

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