Combinatorial proofs of some Bell number formulas

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Abstract

In this note, we provide bijective proofs of some identities involving the Bell number, as previously requested. Our arguments may be extended to yield a generalization in terms of complete Bell polynomials. We also provide a further interpretation for a related difference of Catalan numbers in terms of the inclusion-exclusion principle.

1 Introduction

By a *partition* of a set, we will mean a collection of pairwise disjoint subsets, called *blocks*, whose union is the set. Let $\mathcal{P}_n$ denote the set of all partitions of $[n] = \{1, 2, \ldots, n\}$. Recall that the cardinality of $\mathcal{P}_n$ is given by the n-th Bell number b_n; see A000110 in [7]. In what follows, if m and n are positive integers, then let $[m, n] = \{m, m+1, \ldots, n\}$ if $m \leqslant n$, with $[m, n] = \varnothing$ if $m > n$. Throughout, the binomial coefficient is given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ if $0 \leqslant k \leqslant n$, with $\binom{n}{k}$ taken to be zero otherwise.

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In [6], combinatorial proofs were sought for some identities involving Bell numbers and binomial coefficients. Relation (2.1) below appears in a slightly different though equivalent form as Theorem 4.4 in [6], where an algebraic proof was given, and identities (2.2), (2.3), and (2.4) appear as Corollary 4.5. It is the purpose of this note to provide the requested bijective proofs of (2.1)–(2.4). Our proof may be extended to obtain a more general relation involving Bell polynomials. In the final section, we provide a further interpretation for the related alternating sum $\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}c_i$, where c_i denotes the i-th Catalan number. In particular, we show that it can be thought of in terms of the inclusion-exclusion principle, as requested in [6].

2 Bijective proof of identities

We first provide a bijective proof of a formula relating an alternating sum to a positive one in the spirit of [2]. (See also [1, p. 59].)

Theorem 2.1. *If $0 \leqslant j \leqslant n$, then*

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i}b_{n+1-i} = \sum_{k=0}^{n-j}\binom{n-j}{k}b_{n-k}. \tag{2.1}$$

Proof. Given $0 \leqslant j \leqslant n$, let $\mathcal{A} = \mathcal{A}_{n,j}$ denote the set of ordered pairs $\lambda = (S, \pi)$, where $S \subseteq [j]$ and π is a partition of $[n+1]-S$. Define the sign of λ to be $(-1)^{|S|}$. Note that the left-hand side of (2.1) gives the total weight of all the members of $\mathcal{A}$. We will define a sign-changing involution of $\mathcal{A}$ whose set of survivors has weight given by the right-hand side of (2.1). To do so, let ℓ_0 denote the largest $\ell \in [j]$, if it exists, such that either $\ell \in S$ or ℓ occurs as a singleton block within π. Let λ' denote the member of $\mathcal{A}$ obtained by either removing ℓ_0 from S and adding it to π as the singleton block $\{\ell_0\}$ if $\ell_0 \in S$ or, vice-versa, if ℓ_0 occurs within π as a singleton. Observe that the mapping

$\lambda \mapsto \lambda'$ defines a sign-changing involution of $\mathcal{A} - \mathcal{A}^*$, where $\mathcal{A}^*$ denotes the subset of $\mathcal{A}$ consisting of all λ of the form $\lambda = (\varnothing, \pi)$ in which π is a member of $\mathcal{P}_{n+1}$ having no singleton blocks in $[j]$. Note that each member of $\mathcal{A}^*$ has positive sign.

For example, suppose $n = 8$, $j = 4$, and $\lambda = (S, \pi) \in \mathcal{A}_{8,4}$, with $S = \{1, 3\}$ and $\pi = \{2\}, \{4, 5\}, \{6, 8, 9\}, \{7\}$. Then $\ell_0 = 3$ and λ would be paired with $\lambda' = (S', \pi')$, where $S' = \{1\}$ and $\pi' = \{2\}, \{3\}, \{4, 5\}, \{6, 8, 9\}, \{7\}$.

To complete the proof, we need to show that the number of partitions π of $[n + 1]$ having no singleton blocks among the elements of $[j]$ is given by $\sum_{k=0}^{n-j} \binom{n-j}{k} b_{n-k}$. To count such π, consider the number k of elements of $[j + 1, n]$ belonging to the block containing $n + 1$, where $0 \leqslant k \leqslant n - j$. Having selected k elements in $\binom{n-j}{k}$ ways to go in the block containing $n + 1$, we then form a partition ρ of the remaining $n - k$ elements of $[n]$ in b_{n-k} ways. Finally, we remove any singletons of ρ belonging to $[j]$ and insert them as elements into the block containing $n + 1$. This yields a partition of $[n + 1]$ containing no singletons in $[j]$. Conversely, given any such member of $\mathcal{P}_{n+1}$, one can remove all of the elements less than or equal j in the block containing $n + 1$ and add them back as singleton blocks. Disregarding the block that now contains $n + 1$ and possibly some elements of $[j + 1, n]$ yields an arbitrary partition of size $n - k$, which completes the proof. □

We can also provide a bijective proof of the following corollary.

Corollary 2.1. *If $j \geqslant 0$, then*

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i} b_{j+1-i} = b_j \tag{2.2}$$

and

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i} b_{j+2-i} = b_j + b_{j+1}. \tag{2.3}$$

If $j \geqslant 2$, then

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i}b_{j-i} = \sum_{k=0}^{j-2}(-1)^k b_{j-1-k}. \tag{2.4}$$

Proof. By the preceding argument, the left-hand side of (2.2) counts the members of $\mathcal{P}_{j+1}$ in which no element of $[j]$ occurs as a singleton. Equivalently, one may remove any singleton blocks from a member of $\mathcal{P}_j$ and form a new block with them together with the element $j+1$. Similarly, the left-hand side of (2.3) counts the members of $\mathcal{P}_{j+2}$ in which no element of $[j]$ occurs as a singleton. On the other hand, one may remove any elements occurring as singletons within a member of $\mathcal{P}_j$ and form a new block with them together with the elements $j+1$ and $j+2$ or remove any singletons in $[j]$ from a member of $\mathcal{P}_{j+1}$ and insert them into a block with $j+2$.

To show (2.4), given $0 \leqslant k \leqslant j-2$, let $\mathcal{C}_{j-1-k}$ denote the subset of $\mathcal{P}_j$ whose members contain no singletons amongst $[j-k]$, with each element of $[j+1-k, j]$ occurring as a singleton, and let $\mathcal{D}_{j-1-k}$ denote the subset of $\mathcal{P}_j$ whose members contain no singletons amongst $[j-1-k]$, with each element of $[j-k, j]$ occurring as a singleton. By the definitions, we have $\mathcal{D}_1 = \varnothing$, $\mathcal{D}_{j-1-k} = \mathcal{C}_{j-2-k}$ if $0 \leqslant k \leqslant j-3$, and

$$b_{j-1-k} = |\mathcal{C}_{j-1-k}| + |\mathcal{D}_{j-1-k}|,$$

by (2.2), since $\mathcal{C}_{j-1-k} \cup \mathcal{D}_{j-1-k}$ is the same as the set of all members of $\mathcal{P}_{j-k}$ in which there are no singletons in $[j-1-k]$. Thus, we have

$$\begin{aligned}\sum_{k=0}^{j-2}(-1)^k b_{j-1-k} &= \sum_{k=0}^{j-2}(-1)^k(|\mathcal{C}_{j-1-k}| + |\mathcal{D}_{j-1-k}|)\\ &= |\mathcal{C}_{j-1}| + (-1)^{j-2}|\mathcal{D}_1| = |\mathcal{C}_{j-1}|.\end{aligned}$$

On the other hand, we have $|\mathcal{C}_{j-1}| = \sum_{i=0}^{j}(-1)^i\binom{j}{i}b_{j-i}$, by the proof of (2.1) above, since $\mathcal{C}_{j-1}$ is the set of all partitions of $[j]$ having no singleton blocks. This completes the proof.

□

3 A generalization

Let $B_n = B_n(t_1, t_2, \ldots, t_n)$ denote the n-th complete Bell polynomial; see, for example, [3]. Recall that B_n is obtained by assigning the weight t_i to each block of a partition $\pi \in \mathcal{P}_n$ of size i, defining the weight of π to be the product of the weights of its blocks, and then summing over all π. It is well-known (see [3]) that B_n is given by $\sum_{r=0}^{n} B_{n,r}(t_1, t_2, \ldots, t_n)$, where

$$B_{n,r}(t_1, t_2, \ldots, t_n) = \sum_{s} \frac{n!}{r_1! r_2! \cdots r_n!} \left(\frac{t_1}{1!}\right)^{r_1} \left(\frac{t_2}{2!}\right)^{r_2} \cdots \left(\frac{t_n}{n!}\right)^{r_n}$$

and the sum is taken over all nonnegative integer solutions $s = (r_1, r_2, \ldots, r_n)$ of $x_1+x_2+\cdots+x_n = r$ and $x_1+2x_2+\cdots+nx_n = n$. Note that $B_n = b_n$ when all weights are unity and $B_n = n!$ when $t_i = (i-1)!$ for all i. Other sequences emerge from other specializations of the t_i; for example, $B_n = L_n$, the n-th Lah number, when $t_i = i!$ for all i and $B_n = d_n$, the n-th derangement number, when $t_1 = 0$ and $t_i = (i-1)!$ for $i \geqslant 2$ (see A000262 and A000166, respectively, in [7]).

The following result generalizes (2.1), reducing to it when $t_i = 1$ for all i.

Theorem 3.1. *If* $0 \leqslant j \leqslant n$, *then*

$$\sum_{i=0}^{j} (-1)^i t_1^i \binom{j}{i} B_{n+1-i}$$
$$= \sum_{k=0}^{n-j} \sum_{\ell=0}^{j} \sum_{r=0}^{j-\ell} (-1)^r t_1^r t_{k+\ell+1} \binom{n-j}{k} \binom{j}{\ell} \binom{j-\ell}{r} B_{n-k-\ell-r}. \tag{3.1}$$

Proof. We extend the reasoning used to show (2.1) above. To the ordered pair (S, π), we now assign the weight

$$(-1)^{|S|} t_1^{|S|+s_1(\pi)} t_2^{s_2(\pi)} t_3^{s_3(\pi)} \cdots,$$

where $s_i(\pi)$ denotes the number of blocks of π of size i. Then the left-hand side of (3.1) gives the total weight of all possible ordered pairs and the involution defined in the first paragraph of the proof of (2.1) above is seen to reverse the weight.

To complete the proof, we argue that the total weight of all members of $\mathcal{P}_{n+1}$ having no singletons in $[j]$ is given by the right-hand side of (3.1). In addition to choosing k elements of $[j+1, n]$ to go in the same block as the element $n+1$ as before, we must now also select ℓ members of $[j]$ to go in this block. Note that this creates a block of size $k+\ell+1$, whence the $t_{k+\ell+1}$ factor. Once this is done, the remaining $n-k-\ell$ members of $[n]$ must be partitioned into sets such that none of the remaining $j-\ell$ members of $[j]$ occur as singletons. Reasoning as before, this may be achieved in $\sum_{r=0}^{j-\ell}(-1)^r t_1^r \binom{j-\ell}{r} B_{n-k-\ell-r}$ ways, which completes the proof. $\square$

Taking various specializations of the t_i yields analogues of (2.1) for other sequences such as $n!$, L_n, or d_n.

4 An interpretation for a Catalan number difference

We address here a related question which concerns a sum comparable to the left-hand side of (2.1) above for the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. Let

$$K_n = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}c_i, \qquad n \geqslant 0.$$

In [6], it was shown by induction and generating functions that K_n enumerates a subset of a structure whose members the authors term *non-interlocking, non-skipping n-columns* (denoted by NINS). The problem of finding a direct argument for this using inclusion-exclusion was mentioned.

In fact, a clearer combinatorial interpretation for the numbers K_n may be given. Suppose $\Pi = B_1/B_2/\cdots \in \mathcal{P}_n$ has its blocks arranged in increasing order of smallest elements. Recall that π may be represented, canonically, as a *restricted growth sequence* $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $i \in B_{\pi_i}$ for each i. See, e.g., [5]. For instance, the partition $\Pi = 126/359/4/78$ would be represented as $\pi = 112321442$. A partition is said to be *non-crossing* (see [4]) if its canonical form contains no subsequences of the form *abab* where $a < b$ (i.e., if it avoids all occurrences of the pattern 1212). By the definitions from [6], an NINS n-column is seen to be equivalent to a non-crossing partition of the same length since, when an n-column is non-skipping, it can be shown that the interlocking property is equivalent to the non-crossing property. Thus, Theorem 3.2 in [6] is equivalent to the following combinatorial interpretation of K_n:

> The number of non-crossing partitions of length n containing no two equal adjacent letters (where the first and last letters are considered adjacent) is given by K_n.

In [6], it was requested to provide a direct combinatorial proof of Theorem 3.2. Using our interpretation for K_n in terms of non-crossing partitions, one may provide such a proof as follows. Note that within a non-crossing partition represented sequentially, any letter whose successor is the same (as well as a possible 1 at the end) is extraneous concerning the containment or avoidance of the pattern 1212. Thus, one may cover any positions containing letters whose successor is the same (as well as a possible 1 at the end) and consider the avoidance problem on the remaining uncovered positions. Suppose one covers exactly $n - i$ positions. Then the remaining letters constitute a partition of size i avoiding the pattern 1212, and it is well-known that such partitions are enumerated by c_i (see [4]). When one uncovers the repetitive letters, no occurrences of 1212 are introduced. That K_n counts all non-crossing partitions of length n having no two adjacent letters the same now follows from an application of the inclusion-exclusion principle. In fact, it is further seen that the sum $\sum_{i=0}^{j}(-1)^i\binom{j}{i}c_{n-i}$ counts all non-crossing partitions of length n in which no letter equals its successor among the first j positions in analogy to Theorem 2.3 in [6].

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