On the cyclability of graphs

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Abstract

Given a graph $G = (V, E)$ and $A_1, A_2, \ldots, A_r$, mutually disjoint nonempty subsets of V where $|A_i| \leq |V|/r$ for each i, we say that G is *spanning equi-cyclable* with respect to $A_1, A_2, \ldots, A_r$ if there exist mutually disjoint cycles $C_1, C_2, \ldots, C_r$ that span G such that C_i contains A_i for every i and C_i contains either $\lfloor |V|/r \rfloor$ vertices or $\lceil |V|/r \rceil$ vertices. Moreover, G is *r-spanning-equi-cyclable* if G is spanning equi-cyclable with respect to $A_1, A_2, \ldots, A_r$ for every such mutually disjoint nonempty subsets of V. We define the *spanning equi-cyclability* of G to be r if G is k-spanning equi-cyclable for $k = 1, 2, \ldots, r$ but is not $(r + 1)$-spanning-equi-cyclable. Another approach on the restriction of the A_i's is the following. We say that $G = (V, E)$ is *r-cyclable of order t* if it is cyclable with respect to $A_1, A_2, \ldots, A_r$ for any r nonempty mutually disjoint subsets $A_1, A_2, \ldots, A_r$ of V such that $|A_1 \cup A_2 \cup \ldots \cup A_r| \leq t$. The *$r$-cyclability* of G is t if G is r-cyclable of order k for $k = r, r+1, \ldots, t$ but is not r-cyclable of order $t+1$. On the other hand, the *cyclability of G of order t* is r if G is k-cyclable of order t for $k = 1, 2, \ldots, r$ but is not $(r + 1)$-cyclable of order t. In this paper, we study sufficient conditions for spanning equi-cyclability and r-cyclability of order t as well other related problems.

Keywords: Hamiltonian, cyclability

1 Introduction

Hamiltonicity is a well-studied problem. A number of variations have been developed. Research efforts have been dedicated to pancyclicity [3, 10], super spanning connectivity [1, 16, 17] and Hamilton decompositions [2, 18, 19] among many other areas. Until the 1970's, the interest in Hamiltonian cycles had been centered on their relationship to the 4-color problem. More recently, the study of Hamiltonian cycles in general graphs has been fueled by the issue of complexity and practical applications. In particular, Hamiltonian cycles is a major requirement to design an interconnection network. The Hamiltonian condition can be adjusted in a number of ways. On the one hand, one can strengthen the condition to include a prescribed k vertices in a specific order, this is

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the k-ordered Hamiltonian problem [7,8,12,13,15,20,23]. On the other hand, one can relax the Hamiltonian condition to a union of r cycles.

In this paper, we study a variation that is a mixture of relaxation and strengthening of the Hamiltonian problem. On the one hand, we allow the graph to be spanned by several cycles. On the other hand, each must contain a prescribed set of vertices. This concept can be applied to identify the faulty processors and other related issues in interconnection networks [5,9,11,14].

Given a graph $G = (V, E)$ and $A_1, A_2, \ldots, A_r$, mutually disjoint nonempty subsets of V, we say that G is *cyclable* with respect to $A_1, A_2, \ldots, A_r$ if there exist mutually disjoint cycles $C_1, C_2, \ldots, C_r$ such that C_i contains A_i for every i. If, in addition, $C_1, C_2, \ldots, C_r$ span G, then G is *spanning cyclable* with respect to $A_1, A_2, \ldots, A_r$. A natural question is to ask whether a graph G is (spanning) cyclable with respect to $A_1, A_2, \ldots, A_r$ for every such mutually disjoint nonempty subsets of V, and to find the largest r. However, this is not a very good question unless restrictions are imposed. For example, no graph can be cyclable with respect to A_1, A_2 for every such disjoint nonempty subsets of V. To see this, we simply pick two vertices u and v and set A_1, A_2 to be $\{u, v\}$ and $V \setminus \{u, v\}$. So one may want the restriction $|A_i| \neq 2$. However, even with this restriction, no non-complete graph can be cyclable with respect to A_1, A_2 for every such disjoint nonempty subsets of V. What if we require $|A_i| \geq 4$? Then unless every four vertices is on a 4-cycle, it is not feasible for $r = 2$. Another fundamental problem is whether we treat the graph K_1 as Hamiltonian. If not, then again it is not feasible as A_1 can be a singleton and $A_2 = V \setminus A_1$. However, it is common to not consider K_1 as Hamiltonian. So we need some sensible conditions. In fact, after imposing certain conditions, we may want to require cycles with more properties. We now give one such definition. Given a graph $G = (V, E)$ and $A_1, A_2, \ldots, A_r$, mutually disjoint nonempty subsets of V where $|A_i| \leq |V|/r$ for each i, we say that G is *spanning equi-cyclable* with respect to $A_1, A_2, \ldots, A_r$ if there exist mutually disjoint cycles $C_1, C_2, \ldots, C_r$ that span G such that C_i contains A_i for every i and C_i contains either $\lfloor |V|/r \rfloor$ vertices or $\lceil |V|/r \rceil$ vertices. Moreover, G is *r-spanning-equi-cyclable* if G is spanning equi-cyclable with respect to $A_1, A_2, \ldots, A_r$ for every such mutually disjoint nonempty subsets of V. We define the *spanning equi-cyclability* of G to be r if G is k-spanning equi-cyclable for $k = 1, 2, \ldots, r$ but is not $(r + 1)$-spanning-equi-cyclable. Clearly the spanning equi-cyclability of G is at most $|V|/3$; otherwise, one of the required cycle must have length 2, which is impossible.

Another approach on the restriction of the A_i's is the following. We say that $G = (V, E)$ is *r-cyclable of order t* if it is cyclable with respect to $A_1, A_2, \ldots, A_r$ for any r nonempty mutually disjoint subsets $A_1, A_2, \ldots, A_r$ of V such that $|A_1 \cup A_2 \cup \ldots \cup A_r| \leq t$. Clearly $r \leq t$. Now, we have two parameters r and t. We can fix one of them and find the optimal value for the other. The *r-cyclability* of G is t if G is r-cyclable of order k for $k = r, r + 1, \ldots, t$ but is not r-cyclable of order $t + 1$. On the other hand, the *cyclability of G of order t* is r if G is k-cyclable of order t for $k = 1, 2, \ldots, r$ but is not $(r + 1)$-cyclable of order t. This restriction removes the potential problem of K_1 and K_2 not being Hamiltonian. This simply implies $t \leq |V| - 3$. The spanning version can be defined similarly as follows. A graph $G = (V, E)$ is *r-spanning-cyclable of order t* if it is spanning cyclable with respect to $A_1, A_2, \ldots, A_r$ for any r nonempty mutually disjoint subsets $A_1, A_2, \ldots, A_r$ of V such that $|A_1 \cup A_2 \cup \ldots \cup A_r| \leq t$.

The *r-spanning-cyclability* of G is t if G is r-spanning-cyclable of order k for $k = r, r+1, \dots, t$ but is not r-spanning-cyclable of order $t+1$. On the other hand, the *spanning cyclability of G of order t* is r if G is k-cyclable of order t for $k = 1, 2, \dots, r$ but is not $(r+1)$-cyclable of order t.

Our goal is to study sufficient conditions for these types of problems and we aim for statements that are similar to the classical results of Dirac [6], Ore [21] and Bondy–Chvátal [4]. For example, the following is a result that we will prove: *Let G be a graph with $n \geq 6$ vertices. If $\deg_G(u) + \deg_G(v) \geq 2n - \lfloor \frac{n}{2} \rfloor$ for every pair of nonadjacent vertices u and v in G, then G is 2-spanning-cyclable of order* 2. The statement of Ore's Theorem is: *Let G be a graph with $n \geq 3$ vertices. If $\deg_G(u) + \deg_G(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G is Hamiltonian.* Besides sufficient conditions, one can also consider extremal cases. For example, if we delete $n-2$ edges from K_n, it is possible for the resulting graph to be non-Hamiltonian as one can simply delete $n-2$ edges incident to a singe vertex, leaving a vertex of degree 1 in the resulting graph. So we can delete at most $n-3$ edges. Indeed this is guaranteed by Ore's Theorem as in such resulting graph, $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v. Now if we delete $n-6$ edges from K_n, is it still possible for the resulting graph to be 2-spanning-cyclable of order 2? It certainly no longer satisfies the condition $\deg(u) + \deg(v) \geq 2n - \lfloor \frac{n}{2} \rfloor$ for every pair of nonadjacent vertices u and v in the resulting graph if we delete $n-6$ edges incident to a single vertex, leaving a vertex of degree 5 in the resulting graph. However, one can show that the resulting graph is 2-spanning-cyclable of order 2. We study these extreme cases in this setting in Section 3.

2 Sufficient conditions of the classical type

In this section we give a number of sufficient conditions to the problems that we are interested in. They are in the spirit of Dirac, Ore and Bondy–Chvátal. These results are generalizations of corresponding classical results. Indeed, one can modify the standard book proof of these classical results to prove the new results. More interestingly, one can prove these new results by applying the corresponding classical results directly, and this will be our method of choice.

2.1 Dirac type results

We first state the classical sufficient condition on Hamiltonicity.

Theorem 2.1 (Dirac, [6]). *Let G be a graph with $n \geq 3$ vertices and $\delta(G) \geq n/2$. Then G is Hamiltonian.*

Theorem 2.2. *Let G be a graph with n vertices. Suppose $\lfloor n/r \rfloor \geq 3$. If $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{r} \rfloor$, then G is r-spanning-equi-cyclable.*

Proof. Let $A_1, A_2, \dots, A_r$ be mutually disjoint nonempty subsets of V where $|A_i| \leq n/r$. Since $\lfloor n/r \rfloor \geq 3$, we can find a partition of V, $\{B_1, B_2, \dots, B_r\}$, such that $A_i \subseteq B_i$ and $\lfloor n/r \rfloor \leq |B_i| \leq \lceil n/r \rceil$. We note that $|B_i| \geq 3$. Now for each i, let

G_i be the subgraph of G induced by B_i. We claim that G_i has a Hamiltonian cycle by applying Dirac's Theorem. So we need to check that $\delta(G_i) \geq |B_i|/2$. But

$$\delta(G_i) \geq n - \frac{1}{2}\left\lfloor \frac{n}{r} \right\rfloor - (n - |B_i|) = |B_i| - \frac{1}{2}\left\lfloor \frac{n}{r} \right\rfloor \geq \frac{|B_i|}{2}.$$

So each G_i has a Hamiltonian cycle and the result follows. □

Corollary 2.3. *Let G be a graph with $n \geq 3$ vertices. Then the spanning equi-cyclability of G is at least $\frac{n}{2(n-\delta(G))}$.*

Proof. Let $r \leq \frac{n}{2(n-\delta(G))}$. We claim that G is r-spanning-equi-cyclable. If G is complete, then the result is clearly true. So we may assume that G is not a complete graph. Then $\delta(G) \leq n-2$. So $n/r \geq 2(n-\delta) \geq 4$. Thus $\lfloor n/r \rfloor \geq 4$ and we may apply Theorem 2.2 as $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{r} \rfloor$. □

Theorem 2.4. *Let $G = (V, E)$ be a graph with n vertices. Let $A_1, A_2, \ldots, A_r$ be mutually disjoint nonempty subsets of V such that $|A_1| + |A_2| + \cdots + |A_r| = \beta$. Let*

$$M = \max_{\substack{b_1+b_2+\ldots+b_r=n-\beta \\ b_1,b_2,\ldots,b_r \geq 0}} \min\{|A_1| + b_1, |A_2| + b_2, \ldots, |A_r| + b_r\}.$$

If $M \geq 3$ and $\delta(G) \geq n - \frac{1}{2}M$, then G is spanning cyclable with respect to $A_1, A_2, \ldots, A_r$.

Proof. We extend $A_1, A_2, \ldots, A_r$ to $B_1, B_2, \ldots, B_r$ such that $\{B_1, B_2, \ldots, B_r\}$ is a partition of V and $A_i \subseteq B_i$ for every i. (We note that the max–min definition is simply to add the remaining vertices to $A_1, A_2, \ldots, A_r$ to form $B_1, B_2, \ldots, B_r$ such that the B_i's have about the same size as much as possible and M is the size of the smallest B_i.) Now for each i, let G_i be the subgraph of G induced by B_i. We claim that G_i has a Hamiltonian cycle by applying Dirac's Theorem. So we need to check that $\delta(G_i) \geq |B_i|/2$. But

$$\delta(G_i) \geq n - \frac{1}{2}M - (n - |B_i|) = |B_i| - \frac{1}{2}M \geq \frac{|B_i|}{2}.$$

So each G_i has a Hamiltonian cycle and the result follows. □

As an example of using Theorem 2.4, we consider the following special case.

Corollary 2.5. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices and let t be an integer with $2 \leq t \leq n-2$.*

1. *If $t \leq \lfloor \frac{n}{2} \rfloor + 1$ and $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{2} \rfloor$, then G is 2-spanning-cyclable of order t. (In other words, if $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{2} \rfloor$, then G is 2-spanning-cyclable of order $\lfloor \frac{n}{2} \rfloor + 1$.)*

2. *If $t > \lfloor \frac{n}{2} \rfloor + 1$ and $\delta(G) \geq n - \frac{1}{2}(1 + n - t)$, then G is 2-spanning-cyclable of order t.*

Proof. Let A and B be two nonempty disjoint subsets of V such that $|A| + |B| \leq t$ and we apply Theorem 2.4. If $|A|, |B| \leq \lceil \frac{n}{2} \rceil$, then $M = \lfloor n/2 \rfloor$ and so we need $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{2} \rfloor$. Now suppose $|A| \leq |B|$ and $|B| \geq n/2 + 1$. Since $n \geq 6$ and $t \leq n - 2$, $|M| \geq 3$ as required. Now, the minimum for M occurs when $|A| = 1$ and $|B| = t - 1$. So $M = 1 + n - t$ and we need $\delta(G) \geq n - \frac{1}{2}(1 + n - t)$. Since $n - \frac{1}{2}(1 + n - t)$ is the maximum among all pairs of A and B such that $|A| + |B| \leq t$, we are done as $(1 + n - t) \leq \lfloor \frac{n}{2} \rfloor$ if $t > \lfloor \frac{n}{2} \rfloor + 1$. □

Corollary 2.6. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices. If $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{2} \rfloor$, then the 2-spanning-cyclability of G is at least $2\delta(G) - n + 1$.*

We now turn to the problem of fixing t, that is, we consider the (spanning)-cyclability of order t.

Corollary 2.7. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices and $2 \leq t \leq n - 3$. Then the spanning-cyclability of G of order t is at least*

$$\min\left\{\frac{n}{2(n - \delta(G))}, \frac{n-t}{2} + 1, \frac{n-t}{2n - 2\delta(G) - 1} + 1\right\}$$

Proof. Given r and t, to minimize M in Theorem 2.4, we set $|A_1| = |A_2| = \cdots = |A_{r-1}| = 1$ and $|A_r| = t - r + 1$. If $t - r + 1 \leq n/r$, then it is sufficient to check that $\delta(G) \geq n - \frac{1}{2}\lfloor \frac{n}{r} \rfloor$, which is true as $r \leq \frac{1}{2}\frac{n}{n - \delta(G)}$. Otherwise, $M = 1 + \lfloor \frac{n-t}{r-1} \rfloor$ provided that $\frac{n-t}{r-1} \geq 2$. Now, $\frac{n-t}{r-1} \geq 2$ is true as $r \leq \frac{n-t}{2} + 1$ and $\delta(G) \geq n - \frac{1}{2}\left(1 + \lfloor \frac{n-t}{r-1} \rfloor\right)$ is true as $r \leq \frac{n-t}{2n - 2\delta(G) - 1} + 1$. □

2.2 Ore type results

Theorem 2.8 (Ore, [21]). *Let G be a graph with $n \geq 3$ vertices. If $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G is Hamiltonian.*

Theorem 2.9. *Let G be a graph with n vertices. Suppose $\lfloor n/r \rfloor \geq 3$. If $\deg(u) + \deg(v) \geq 2n - \lfloor \frac{n}{r} \rfloor$ for every pair of nonadjacent vertices u and v in G, then G is r-spanning-equi-cyclable.*

Proof. Let $A_1, A_2, \ldots, A_r$ be mutually disjoint nonempty subsets of V where $|A_i| \leq n/r$. Clearly, we can find a partition $\{B_1, B_2, \ldots, B_r\}$ of V such that $A_i \subseteq B_i$ and $\lfloor n/r \rfloor \leq |B_i| \leq \lceil n/r \rceil$. We note that $|B_i| \geq 3$. Now for each i, let G_i be the subgraph of G induced by B_i. We claim that G_i has a Hamiltonian cycle by applying Ore's Theorem. Let u and v be nonadjacent vertices in G_i. We need to check that $\deg_{G_i}(u) + \deg_{G_i}(v) \geq |B_i|$. But

$$\begin{aligned}
\deg_{G_i}(u) + \deg_{G_i}(v) &\geq 2n - \left\lfloor \frac{n}{r} \right\rfloor - 2(n - |B_i|) \\
&= 2|B_i| - \left\lfloor \frac{n}{r} \right\rfloor \\
&= |B_i| + \left(|B_i| - \left\lfloor \frac{n}{r} \right\rfloor\right) \\
&\geq |B_i|.
\end{aligned}$$

So each G_i has a Hamiltonian cycle and the result follows. ☐

Corollary 2.10. *Let G be a graph with $n \geq 3$ vertices. Let α be the minimum of* $\deg(u) + \deg(v)$ *over all pairs of nonadjacent vertices u and v. Then the spanning equi-cyclability of G is at least $\frac{n}{2n-\alpha}$.*

Theorem 2.11. *Let $G = (V, E)$ be a graph with n vertices. Let $A_1, A_2, \ldots, A_r$ be mutually disjoint nonempty subsets of V such that $|A_1| + |A_2| + \cdots + |A_r| = \beta$. Let*

$$M = \max_{\substack{b_1+b_2+\ldots+b_r=n-\beta \\ b_1,b_2,\ldots,b_r \geq 0}} \min\{|A_1| + b_1, |A_2| + b_2, \ldots, |A_r| + b_r\}.$$

If $M \geq 3$ and $\deg(u) + \deg(v) \geq 2\left(n - \frac{1}{2}M\right)$ *for every nonadjacent vertices u and v in G, then G is spanning cyclable with respect to $A_1, A_2, \ldots, A_r$.*

Proof. We extend $A_1, A_2, \ldots, A_r$ to $B_1, B_2, \ldots, B_r$ such that $\{B_1, B_2, \ldots, B_r\}$ is a partition of V and $A_i \subseteq B_i$ for every i. (We note that the max min definition is simply to add the remaining vertices to $A_1, A_2, \ldots, A_r$ to form $B_1, B_2, \ldots, B_r$ such that the B_i's have about the same size as much as possible and M is the size of the smallest B_i.) Now for each i, let G_i be the subgraph of G induced by B_i. We claim that G_i has a Hamiltonian cycle by applying Ore's Theorem. Let u and v be nonadjacent vertices in G_i; we need to check that $\deg_{G_i}(u) + \deg_{G_i}(v) \geq |B_i|$. But

$$\deg_{G_i}(u) + \deg_{G_i}(v) \geq 2\left(n - \frac{1}{2}M\right) - 2(n - |B_i|) = 2|B_i| - M \geq |B_i|.$$

So each G_i has a Hamiltonian cycle and the result follows. ☐

Corollary 2.12. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices and let t be an integer with $2 \leq t \leq n - 2$.*

1. *If $t \leq \lfloor \frac{n}{2} \rfloor + 1$ and* $\deg(u) + \deg(v) \geq 2n - \lfloor \frac{n}{2} \rfloor$ *for every nonadjacent vertices u and v in G, then G is 2-spanning-cyclable of order t. (In other words, if* $\deg(u) + \deg(v) \geq 2n - \lfloor \frac{n}{2} \rfloor$ *for every nonadjacent vertices u and v in G, then G is 2-spanning-cyclable of order $\lfloor \frac{n}{2} \rfloor + 1$.)*
2. *If $t > \lfloor \frac{n}{2} \rfloor + 1$ and* $\deg(u) + \deg(v) \geq 2n - (1 + n - t) = n + t - 1$ *for every nonadjacent vertices u and v in G, then G is 2-spanning-cyclable of order t.*

We note that the example given in Section 1 is the above corollary with $t = 2$. One may wonder whether Corollary 2.12 is tight; in particular, it may seem suboptimal that for $t \leq \lfloor \frac{n}{2} \rfloor + 1$.

Corollary 2.13. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices. Let α be the minimum of* $\deg(u) + \deg(v)$ *over all pairs of nonadjacent vertices u and v in G. Then the 2-spanning-cyclability of G is at least $\alpha - n + 1$.*

Corollary 2.14. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices and $2 \leq t \leq n - 3$. Let α be the minimum of* $\deg(u) + \deg(v)$ *over all pairs of nonadjacent vertices u and v in G. Then the 2-spanning-cyclability of G of order t is at least* $\min\left\{\frac{n}{2n-\alpha}, \frac{n-t}{2} + 1, \frac{n-t}{2n-\alpha-1} + 1\right\}$.

2.3 Bondy–Chvátal type results

Theorem 2.15 (Bondy–Chvátal, [4]). *Let u and v be nonadjacent vertices in a graph G with $n \geq 3$ vertices such that* $\deg(u) + \deg(v) \geq n$. *Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

Theorem 2.16. *Let G be a graph with n vertices. Suppose $\lfloor n/r \rfloor \geq 3$. If u and v are nonadjacent vertices in G such that* $\deg(u) + \deg(v) \geq 2n - \lfloor \frac{n}{r} \rfloor$, *then G is r-spanning-equi-cyclable if and only if $G + uv$ is r-spanning-equi-cyclable.*

Proof. Let u and v be nonadjacent vertices in G such that $\deg(u)+\deg(v) \geq 2n - \lfloor \frac{n}{r} \rfloor$. Clearly if G is r-spanning-equi-cyclable, then $G + uv$ is r-spanning-equi-cyclable. So we assume that $G + uv$ is r-spanning-equi-cyclable. Let $A_1, A_2, \ldots, A_r$ be mutually disjoint nonempty subsets of V where $|A_i| \leq n/r$. Clearly, we can find a partition $\{B_1, B_2, \ldots, B_r\}$ of V such that $A_i \subseteq B_i$ and $\lfloor n/r \rfloor \leq |B_i| \leq \lceil n/r \rceil$. We note that $|B_i| \geq 3$. Now for each i, let G_i be the subgraph of G induced by B_i. We consider two cases. The first case is when u and v are in different G_i's, say, G_1 and G_2. Since $G+uv$ is r-spanning-equi-cyclable and the B_i's partitioned V, each of $G_1, G_2, \ldots, G_r$ is Hamiltonian and we are done. The second case is when u and v are in the same G_i, say, G_1. Since $G + uv$ is r-spanning-equi-cyclable, each of $G_1 + uv, G_2, \ldots, G_r$ is Hamiltonian. Since

$$\begin{aligned} \deg_{G_i}(u) + \deg_{G_i}(v) &\geq 2n - \left\lfloor \frac{n}{r} \right\rfloor - 2(n - |B_i|) \\ &= 2|B_i| - \left\lfloor \frac{n}{r} \right\rfloor \\ &= |B_i| + \left(|B_i| - \left\lfloor \frac{n}{r} \right\rfloor\right) \\ &\geq |B_i|, \end{aligned}$$

we may apply the Bondy–Chvátal's Theorem to conclude that G_1 is Hamiltonian, so we are done. □

We can now define the *closure* of a graph G with n vertices and $\lfloor n/r \rfloor \geq 3$ to be the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $2n - \lfloor \frac{n}{r} \rfloor$ until no such pair remains.

Corollary 2.17. *Let G be a graph with n vertices. Suppose $\lfloor n/r \rfloor \geq 3$. Then a graph is r-spanning-equi-cyclable if and only if its closure is r-spanning-equi-cyclable. In particular, if the closure of a graph is complete, then the graph is r-spanning-equi-cyclable.*

As an example of 2-spanning-cyclability, we have the following result whose proof is the same as the proof of Theorem 2.16 since $|A_1| + |A_2| \leq \lfloor \frac{n}{2} \rfloor + 1$ implies $|A_1|, |A_2| \leq \lfloor \frac{n}{2} \rfloor$.

Theorem 2.18. *Let $G = (V, E)$ be a graph with $n \geq 6$ vertices. If u and v are nonadjacent vertices in G such that* $\deg(u)+\deg(v) \geq 2n - \lfloor \frac{n}{2} \rfloor$, *then G is 2-spanning-cyclable of order $\lfloor \frac{n}{2} \rfloor + 1$ if and only if $G + uv$ is 2-spanning-cyclable of order $\lfloor \frac{n}{2} \rfloor + 1$.*

One can then define an appropriate closure to obtain a result similar to Corollary 2.17.

3 Sufficient conditions of the extremal type

In this section we study extremal cases that are not covered by our results in Section 2 and provide sufficient conditions based on the number of edges for a graph to be 2-spanning-cyclable of order k for $k \leq n-1$. We start with the following less well-known result of Ore [22].

Theorem 3.1 (Ore, [22]). *Let G be a graph n vertices. If G has at least $(n-1)(n-2)/2+2$ edges, then G is Hamiltonian.*

One may feel that Theorem 3.1 is not a strong result as such a graph is almost complete as only $n-3$ edges are missing. However, this result is tight as there are non-Hamiltonian graphs on n vertices with $(n-1)(n-2)/2+1$ edges. Our goal is to find similar tight extremal result for 2-spanning-cyclable graph. We note that although Theorem 3.1 can be proved easily from Theorem 2.8, the results in this section does not follow from the Ore type result given in Section 2.2. First we have the following easy bounds on 2-spanning-cyclability:

Theorem 3.2. *Let G be a connected graph on at least six vertices. If G is 2-spanning-cyclable of order t, then $t \leq \min\{\delta(G)-1, \kappa(G)\}$.*

Proof. Clearly if G is 2-spanning-cyclable, then $\kappa(G) \geq 2$. Henceforth, we may assume that $\kappa(G) \geq 2$. Let u be any vertex of G with $\deg_G(u) = \delta(G)$, where $\delta(G) \geq 2$ since G is connected. Set $A_1 = N_G(u) \setminus \{w\}$ for some $w \in N_G(u)$ and $A_2 = \{u\}$. Since $\deg_{G-A_1}(u) = 1$, there is no cycle of $G - A_1$ containing A_2, thus G is not 2-spanning-cyclable of order $\delta(G)$. Hence the 2-spanning-cyclability of G is at most $\delta(G)-1$.

If G is isomorphic to the complete graph K_n for some integer n, then $\kappa(G) = \delta(G) = n-1$, thus 2-spanning-cyclability of G is at most $\delta(G)-1 < \kappa(G)$. Assume that G is not complete. Then G has a set S such that $|S| = \kappa(G)$ and $G-S$ has at least two components. Pick one vertex each from two components, let these vertices be x and y. Set $A_1 = \{x, y\}$ and $A_2 = S \setminus \{z\}$ for some vertex $z \in S$. Obviously, there is no cycle of $G - A_2$ containing A_1, thus G is not 2-spanning-cyclable of order $\kappa(G)+1$, hence the 2-spanning-cyclability of G is at most $\kappa(G)$.

Thus the 2-spanning-cyclability of G is at most $\min\{\delta(G)-1, \kappa(G)\}$. □

We also need the following easy result.

Lemma 3.3. *If G is a graph on n vertices with $|E(G)| \geq \frac{(n-1)(n-2)}{2}+2$, then every vertex of G is in a cycle with exactly three vertices.*

Proof. Let u be a vertex of G, and suppose that u is not in a cycle of three vertices. Let d be the degree of u. Since $|E(G)| \geq \frac{(n-1)(n-2)}{2}+2$, we have $d \geq 2$. Set $N_G(u) = \{x_1, x_2, \ldots, x_d\}$. Since u is not in a triangle, we have $(x_i, x_j) \notin E(G)$ for every $1 \leq i < j \leq d$. Thus $|E(G)| \leq \frac{n(n-1)}{2} - (n-1-d) - \frac{d(d-1)}{2} = \frac{(n-1)(n-2)}{2} + d - \frac{d(d-1)}{2}$. Since $d \geq 2$, we have $d - \frac{d(d-1)}{2} \leq 1$. Thus $|E(G)| \leq \frac{(n-1)(n-2)}{2} + 1 < \frac{(n-1)(n-2)}{2} + 2$, which is a contradiction. □

It is clear that if x and y are two distinct vertices of G, and there are two disjoint cycles in G such that one contains x and the other contains y, then we must have $|(N_G(x) \cup N_G(y)) - \{x, y\}| \geq 4$, and G must have at least six vertices. Consider the following example: Let G be the complete graph K_6 with two adjacent edges deleted. Then we have $|E(G)| = 13$ and $|(N_G(x_1) \cup N_G(x_2)) - \{x_1, x_2\}| = 3$, hence G is not 2-spanning-cyclable of order 2. However, G is Hamiltonian by Theorem 3.1.

Consider another example: Let G be the graph obtained from K_n by deleting $n-3$ edges incident to the same vertex v. Then $\deg_G(v) = 2$, so Theorem 3.2 implies that G is not 2-spanning-cyclable of order 2 even though it has $\frac{(n-1)(n-2)}{2} + 2$ edges.

Thus the following result is tight:

Theorem 3.4. *If G is a graph on n vertices where $n \geq 7$, and G has at least $\frac{(n-1)(n-2)}{2} + 3$ edges, then G is 2-spanning-cyclable of order 2.*

Proof. Set $A_1 = \{x\}$ and $A_2 = \{y\}$, where x and y are any two distinct vertices of G, and consider the following cases.

Case 1. $\min\{\deg_G(x), \deg_G(y)\} \leq n-3$.

Without loss of generality we may assume that $\deg_G(x) \leq n-3$. Set $H = G - \{y\}$. We have

$$|E(H)| \geq \frac{(n-1)(n-2)}{2} + 3 - (n-1) = \frac{(n-2)(n-3)}{2} + 2,$$

hence by Lemma 3.3, there is a cycle C_1 of length 3 in H containing x. Set $Z = G - V(C_1)$. Then since $\deg_G(x) \leq n-3$,

$$|E(Z)| \geq \frac{(n-1)(n-2)}{2} + 3 - (3n-8) = \frac{(n-4)(n-5)}{2} + 2.$$

Thus by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z, so C_1 and C_2 are the desired cycles.

Case 2. $\deg_G(x) = n-2$ and $\deg_G(y) = n-2$.

Set $H = G - \{y\}$.

Case 2.1. $(x, y) \notin E(G)$.

Set $N_G(x) = \{z_1, z_2, \ldots, z_{n-2}\}$. Since $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 3$, we have $\deg_G(z_i) \geq 3$ for every $1 \leq i \leq n-2$. Without loss of generality, we may assume that $\deg_G(z_1) \leq \deg_G(z_i)$ for every $2 \leq i \leq n-2$ and assume that $(z_1, z_2) \in E(G)$.

If $\deg_G(z_1) = n-1$, then $\deg_G(z_i) = n-1$ for every $1 \leq i \leq n-2$, thus G is isomorphic to the complete graph minus an edge, hence the desired cycles can be found easily. Now assume $\deg_G(z_1) \leq n-2$. Set $C_1 = \langle x, z_1, z_2, x \rangle$ to be a cycle of length 3, and let $Z = G - V(C_1)$. Then

$$|E(Z)| \geq \frac{(n-1)(n-2)}{2} + 3 - (2(n-2) + (n-1) - 3) = \frac{(n-4)(n-5)}{2} + 2,$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z, so C_1 and C_2 are the desired cycles.

Case 2.2. $(x, y) \in E(G)$.

Set $N_G(x) = \{z_1, z_2, \ldots, z_{n-2}\}$ where $y = z_{n-2}$, and let z_{n-1} be the last vertex of G.

Since $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 3$, $\deg_G(z_i) \geq 3$ for every $1 \leq i \leq n-2$. Without loss of generality, we may assume that $\deg_G(z_1) \leq \deg_G(z_i)$ for every $2 \leq i \leq n-3$.

In the case when $deg_G(z_1) = n-1$, the graph G is the complete graph minus two edges. We can easily find the desired cycle.

Next let us assume that $\deg_G(z_1) \leq n-2$. Since $\deg_G(x) = n-2$ and $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 3$, we must have $\deg_G(z_1) \geq 4$. Thus x and z_1 must have at least one common neighbor different from y, we may assume it is z_2. Set $C_1 = \langle x, z_1, z_2, x \rangle$ and $Z = G - V(C_1)$. Then

$$|E(Z)| \geq \frac{(n-1)(n-2)}{2} + 3 - (2(n-2) + (n-1) - 3) = \frac{(n-4)(n-5)}{2} + 2$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z, so C_1 and C_2 are the desired cycles.

Case 3. $\min\{\deg_G(x), \deg_G(y)\} \geq n-2$ and $\max\{\deg_G(x), \deg_G(y)\} = n-1$.
Without loss of generality, we may assume that $\deg_G(x) = n-1$. Let z be a vertex of minimum degree in G excluding x and y. Since $\deg_G(z) \geq 3$, vertex z has a neighbor different from x and y.

If $\deg_G(z) = n-1$, then every vertex has degree $n-1$ except maybe y, so we can find the needed cycles as in Case 2.2. So let us assume $\deg_G(z) \leq n-2$.

Case 3.1. $\deg_G(z) \leq n-3$.
Let w be a vertex adjacent to z different from x and y. Set $C_1 = \langle x, z, w, x \rangle$ and $Z = G - C_1$. Then

$$|E(Z)| \geq \frac{(n-1)(n-2)}{2} + 3 - (2(n-1) + (n-3) - 3) = \frac{(n-4)(n-5)}{2} + 2,$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z, so C_1 and C_2 are the desired cycles.

Case 3.2. $\deg_G(z) = n-2$.
If every neighbor of z apart from x and possibly y has degree $n-1$, we can find the needed cycles as in Case 2.2. So let us assume that there is a vertex w adjacent to z such that $\deg_G(w) \leq n-2$. Set $C_1 = \langle x, z, w, x \rangle$ and $Z = G - C_1$. Then

$$|E(Z)| \geq \frac{(n-1)(n-2)}{2} + 3 - ((n-1) + 2(n-2) - 3) = \frac{(n-4)(n-5)}{2} + 2,$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z, so C_1 and C_2 are the desired cycles.

This covers all cases, finishing the proof. □

To generalize Theorem 3.4 we need the following lemma:

Lemma 3.5. *Let A be a vertex subset of G such that $2 \leq |A| \leq |V(G)| - 1$, and let C be a cycle of minimum length in G among cycles containing A. If $|V(C)| \geq 4$ and $|V(C)| > |A|$, then $|E(C)| \leq \frac{|V(C)|(|V(C)|-1)}{2} - (|V(C)| - |A|)$.*

Proof. Let $C = \langle x_1, x_2, \ldots, x_c, x_1 \rangle$ where $x_1 \in A$. Set $S = \{i \mid x_i \in A \text{ for } 1 \leq i \leq c\}$ and let $S = \{j_1, j_2, \ldots, j_a\}$ where $a = |A|$ and $j_i < j_{i+1}$ for every $1 \leq i \leq a-1$.

Note that $j_1 = 1$. Set $S' = \{j_i \mid x_{j_i+1} \neq x_{j_{i+1}} \text{ for } 1 \leq i \leq a-1\} \cup \{j_a \mid j_a \neq c\}$. Since $|V(C)| > |A|$, we have $S' \neq \emptyset$.

For some $j_i \in S' - \{j_a\}$, suppose that there is an edge $(x_{j_{i+1}}, x_k) \in E(G)$ for some $j_i \leq k \leq j_{i+1} - 2$. Since the length of C is at least 4, $\langle x_1, x_2, \ldots, x_k, x_{j_{i+1}}, x_{j_{i+1}+1}, \ldots, x_c, x_1\rangle$ is also a cycle in G containing A, and it is shorter than C, which is a contradiction. Thus $(x_{j_{i+1}}, x_k) \notin E(G)$ for every $j_i \leq k \leq j_{i+1} - 2$. Similarly, $(x_{j_1}, x_k) \notin E(G)$ for every $j_a \leq k \leq c-1$. Thus $|E(C)| \leq \frac{|V(C)|(|V(C)|-1)}{2} - (|V(C)| - |A|)$. □

Theorem 3.6. *If G is a graph on n vertices where $n \geq 7$, and G has at least $\frac{(n-1)(n-2)}{2} + k$ edges for $4 \leq k \leq n-1$, then G is 2-spanning-cyclable of order $k-1$.*

Proof. Let A and B be any two disjoint sets of vertices of G with $|A| = a$ and $|B| = b$ and $a + b = k - 1$. Without loss of generality, we may assume that $a \leq b$. Since $k \geq 4$, we have $b \geq 2$. Set $H = G - A$. We have

$$\begin{aligned} |E(H)| &\geq \frac{(n-1)(n-2)}{2} + k - \left(\frac{a(a-1)}{2} + a(n-a)\right) \\ &= \frac{(n-a-1)(n-a-2)}{2} + k - a \\ &\geq \frac{(n-a-1)(n-a-2)}{2} + 2, \end{aligned}$$

thus by Theorem 3.1, graph H is Hamiltonian. Let C_1 be a cycle of minimum length among cycles in H containing B. Set $c = |V(C_1)|$, so $b \leq c$. Let $Z = G - C_1$. We consider the following cases.

Case 1. $c = b$.

Since $a + b = k - 1$, we have $k - c = a + 1$, thus

$$\begin{aligned} |E(Z)| &\geq \frac{(n-1)(n-2)}{2} + k - \left(c(n-c) + \frac{c(c-1)}{2}\right) \\ &= \frac{(n-c-1)(n-c-2)}{2} + k - c \\ &\geq \frac{(n-c-1)(n-c-2)}{2} + 2, \end{aligned}$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z containing A, so C_1 and C_2 are the desired cycles.

Case 2. $c > b$ and $c \geq 4$.

By Lemma 3.5, $|E(C_1)| \leq \frac{c(c-1)}{2} - (c - b)$, thus

$$\begin{aligned} |E(Z)| &\geq \frac{(n-1)(n-2)}{2} + k - \left(c(n-c) + \left(\frac{c(c-1)}{2} - (c-b)\right)\right) \\ &= \frac{(n-c-1)(n-c-2)}{2} + k - b \\ &= \frac{(n-c-1)(n-c-2)}{2} + a + 1 \\ &\geq \frac{(n-c-1)(n-c-2)}{2} + 2, \end{aligned}$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z containing A, so C_1 and C_2 are the desired cycles.
Case 3. $c > b$ and $c = 3$.
Then we have $b = 2$, hence $k = a + b + 1$ implies $4 \leq k \leq 5$.
Case 3.1. $k = 5$.
We get

$$\begin{aligned}|E(Z)| &\geq \frac{(n-1)(n-2)}{2} + k - (3(n-3)+3) \\ &= \frac{(n-4)(n-5)}{2} + k - 3 \\ &= \frac{(n-c-1)(n-c-2)}{2} + a + 1 \\ &\geq \frac{(n-c-1)(n-c-2)}{2} + 2,\end{aligned}$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z containing A, so C_1 and C_2 are the desired cycles.
Case 3.2. $k = 4$.
We have $a = 1$; set $B = \{p, q\}$.
Case 3.2.1. $\min\{\deg_G(p), \deg_G(q)\} \leq n - 2$.
Then

$$|E(Z)| \geq \frac{(n-1)(n-2)}{2} + k - (2(n-3) + (n-4) + 3) = \frac{(n-4)(n-5)}{2} + 2,$$

so by Theorem 3.1, there is a Hamiltonian cycle C_2 of Z containing A, so C_1 and C_2 are the desired cycles.
Case 3.2.2. $\deg_G(p) = \deg_G(q) = n - 1$.
If every vertex in G has degree $n-1$, then the claim trivially holds. Otherwise there is a vertex w in $G - A$ with $\deg_G(w) \leq n-2$. Redefine $C_1 = \langle p, w, q, p\rangle$ and $Z = G - C_1$. Then we can find the desired cycles just like in Case 3.2.1.

This covers all cases, finishing the proof. □

According to Theorem 3.1, Theorem 3.4, and Theorem 3.6, we have the following result.

Theorem 3.7. *If G is a graph on n vertices where $n \geq 7$, and G has at least $\frac{(n-1)(n-2)}{2} + k$ edges for $3 \leq k \leq n-1$, then G is 2-spanning-cyclable of order $k-1$.*

If a graph on n vertices has $\frac{(n-1)(n-2)}{2} + k$ edges, then its minimum degree can be k (if all missing edges are incident to one vertex), so Theorem 3.7 is best possible by Theorem 3.2.

4 Conclusion

As noted earlier in this paper, the classical Ore's Theorem 2.8 implies the extremal version, Theorem 3.1, immediately. In this paper, we presented a number of results of these types for the spanning-cyclable problem. It is worth noting that Corollary 2.12 is the 2-spanning-cyclable version of Theorem 2.8 and Theorem 3.7 is the 2-spanning-cyclable version of Theorem 3.1. However, Corollary 2.12 does not imply Theorem 3.7 as the condition in Corollary 2.12 does not have t as a parameter for Case (1). (In other words, Case (1) is really a statement for $t = \lfloor \frac{n}{2} \rfloor + 1$.) However, for $t > \frac{n}{2} + 1$, Corollory 2.12 does imply Theorem 3.7.

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