A SHORT NOTE INTRODUCING GRAFTING NUMBERS AND THEIR CONNECTION TO CATALAN NUMBERS

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ABSTRACT. Catalan Numbers and their generalizations are found throughout the field of Combinatorics. This paper describes their connection to numbers whose digits appear in the number's own p^{th} root. These are called Grafting Numbers and they are defined by a class of polynomials given by the Grafting Equation: $(x+y)^p = b^a x$. A formula that solves for x in these polynomials uses a novel extension to Catalan Numbers and will be proved in this paper. This extension results in new sequences that also solve natural extensions to previous Combinatorics problems. In addition, this paper will present computationally verified conjectures about formulas and properties of other solutions to the Grafting Equation.

1. INTRODUCTION

The concept of *Grafting Numbers*(*GNs*) is introduced by Parker (2012) [7]. In his article and video, he refers to integers where the digits of the number appear in its square root before or just after the decimal point (i.e. $\sqrt{98}$ = **9.8**9949...). To quote Parker, "the root grows out from the number itself," hence the name. These integers will be referred to in this paper as *Grafting Integers*(*GIs*) and, since they involve square roots and are represented in base-10, they are base-10 2nd-order GIs. A list of these GIs that are less than 10,000 is shown in Table 1 (Note: 10,000 is also a GI). OEIS [6] entry #A074841 by Lusch has code using string manipulation to find GIs, although his criteria is more strict than what is described in this paper.

A couple more 2nd-order GIs that illustrate an important pattern are 76,394 (2**76.394**645...) and 7,639,321 (2**763.9321**6...). Parker showed that this sequence of digits corresponds to $3-\sqrt{5} = 0.7639320225002...$, which makes sense because this number is a solution to the equation $\sqrt{10x} = x+2$. So $3-\sqrt{5}$ is a GN, and Parker showed that it generates GIs using the

n	$\sqrt{n}$	n	$\sqrt{n}$
0	**0**	764	**27.6405499...**
1	**1**	765	**27.65863...**
8	2.8284...	5,711	75.**5711585**...
77	8.**77**496...	5,736	75.**7363849**...
98	**9.89949**...	9,797	98.**9797959**...
99	**9.94987**...	9,998	**99.9899995**...
100	**10.0**	9,999	**99.9949987**...

TABLE 1. The 2nd-order GIs in the range $[0, 10^4)$.

following equation for $k \geq 0$.

$$\left\lceil \left(3 - \sqrt{5}\right) 10^{2k+1} \right\rceil \tag{1}$$

The other solution, $3 + \sqrt{5}$ does not create any GIs because it is greater than 1, so adding 2 to it changes its first digit. A similar equation, $\sqrt{10x} = x + 1$ yields another GN: $4 - \sqrt{15} = 0.127016653792\ldots$, and investigation into possible GIs using this number shows that $\sqrt{127} = \mathbf{11.2694}\ldots$ and $\sqrt{12,702} = \mathbf{112.7031}\ldots$ don't work as GIs. The first GI in this sequence is 127,016,654 (**11270.1665471**...). So a generalization of Equation (1) will only generate *potential GIs* because the rounding does not always work.

A first possible construction of an equation to define GNs is $\sqrt{10x} = x+y$ or equivalently $(x + y)^2 = 10x$ using an integer constant $y \geq 0$. Upon investigation, for $y > 2$, this equation does not have any real solutions. Looking at the GI, 77, and attempting to reverse engineer an equation to find the GN yields $(x+8)^2 = 10^2x$ and another GI, 5,711, yields $(x+75)^2 = 10^4x$. These equations both have solutions (**0.7689437**...and **0.57109910**...), and this process leads to the following equation for $a > 0$:

$$(x + y)^2 = 10^a x \tag{2}$$

The parameter a is an integer that can be viewed to represent the amount that the decimal point is shifted in base-10. This equation can then be generalized to apply to any integer base, $b > 1$, and any integer power, $p > 1$. So, in the most general form, GNs are defined as solutions to a class of polynomials defined by the *Grafting Equation*:

$$(x + y)^p = b^a x \tag{3}$$

When the number, x, represented in base b, is added to a constant, y, the p^{th} power of the result is equal to x with the decimal point shifted a units to the right. If $0 \leq x \leq 1$ then all the digits of x will appear in the results of both sides of the equation, and x is called a *Complete Grafting Number*. If $x > 1$ then the digits to the left of the decimal point will be altered but the remaining digits will appear on both sides, and x is called a *Partial Grafting Number*. Equation (1) can be generalized as well to generate potential GIs from x, a solution to a specific Grafting Equation.

$$(4) \qquad \textit{Potential GIs} = \lceil b^{pk+a}x \rceil \text{ or } \lfloor b^{pk+a}x \rfloor \; \textit{for } k \geq 0$$

In Table 1, 764 and 765 are both GIs and so are 99 and 98. So adding or subtracting 1 to GIs sometimes yields another one. The presence or absence of GIs is related to rounding error and will not be further discussed in this paper. This paper will describe the link between Grafting Numbers and Catalan Numbers. It will then use this relationship to prove one of the solutions to the Grafting Equation and provide conjectures for two other real solutions.

1.1. **Catalan Numbers and generalizations.** Described as the longest entry in the Online Encyclopedia of Integer Sequences (OEIS #A000108) [6], the Catalan Numbers, occasionally called Segner Numbers, are related to a very diverse set of Combinatorics problems. The first few terms are as follows:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

The n^{th} Catalan Number, C_n where $n \geq 0$, can be defined in a few equivalent ways. Explicitly as:

$$C_n = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

or recursively as:

$$C_0 = 1; \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

Larcombe and Wilson [5] give a detailed look at the discovery of Catalan Numbers and their relation to three famous problems. The first is the number of ways to subdivide an $(n+2)$-gon into n triangles using non-intersecting diagonals. The second is the number of ways to apply n binary operations on $n+1$ variables. The third is the number of lattice paths from $(0,0)$ to (n,n) without rising above the diagonal line that connects those two points. The solution to each of these problems and countless more is given by C_n.

Hilton and Pedersen [3] describe extensions to each of the three main problems using a variable $p \geq 2$. Instead of subdividing into triangles, they consider subdividing into $(p+1)$-gons; instead of applying binary operations, they apply p-ary operations; and instead of paths from $(0,0)$ to (n,n) they count from $(0,-1)$ to $(n,(p-1)n-1)$. The reason for the seemingly odd use of $(0,-1)$ instead of $(0,0)$ is that it helps out with the proofs used in this paper. Using bijection proofs and previously shown results by Klarner [4], who calculates the number of p-ary trees with n source nodes, they show that these new problems are all equivalent and can be counted by the following explicit formula:

$$_pC_n = \frac{1}{n}\binom{pn}{n-1} = \frac{1}{(p-1)\,n+1}\binom{pn}{n}$$

or recursively as:

$$_pC_0 = 1; \quad _pC_{n+1} = \sum_{i_1+i_2+\ldots+i_p=n} {_pC_{i_1}} \cdot {_pC_{i_2}} \cdots {_pC_{i_p}}$$

Note that $_2C_n$ is equivalent to C_n. These generalized Catalan Numbers are sometimes called Pfaff-Fuss-Catalan Numbers or p-Raney Sequences [9]. Hilton and Pedersen then show the following lemma:

Lemma 1. *If a power series $S\,(w)$ is defined as:*

$$S\,(w) = 1 + \sum_{n=1}^{\infty} {_pC_n} w^n \tag{5}$$

then the following is true:

$$wS^p = S - 1 \tag{6}$$

Proof. The proof follows directly from the Lagrange Inversion formula in Pólya and Szegö [8]. Equation (6) can be rewritten as follows:

$$\begin{aligned} &Let\ z = S - 1, \text{ and} \\ &Let\ \varphi(z) = S^p = (z+1)^p \\ &So\ w = \frac{z}{\varphi(z)} \end{aligned}$$

For $f(z) = S = 1 + z$, the Lagrange Inversion formula states that:

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1} f'(x) \left[\varphi(x)\right]^n}{dx^{n-1}} \right]_{x=0}$$

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\frac{d^{n-1} (x+1)^{pn}}{dx^{n-1}} \right]_{x=0}$$

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[\prod_{i=0}^{n-2} [pn - i] \, (x+1)^{pn-n+1} \right]_{x=0}$$

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{w^n \prod_{i=0}^{n-2} [pn - i]}{n!}$$

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{w^n (pn)!}{n!(pn - n + 1)!}$$

$$f(z) = 1 + \sum_{n=1}^{\infty} {}_pC_n w^n = S$$

□

2. Connecting Grafting Numbers and Catalan Numbers

Equation (2) can be solved using the quadratic formula to provide all of the base-10 2nd-order GNs. The result is:

$$x = \left(\frac{10^a}{2} - y \right) \pm \sqrt{10^a \left(\frac{10^a}{4} - y \right)} \tag{7}$$

Table 2 provides the values for the smaller of the two solutions shown in the equation above for $1 \le y \le 5$ and $1 \le a \le 4$. What becomes apparent when looking horizontally across this table is that sequences of numbers emerge. When looking at the actual square root form of the numbers on the top row: $4 - \sqrt{15}$, $49 - \sqrt{2400}$, $499 - \sqrt{249000}$, and $4999 - \sqrt{24990000}$, it

y	a=1	a=2	a=3	a=4
1	0.127016653	0.0102051443	0.001002005014	0.0001000200050014
2	0.763932022	0.0416847669	0.004016080451	0.0004001600800448
3		0.0958424018	0.009054408433	0.0009005404053405
4		0.1742430504	0.016129294510	0.0016012812814353
5		0.2786404500	0.025253169417	0.0025025031293815

TABLE 2. The first several digits of the 2nd-order GNs in the range y=[1, 5], a=[1,4].

does not seem like these numbers should necessarily be related at all. When considering the Grafting Equation though, it makes sense for GNs with the same y value and different a values to be linked because $b = 10, a = 2$ is the same as $b = 100, a = 1$. When base-100 is represented in base-10, each number would be represented by two digits, and for base-1000, each number would get three digits. The effect of this is a spreading out of the emergent sequence allowing it to be recognized more easily.

Remarkably, for $y = 1$ the sequence that emerges is C_n for $n \geq 1$. This is due to the property of the generating function of the Catalan Numbers shown in Lemma 1 for $p = 2$ and $w = \frac{1}{10^a}$:

$$S(w) = 1 + \sum_{n=1}^{\infty} C_n w^n$$

$$wS^2 = S - 1$$

$$Let\ x = S - 1 = \sum_{n=1}^{\infty} C_n w^n$$

$$w(x+1)^2 = x$$

$$\frac{1}{10^a}(x+1)^2 = x$$

$$(x+1)^2 = 10^a x$$

The final line is equivalent to Equation (2) for $y = 1$. For other values of y, analysis of the emergent sequences yields the following formula for the smaller solution to Equation (2) for $y > 0$.

$$Sol_1(a, y) = \sum_{n=1}^{\infty} \left[y^{n+1} C_n \left(\frac{1}{10^a} \right)^n \right] \tag{8}$$

When extending to $p > 2$, a few of the solutions to the Grafting Equation for $y = 1$, $a = 5$, $b = 10$ are shown in Table 3. The emergent sequences correspond to ${}_pC_n$ for $n \geq 1$. Some sequences for other values of p and y

p	y=1, a=5, b=10
3	0.00001 00003 00012 00055 00273...
4	0.00001 00004 00022 00140 00969...
5	0.00001 00005 00035 00285 02530...
6	0.00001 00006 00051 00506 05481...

TABLE 3. Solutions to the Grafting Equation for $y = 1$, $a = 5$, $b = 10$, and $p = [3, 6]$.

are shown in Appendix A. Analysis of these sequences led to the complete formula for $a > 0$, $y > 0$, $b > 1$, and $p > 1$:

$$Sol_1\left(a, y, b, p\right) = \sum_{n=1}^{\infty}\left[y^{(p-1)n+1}{}_pC_n\left(\frac{1}{b^a}\right)^n\right] \tag{9}$$

Theorem 1. *If x satisfies the Grafting Equation then one solution for x can be calculated by Equation* (9).

Proof. Rearranging the y term in Equation 9 yields:

$$Sol_1\left(a, y, b, p\right) = y\sum_{n=1}^{\infty}\left[{}_pC_n\left(\frac{y^{p-1}}{b^a}\right)^n\right]$$

$$Let\ w = \left(\frac{y^{p-1}}{b^a}\right)$$

$$Sol_1\left(a, y, b, p\right) = y\sum_{n=1}^{\infty}\left[{}_pC_n w^n\right]$$

Comparing this formula to Lemma 1, it follows that $S = 1 + \frac{Sol_1}{y}$ and $wS^p = S - 1$. This yields:

$$w\left(1+\frac{Sol_1}{y}\right)^p = \left(1+\frac{Sol_1}{y}\right) - 1$$

$$w\left(\frac{Sol_1+y}{y}\right)^p = \frac{Sol_1}{y}$$

$$\frac{w}{y^p}\left(Sol_1+y\right)^p = \frac{Sol_1}{y}$$

$$\frac{w}{y^{p-1}}\left(Sol_1+y\right)^p = Sol_1$$

$\left(\frac{y^{p-1}}{b^a}\right)$ is then substituted back in for w:

$$\frac{\left(\frac{y^{p-1}}{b^a}\right)}{y^{p-1}}\left(Sol_1+y\right)^p = Sol_1$$

$$\frac{1}{b^a}\left(Sol_1+y\right)^p = Sol_1$$

$$\left(Sol_1+y\right)^p = b^a Sol_1$$

The last line is equivalent to the Grafting Equation and completes the proof. □

3. Discussion

3.1. Extension to Catalan Numbers. Equation (9) uses a new extension to Pfaff-Fuss-Catalan Numbers, referred to in this paper as y-p-Catalan Numbers, to account for differing y values.

Definition 1 (y-p-Catalan Numbers). *For integers $y > 0$, $p > 1$, and $n \geq 0$, y-p-Catalan Numbers are defined in three equivalent ways below:*

$$\begin{aligned} {}_{y,p}C_n &= y^{(p-1)n+1}\,{}_pC_n \\ &= \frac{y^{(p-1)n+1}}{n}\binom{pn}{n-1} \\ &= \frac{y^{(p-1)n+1}}{(p-1)\,n+1}\binom{pn}{n} \end{aligned}$$

In addition to providing a solution to the Grafting Equation, this sequence can count new, yet natural, extensions to some of the Catalan Number problems discussed by Hilton and Pedersen [3]. All of these extensions involve elements that can be any of y different values or colors with duplicates being allowed. The repeated application of a p-ary operator problem uses $(p-1)n+1$ variables [3]. If these variables are allowed to be any of y different values, the number of ways to do this is ${}_{y,p}C_n$. An example is shown below for $p = 2$, $y = 2$ using a and b to represent the different values.

$n = 1, \quad {}_{2,2}C_1 = 4$

(aa) (ba)
(ab) (bb)

$n = 2, \quad {}_{2,2}C_2 = 16$

$((aa)a)$ $((ba)a)$ $(a(aa))$ $(b(aa))$
$((aa)b)$ $((ba)b)$ $(a(ab))$ $(b(ab))$
$((ab)a)$ $((bb)a)$ $(a(ba))$ $(b(ba))$
$((ab)b)$ $((bb)b)$ $(a(bb))$ $(b(bb))$

$n = 3, \quad {}_{2,2}C_3 = 80 \quad \ldots$

This problem provides a proof for the recursive definition for Pfaff-Fuss-Catalan Numbers and can easily be extended to y-p-Catalan Numbers as well.

Claim 1. *y-p-Catalan Numbers can be defined recursively in the same way as the original Pfaff-Fuss-Catalan Numbers with the initial term set to y. Namely:*

$${}_{y,p}C_0 = y; \quad {}_{y,p}C_{n+1} = \sum_{i_1+i_2+\ldots+i_p=n} {}_{y,p}C_{i_1} \cdot {}_{y,p}C_{i_2} \cdots {}_{y,p}C_{i_p}$$

Proof. Consider any ordering of $n + 1$ applications of a p-ary operator, P, with variables that are any of y different values. This ordering can be written in the following form: $P(G_1, G_2, \cdots, G_p)$, where G_1 through G_p are groups of applications of P. Since this form shows the 1 final application of P, the groups must contain between 0 and n applications of P, which adds up to n. If a specific group is just a single variable, hence the number of applications of P is 0, then the number of combinations possible equals the number of values that variable can take: y, which is the initial condition. The total number of combinations for $n+1$ applications of P is the product of the number of combinations of G_1 through G_p. □

A related problem to the application of a p-ary operator is the number of arrangements of a p-ary tree with n internal nodes. The bijection between the two problems maps the internal nodes to the brackets and the leaf nodes to the variables so these trees will have $(p-1)n+1$ leaf nodes. If each of these leaf nodes can have one of y different values, this new problem is also counted by ${}_{y,p}C_n$. Lastly, a more strained example is the dissection of a polygon into n $(p+1)$-gons which uses $((p-1)n+2)$-gons as the source polygons [3]. If each edge other than the base of the polygon is colored any of y different colors, the number of possible dissection+colorings is ${}_{y,p}C_n$.

3.2. **Occurrences of y-p-Catalan Numbers in Literature.** As already mentioned, the y-p-Catalan Numbers for $y = 1$ match the extended Catalan numbers discussed by Hilton and Pedersen as well as in numerous other papers. Very few of the sequences for greater values of y, shown in Appendix A, are found in the literature. For $y = 2$ and $p = 2$, a paper by Guo and Sit [2] shows that this sequence counts the number of Rota-Baxter words with n pairs of brackets. Tarau and Luderman [10] prove that a formula equivalent to ${}_{y,p}C_n$ for $p = 2$ counts the number of Leaf-DAGs (directed acyclic graphs where only the leaf nodes can have multiple incoming edges) with n binary operation (internal) nodes and y primary inputs (possible values for the leaves). This counting problem is equivalent to counting p-ary trees as described above for $p = 2$. The OEIS entry numbers for some of the sequences are shown in the table below. For $y = 2$ and $p = 3$, the OEIS entry references a paper by Bousquet-Mélou [1] although the link between the paper and the sequence is unclear.

y	p	OEIS Number
2	2	A025225
	3	A098272
3	2	A025226

TABLE 4. OEIS [6] entries for y-p-Catalan Number sequences.

3.3. Other Solutions to the Grafting Equation.

Claim 2. *The Grafting Equation for $y > 0$ has at most two positive real solutions.*

Proof. When converted into standard polynomial form and the binomial theorem is applied, the Grafting Equation becomes:

$$x^p + px^{p-1}y + \binom{p}{2}x^{p-2}y^2 + \cdots + \binom{p}{p-2}x^2y^{p-2} + \left(py^{p-1} - b^a\right)x + y^p$$

Every term is positive except for possibly the x term so using Descartes' Rule of Signs:

$$\text{if } \left(py^{p-1} - b^a\right) < 0 \text{ then at most 2 positive real roots}$$
$$\text{if } \left(py^{p-1} - b^a\right) \geq 0 \text{ then no positive real roots}$$

□

Extensive analysis of emergent sequences for the larger of the two positive solutions has led to the following conjecture about an equation for the larger positive solution, which defines an extension of binomial coefficient notation to handle multifactorials:

Definition 2 (Multifactorials).

$$n!^{(m)} = \begin{cases} 1 & \textit{if } 0 \leq n < m \\ n\left((n-m)!^{(m)}\right) & \textit{if } n \geq m \end{cases}$$

Definition 3 (Binomial Coefficients Using Multifactorials).

$$\binom{n}{k}^{(m)} = \frac{n!^{(m)}}{k!^{(m)}\,(n-k)!^{(m)}}$$

Conjecture 1. *If x satisfies the Grafting Equation then a second solution for x can be calculated by the following formula:*

$$Sol_2\,(a,y,b,p) = b^{\frac{a}{p-1}} - \frac{py}{p-1} - \sum_{n=1}^{\infty}\left[\frac{y^{n+1}}{(n+1)(p-1)}\binom{pn}{n}^{(p-1)}\left(\frac{1}{b^{\frac{a}{p-1}}}\right)^n\right]$$

Analysis of the divergence of the infinite sums in Sol_1 and Sol_2 leads to the following conjecture:

Conjecture 2. *The infinite sums in $Sol_1\,(a,y,b,p)$ and $Sol_2\,(a,y,b,p)$ diverge iff the respective solution to the Grafting Equation is non-real for those values of a, y, b, and p.*

For odd values of p, there seems to always be a negative root and analysis has led to the following conjecture about a formula for this third solution which is nearly the same form as Sol_2:

Conjecture 3. *If x satisfies the Grafting Equation and p is odd, then a third solution for x can be calculated by the following formula:*

$$Sol_3\left(a,y,b,p\right) = -b^{\frac{a}{p-1}} - \frac{py}{p-1} - \sum_{n=1}^{\infty}\left[\frac{y^{n+1}}{(n+1)(p-1)}\binom{pn}{n}^{(p-1)}\left(\frac{-1}{b^{\frac{a}{p-1}}}\right)^{n}\right]$$

This equation does diverge for large enough values of y even though there is always a negative root so it does not provide a complete solution.

4. Conclusion

Grafting Numbers are solutions to a class of polynomials defined by the Grafting Equation: $(x+y)^p = b^a x$. A formula using a new extension to Catalan Numbers is proved to provide a solution to the Grafting Equation. This extension also counts new, yet natural, extensions to well-known Combinatorics problems. Finally, computational analysis of the patterns for two other solutions to the Grafting Equation provide conjectured formulas for these solutions.

5. Future Research

Future research paths include predicting the occurrence of Grafting Integers and proving the conjectured formulas for the 2nd and 3rd real solutions to the Grafting Equation. Also, the infinite sums in the solutions described in this paper converge slowly for certain values of y. Performing a series acceleration transformation to these sums could increase the speed at which high precision Grafting Numbers are computed. Another avenue of research involves extending the Catalan Triangle to include y-p-Catalan Numbers.

6. Acknowledgements

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APPENDIX A. SELECTED Y-P-CATALAN SEQUENCES

y	Sequences for $p = 2$
1	1, 1, 2, 5, 14, 42, ...
2	2, 4, 16, 80, 448, 2688, ...
3	3, 9, 54, 405, 3402, 30618, ...
4	4, 16, 128, 1280, 14336, 172032, ...
5	5, 25, 250, 3125, 43750, 656250, ...

y	Sequences for $p = 3$
1	1, 1, 3, 12, 55, 273, ...
2	2, 8, 96, 1536, 28160, ...
3	3, 27, 729, 26244, 1082565, ...
4	4, 64, 3072, 196608, 14417920, ...
5	5, 125, 9375, 937500, 107421875, ...

y	Sequences for $p = 4$
1	1, 1, 4, 22, 140, 969, ...
2	2, 16, 512, 22528, 1146880, ...
3	3, 81, 8748, 1299078, 223205220, ...
4	4, 256, 65536, 23068672, 9395240960, ...
5	5, 625, 312500, 214843750, 170898437500, ...

y	Sequences for $p = 5$
1	1, 1, 5, 35, 285, 2530, ...
2	2, 32, 2560, 286720, 37355520, ...
3	3, 243, 98415, 55801305, 36804946455, ...
4	4, 1024, 1310720, 2348810240, 4896262717440, ...
5	5, 3125, 9765625, 42724609375, 217437744140625, ...