

All face 2-colorable d -angulations are Grünbaum colorable

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Dedicated to the memory of our colleague, Dan Archdeacon.

Abstract

A d -angulation of a surface is an embedding of a 3-connected graph on that surface that divides it into d -gonal faces. A d -angulation is said to be Grünbaum colorable if its edges can be d -colored so that every face uses all d colors. Up to now, the concept of Grünbaum coloring has been related only to triangulations ($d = 3$), but in this note, this concept is generalized for an arbitrary face size $d \geq 3$. It is shown that the face 2-colorability of a d -angulation P implies the Grünbaum colorability of P . Some wide classes of triangulations have turned out to be face 2-colorable.

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1 Terminology and notation

In this note we consider only *simple* graphs—that is, graphs without loops or parallel edges, and we embed the graphs only on *closed* surfaces—that is, surfaces without boundaries, such as a sphere. We mainly follow the standard terminology and notation of graph theory ([10]).

A d -angulation P of a surface means a d -gonal embedding of a 3-connected graph $G = G(P)$ on that surface—that is, an embedding each face of which is bounded by a simple cycle of G with fixed length $d \geq 3$. Combinatorially, P is defined by the triple of sets $V(P)$, $E(P)$, and $F(P)$ of vertices, edges, and faces, respectively. The *dual graph* $G^*(P)$ is defined to be the graph the vertex sets of which corresponds to $F(P)$ and in which two vertices are adjacent if and only if the corresponding faces of P are adjacent. Notice that $G^*(P)$ is d -regular—that is, the degree of each vertex is equal to d .

In this note we only consider d -angulations P whose dual graphs are simple graphs, and therefore we will suppose that $G(P)$ is 3-connected, which ensures the simplicity of $G^*(P)$ in the following two important cases:

- (i) $d = 3$,
- (ii) the carrier surface is a sphere.

In case ii, by Steinitz's Theorem, every 3-connected planar graph G is the 1-skeleton of a convex polytope (in 3-space) with boundary complex P , and the dual graph $G^*(P)$ appears to be the 1-skeleton of the dual polytope with boundary complex P^* .

A *vertex, edge, or face k -coloring* of a d -angulation P is a surjection of the set $V(P)$, $E(P)$, or $F(P)$ onto a set of k distinct colors such that the images of adjacent vertices, edges, or faces are different, respectively. Especially, any edge 3-coloring of a 3-regular graph is called a *Tait coloring*. The *vertex, edge, and face chromatic numbers* of P are defined to be the smallest values of k possible to obtain corresponding k -colorings, and are denoted by $\chi(P)$, $\chi'(P)$, and $\chi''(P)$, respectively. The numbers $\chi(P)$ and $\chi'(P)$ are also called the vertex and edge chromatic numbers of the graph $G(P)$ itself and denoted as $\chi(G(P))$ and $\chi'(G(P))$, respectively.

Clearly, any face k -coloring of an arbitrary d -angulation P corresponds to some vertex k -coloring of the dual graph $G^*(P)$, and conversely, whence

$$\chi''(P) = \chi(G^*(P)).$$

Interestingly, since $G^*(P)$ is d -regular, it follows that

$$\chi(G^*(P)) \in \{2, 3, \dots, d, d + 1\},$$

and that

$$\chi'(G^*(P)) \in \{3, \dots, d, d + 1\}$$

by Vizing's Theorem [18].

2 Grünbaum colorings

A *Grünbaum coloring* is a coloring of the edges of a d -angulation P with d colors such that for each face f all d colors occur at the edges incident to f . Up to now, the concept of Grünbaum coloring has been related only to triangulations—that is, the case $d = 3$, but in this note we will generalize this concept for an arbitrary face size $d \geq 3$. If T is a triangulation, then $\chi'(G^*(T)) \in \{3, 4\}$ by Vizing's Theorem. The equality $\chi'(G^*(T)) = 3$ means that $G^*(T)$ is Tait colorable, which, in the dual form, means that T is Grünbaum colorable.

Conjecture 1 (Grünbaum [9], 1969). Every triangulation T of an orientable surface is Grünbaum colorable—that is, $\chi'(G^*(T)) = 3$.

Conjecture 1 stood for 40 years, until Kochol [11] constructed infinite families of counterexamples on orientable surfaces with genus g for all $g \geq 5$. Here we put forward another conjecture about triangulations by strengthening the vacuous restriction $\chi''(T) \leq 4$ (which obviously holds for any T) to the restriction $\chi''(T) \leq 3$:

Conjecture 2. Every triangulation T of an orientable surface with $\chi''(T) \leq 3$ is Grünbaum colorable.

Conjecture 2 is false if extended to the nonorientable case. For instance, it is not hard to verify that the minimal triangulation T_{\min} of the projective plane by the complete 6-graph $G = K_6$ is face 3-colorable. However, for this particular instance, $G^*(T_{\min})$ turns out to be the Petersen Graph [15] which cannot be decomposed into the sum of three 1-factors (see [10], [14], [15]). Therefore, by Lemma 1 below, $G^*(T_{\min})$ has edge chromatic number at least 4. (This number is in fact equal to 4, which, by the way, easily implies the non-Hamiltonicity of the Petersen Graph—these are excellent creative exercises for a college course on Discrete Mathematics!)

In Section 3 we establish (Theorem 2) that for Grünbaum colorability of a d -angulation P (that is, for the equality $\chi'(G^*(P)) = d$ to hold), it suffices that $\chi''(P) = 2$, without the orientability restriction. In Sections 4 and 5 we establish the face 2-colorability in some known, and quite wide, classes of triangulations.

3 Key Theorem

Let P be a d -angulation of an orientable or nonorientable surface (whose dual graph is a simple graph). Since the dual graph $G^*(P)$ is d -regular, the following lemma is obvious.

Lemma 1. *In order for the equality $\chi'(G^*(P)) = d$ to hold, it is necessary and sufficient that the graph $G^*(P)$ be 1-factorable—that is, be the sum of d one-factors.*

In a classical article, König [12] (also see [14]) proved that each bipartite d -regular graph expands to the sum of d one-factors. Since a graph is bipartite if and only if it is vertex 2-colorable, we get the following reformulation of König's Theorem:

Theorem 1 (König). *If $\chi(G^*(P)) = 2$, then $G^*(P)$ is 1-factorable.*

By a combination of Lemma 1 and Theorem 1, we obtain our key theorem which states that each face 2-colorable d -angulation of an orientable or nonorientable surface is Grünbaum colorable:

Theorem 2 (Key Theorem). *If $\chi(G^*(P)) = 2$, then $\chi'(G^*(P)) = d$.*

Dual formulation: *If $\chi''(P) = 2$, then P is Grünbaum colorable.*

As a particular case of Theorem 2, when $d = 3$, we can state that Conjecture 1 certainly holds for all face 2-colorable triangulations of orientable and nonorientable surfaces. Notice that in Theorem 2 the face chromaticity condition is only minimally strengthened in comparison to that in Conjecture 2.

4 Triangulations by complete graphs

In this section, we establish the existence of Grünbaum colorable triangulations on orientable and nonorientable surfaces by complete graphs K_n for at least half of the residue classes in the spectrum of possible values of n .

We begin with the orientable case, in which there exists a triangulation by K_n if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$; see [16]. Grannell, Griggs, and Širáň [6] noticed that, when $n \equiv 0$ or $4 \pmod{12}$, such triangulations are not face 2-colorable because for face 2-colorability it is necessary that all vertex degrees should be even, that is, n should be odd. Furthermore, they established that the orientable triangulations constructed by Ringel [16] for all $n \equiv 3 \pmod{12}$ are face 2-colorable. Finally, they established that there exists a face 2-colorable triangulation for each $n \equiv 7 \pmod{12}$ among the orientable triangulations constructed by Youngs [19]. We summarize these results in the following theorem.

Theorem 3 (Ringel [16]; Youngs [19]; Grannell, Griggs, Širáň [6]). *There exists a face 2-colorable triangulation of an orientable surface by the complete graph K_n if and only if $n \equiv 3$ or $7 \pmod{12}$.*

If one triangulation is face 2-colorable and the other is not, the two triangulations are certainly *nonisomorphic*—that is, there is no bijection between their vertex sets that extends to a homeomorphism between the surfaces carrying the triangulations.

Historically, the first examples of pairs of nonisomorphic orientable triangulations with the same complete graph were constructed [19] in 1970. In those examples, the non-isomorphism follows immediately from the fact that one of the triangulations is face 2-colorable while the other is not; see review [3]. After a quarter of a century, in [13], there was constructed an example of *more than two* nonisomorphic orientable triangulations with the same complete graph, namely: there were constructed *three* such triangulations, only one of which is face 2-colorable. In 2000, it was shown [2] (also see [3]) that the number of nonisomorphic orientable triangulations with graph K_n actually grows very rapidly as $n \rightarrow \infty$ even

within the class of face 2-colorable triangulations; for instance, when $n \equiv 7$ or $19 \pmod{36}$, that number is at least $2^{n^2/54 - o(n^2)}$.

The following corollary can be proved by a combination of Theorems 3 and 2.

Corollary 1. *For each $n \equiv 3$ or $7 \pmod{12}$, there exists a Grünbaum colorable orientable triangulation by the complete graph K_n .*

To turn to the nonorientable case, recall [16] that K_n triangulates a nonorientable surface if and only if $n \equiv 0$ or $1 \pmod{3}$, $n \geq 6$ and $n \neq 7$. Thus, under these conditions, n is odd if and only if $n \equiv 1$ or $3 \pmod{6}$, $n \geq 9$. For all these values of n , face 2-colorable triangulations of the corresponding nonorientable surface by the graph K_n are constructed in [16] and [8].

Theorem 4 (Ringel [16]; Grannell, Korzhik [8]). *There exists a face 2-colorable triangulation of a nonorientable surface by the complete graph K_n if and only if $n \equiv 1$ or $3 \pmod{6}$, $n \geq 9$.*

The following corollary can be proved by a combination of Theorems 4 and 2.

Corollary 2. *For each $n \equiv 1$ or $3 \pmod{6}$, $n \geq 9$, there exists a Grünbaum colorable nonorientable triangulation by the complete graph K_n .*

Theorems 3 and 4 guarantee that we have not missed any face 2-colorable triangulations when applying Theorem 2 for obtaining Corollaries 1 and 2 (respectively).

5 Triangulations by tripartite graphs

Firstly, we notice that the existence of an orientable triangulation by the complete tripartite graph $K_{n,n,n}$ was established by Ringel and Youngs [17] for each n . Secondly, the face 2-colorability of each such triangulation was established by Grannell, Griggs, and Knor [4] (also see [3]). A combination of these two results with Theorem 2 leads to the following statement: *For each $n \geq 2$, all triangulations of the corresponding orientable surface by the complete tripartite graph $K_{n,n,n}$ are Grünbaum colorable.* However, as observed by Archdeacon [1], it is very easy to prove this fact directly, even without using the completeness or orientability conditions: if the vertex parts are A, B, C , then color the edges between A and B red, those between B and C blue, and those between A and C green, and we are done!

At first sight, the statement of the preceding paragraph may seem to be subjectless; however, as shown in [5] (see also [3]) in the case n is prime, there exist at least $(n-2)!/(6n)$ nonisomorphic orientable triangulations by $K_{n,n,n}$. Furthermore, [7] provides improved bounds on the number of such triangulations; for instance, when $n \equiv 6$ or $30 \pmod{36}$, there exist at least $n^{n^2/144 - o(n^2)}$ nonisomorphic orientable triangulations by $K_{n,n,n}$.

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