Pre-hull number of Cartesian and strong graph products

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Abstract

Recently introduced invariants copoint pre-hull number and convex pre-hull number are both numerical measures of nonconvexity of a graph G that is a convex space. We consider in this work both on the Cartesian and the strong product of graphs. Exact values in terms of invariants of the factors are presented for the first mentioned product. For the strong product it is shown that such a result does not exists, but but an exact result for trees is proved.

Keywords: pre-hull number, convexity, Cartesian product, strong product

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1 Introduction and preliminaries

The (geodetic) convex hull of some subset of vertices of a finite graph G can be achieved in finite number of steps in which at each step the new set is the union of intervals between all pairs of vertices of a set of previous step. The number of these steps is a numerical measure of nonconvexity of a starting set in a convex space (which is a graph G). The general approach was introduced by Harary and Nieminen in [5] with geodetic iteration number gin$G$ of G. Similar approach was taken recently by Polat and Sabidussi in [9] with an additional restriction to sets: they observed the union of a vertex v and its copoints $C \subset V(G)$, this are maximal convex sets with the property that they do not include v —the (copoint) pre-hull number. An approach that is structurally between both mentioned was taken in [7], where the restriction to copoints was omitted and the union of a vertex v with all convex sets have been treated. Such an invariant is a natural upper bound for the copoint pre-hull number.

Graph products are by now well studied procedures of graph enlargement. They have been investigated with respect to their structure, (non)uniqueness of their factors as well as the decomposition algorithms and their

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complexity. For general results see the book [4]. Other standard approach to graph products is to find some properties of the product with respect to the properties of factors. This approach is used also in this work. For a collection of convexity related results of this type see recent papers [1, 7, 8]. In particular, in [7] the exact convex and copoint pre-hull numbers are given for the lexicographic product.

In the next section we give the exact result for the convex and the copoint pre-hull number of the Cartesian product of graphs with respect to the copoint and the convex pre-hull numbers of both factors. The last section is devoted to the copoint and the convex pre-hull number of the strong product. Only partial results are given for the strong product, despite the new characterization of convex sets of this product, see [8]. In the remainder of this section we give a formal definition of the convex and the copoint pre-hull number.

All graphs considered are simple, undirected, and finite. The shortest path between two vertices u and v of a graph G is called a *u, v-geodesic*. The *distance* $d_G(u, v)$ between $u, v \in V(G)$ is the length of a u, v-geodesic. An *interval* $I(u, v)$ between $u, v \in V(G)$ consists of all vertices that belong to u, v-geodesics in G.

In general a *convexity* on a set X is an algebraic closure system $\mathcal{C}$ on X and elements of $\mathcal{C}$ are called *convex sets*. For general convexities see the book of van de Vel [11]. If we concentrate on graphs we can define the (geodesic) convexity in terms of intervals. Namely, a subset C of $V(G)$ is *(geodesic) convex* if $I(u, v) \subseteq C$ for all $u, v \in C$. Convexity can also be defined with respect to other path properties, see [3] and references there in. In this work we concentrate only on (geodesic) convexity and we omit the term geodesic from now on. Note that if $C \subseteq V(G)$ induce a complete graph in G or if $C = V(G)$, then C is convex for any graph G. We call such sets *trivial* convex sets. Let A be a subset of $V(G)$. The convex hull $\text{ch}(A)$ is the smallest convex set that contains A. Clearly $\text{ch}(A) = A$ if and only if A is convex. For $v \in V(G)$ let C be a maximal convex set with respect to inclusion and with the property that $v \notin C$. We call C a *copoint* of v and v is an *attaching point* of C. The set of all attaching points of C is denoted by $\text{Att}(C)$.

We now define the *pre-hull operator* $\ell : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ for a connected graph G with

$$\ell(A) = \bigcup_{u,v \in A} I(u, v).$$

for every A in a power set operator $\mathcal{P}(V(G))$. Clearly $\ell(A) = A$ if and only if A is a convex set. Thus ℓ is more interesting for nonconvex sets and we can measure with him "how far" is A from being a convex set in G. For this observe that we can express the convex hull of A with

$$\mathrm{ch}(A) = \bigcup_{n \in \mathbb{N}} \ell^n(A),$$

where $\ell^n(A)$ is defined inductively $\ell^n(A) = \ell(\ell^{n-1}(A))$. Let v be an arbitrary vertex of G and let C be any convex set in G. Then $\ell^n(C \cup \{v\})$ must be convex for some $n \in \mathbb{N}_0$ since we deal only with finite graphs. Denote by $r(v; C)$ the smallest such number. In particular $\ell^{r(v;C)}(C \cup \{v\}) = \ell^{r(v;C)+1}(C \cup \{v\})$. Note that $r(v; C) = 0$ if $v \in C$ or $v \notin C$ and $C \cup \{v\}$ is a convex set already. The *convex pre-hull number* of a convex set C is then

$$\mathrm{cph}(G; C) = \max\{r(v; C) : v \in V(G)\}$$

and the *convex pre-hull number* of a graph G is

$$\mathrm{cph}(G) = \max\{\mathrm{cph}(G; C) : C \text{ is convex in G}\}.$$

Polat and Sabidussi have an additional restriction in [9] where they defined the pre-hull number only for copoints C. We will use the term copoint pre-hull number for this. More precisely, let G be a connected graph on at least two vertices and let C be a copoint in G. The *copoint pre-hull number* of C is

$$\mathrm{ph}(G; C) = \max\{r(v; C) : v \in \mathrm{Att}(C)\}$$

and the *copoint pre-hull* number of a graph G is

$$\mathrm{ph}(G) = \max\{\mathrm{ph}(G; C) : C \text{ is a copoint of } G\}.$$

In addition note that we can use maximums in the definition since we are interested only in finite graphs. In case of infinite graphs one must replace maximums with supremums.

The obvious bounds are $0 \leq \mathrm{ph}(G) \leq \mathrm{cph}(G) \leq |V(G)| - 2$. Indeed, the first inequality is a direct consequence of the definition, the second is due to the fact that copoints are convex sets as well, and the third inequality must hold if C is a singleton and at each step of the pre-hull operator we add exactly one vertex. Also the middle inequality can be strict. Namely, it is easy to see that $\mathrm{cph}(T) = 1 > 0 = \mathrm{ph}(T)$ for every tree T on at least 3 vertices. This already asserts that the above upper bound is not very accurate.

2 The Cartesian product

The *Cartesian product* $G \Box H$ of graphs G and H has $V(G \Box H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if $g = g'$ and $hh' \in$

$E(H)$, or $gg' \in E(G)$ and $h = h'$. For $h' \in V(H)$ is $G^{h'} = \{(g,h') \in V(G\Box H) : g \in V(G)\}$ a *G-fiber* in $G\Box H$ and for $g' \in V(G)$ is ${}^{g'}H = \{(g',h) \in V(G\Box H) : h \in V(H)\}$ an *H-fiber*. Note that subgraphs of $G\Box H$ induced by $G^{h'}$ and ${}^{g'}H$ are isomorphic to G and H, respectively. A map $p_G : V(G\Box H) \to V(G)$, defined by $p_G((g,h)) = g$ is called the *projection* of $G\Box H$ to the first factor G. Similarly we define the projection p_H to the second factor H. Projections p_G and p_H can be in a natural way extended from maps on vertices to maps p'_G and p'_H, respectively, between graph $G\Box H$ and G and H, respectively. The distance formula is

$$d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h').$$

Convex sets behave very natural in $G\Box H$ with respect to both factors as can be seen from the next well-known result from [6].

Theorem 2.1 *Let G and H be connected graphs. A subset $C \subseteq V(G\Box H)$ is convex if and only if $C = C_G \times C_H$ where C_G and C_H are convex subsets of $V(G)$ and $V(H)$, respectively.*

Similar holds also for intervals

$$I_{G\Box H}((g,h),(g',h')) = I_G(g,g') \times I_H(h,h'), \tag{1}$$

and the next technical lemma is no surprise.

Lemma 2.2 *Let G and H be connected graphs. If A is a subset of $V(G\Box H)$, then*

$$\ell_{G\Box H}(A) = \ell_G(p_G(A)) \times \ell_H(p_H(A)).$$

Proof. If $(g,h) \in \ell_{G\Box H}(A)$, then $(g,h) \in I_{G\Box H}((g',h'),(g'',h''))$ for some (g',h'), $(g'',h'') \in A$. By (1) we have $g \in I_G(g',g'')$ and $h \in I_H(h',h'')$. Thus $g \in \ell_G(p_G(A))$, $h \in \ell_H(p_H(A))$, and $(g,h) \in \ell_G(p_G(A)) \times \ell_H(p_H(A))$.

Conversely, if $(g,h) \in \ell_G(p_G(A)) \times \ell_H(p_H(A))$, then $g \in I_G(g_1,g_2)$ and $h \in I_H(h_1,h_2)$ for some vertices $g_1, g_2 \in p_G(A)$ and $h_1, h_2 \in p_H(A)$. Hence $(g_1,u_1), (g_1,u_2), (v_1,h_1)$, $(v_2,h_2) \in A$ for some $u_1, u_2 \in V(H)$ and $v_1, v_2 \in V(G)$. But then, again by (1), (g,h) is in at least one of $I_{G\Box H}((g_i,u_i),(v_j,h_j))$ for $i,j \in \{1,2\}$ and thus $(g,h) \in \ell_{G\Box H}(A)$. □

With this result we can derive both copoint and convex pre-hull numbers of the Cartesian product with respect to their copoint and convex pre-hull numbers of their factors.

Theorem 2.3 *If G and H are connected graphs with at least two vertices, then*

$$\begin{aligned} \mathrm{ph}(G\Box H) &= \max\{1, \mathrm{ph}(G), \mathrm{ph}(H)\} \text{ and} \\ \mathrm{cph}(G\Box H) &= \max\{1, \mathrm{cph}(G), \mathrm{cph}(H)\}. \end{aligned}$$

Proof. First note that a copoint C of a vertex (g,h) is either of the form $C = C_g \times V(H)$ or $C = V(G) \times C_h$ for some copoint C_g of g in G or some copoint C_h of h in H, respectively. Indeed, by Theorem 2.1 a copoint is a subproduct and if one factor is not the whole vertex set and the other not a copoint, then we violate the maximality of a copoint.

Graphs G and H have more than one vertex and $\{(g,h)\} \cup C$ is not convex by Theorem 2.1 for any vertex $(g,h) \in V(G\Box H)$ and its copoint C. Hence $\mathrm{ph}(G\Box H) \geq 1$ and we may assume without lost of generality that $\mathrm{ph}(G) \geq \mathrm{ph}(H) \geq 1$. Suppose that $\mathrm{ph}(G)$ is achieved by vertex g and its copoint C_g and let U_i, for $i \in \{1, \dots, \mathrm{ph}(G)\}$, be subsets of $V(G)$ that are added at the i-th step of the pre-hull operator for g and C_g. Thus $C = C_g \times V(H)$ is a copoint of (g,h) in $G\Box H$ for any $h \in V(H)$. In the first step of the pre-hull operator for (g,h) and C in $G\Box H$ all vertices in $\bigcup_{v_j \in U_1} {}^{v_j}H \cup ({}^{g}H - \{(g,h)\})$ are added to $\{(g,h)\} \cup C$. In i-th step, $i \geq 2$, all vertices in $\bigcup_{v_j \in U_i} {}^{v_j}H$ are added by Lemma 2.2. Hence the set $\mathrm{ch}(\{g\} \cup C_g) \times V(H)$ generated by the pre-hull operator for $(g,h) \cup C$ is a convex set and we have $\mathrm{ph}(G\Box H) \geq \max\{1, \mathrm{ph}(G), \mathrm{ph}(H)\}$.

Conversely, let C be any copoint in $G\Box H$ and (g,h) its arbitrary attaching vertex. Either $C = C_g \times V(H)$ or $C = V(G) \times C_h$, where C_g and C_h are copoints of g in G and h in H, respectively. By the description of the pre-hull operator from the previous paragraph and by Lemma 2.2 again, we have $\ell^{r(x;C)}((g,h) \cup C) = \ell^{r(x;C)+1}((g,h) \cup C)$ for $x \in \{g,h\}$, whenever $r(x;C) > 0$. If $r(x;C) = 0$, we have $\ell((g,h) \cup C) = \ell^2((g,h) \cup C)$ for $x \in \{g,h\}$. Since $r(h;C) \leq \mathrm{ph}(H)$ and $r(g;C) \leq \mathrm{ph}(G)$ for any g,h, and C, we have $\mathrm{ph}(G\Box H) \leq \max\{1, \mathrm{ph}(G), \mathrm{ph}(H)\}$ and the equality holds.

For the convex pre-hull number we have immediately $\mathrm{cph}(G\Box H) \geq \mathrm{ph}(G\Box H) \geq 1$. Since the Cartesian product is commutative, we may assume that $\mathrm{cph}(G) \geq \mathrm{cph}(H)$. Observe a vertex g and a convex set C'_g for which $\mathrm{cph}(G)$ is achieved. For a convex set $C = C'_g \times V(H)$ and a vertex (g,h), $h \in V(H)$, we have the same construction by the pre-hull operator as for the copoint pre-hull number. Thus $\mathrm{cph}(G\Box H) \geq \mathrm{cph}(G) \geq \mathrm{cph}(H)$ and together $\mathrm{cph}(G\Box H) \geq \max\{1, \mathrm{cph}(G), \mathrm{cph}(H)\}$. For the other inequality, let C be any convex set of $G\Box H$ and (g,h) an arbitrary vertex. We may assume that $r(g; p_G(C)) \geq r(h; p_H(C))$. But then we have $\ell^{r(g;p_G(C))}((g,h) \cup C) = \ell^{r(g;p_G(C))+1}((g,h) \cup C)$ by Lemma 2.2 again whenever $r(g; p_G(C)) > 0$. If $r(g; p_G(C)) = 0$, we have $\ell((g,h) \cup C) = \ell^2((g,h) \cup C)$. Since $r(g; p_G(C)) \leq \mathrm{cph}(G)$ for any g and any C, we have $\mathrm{cph}(G\Box H) \leq \max\{1, \mathrm{cph}(G), \mathrm{cph}(H)\}$ and the proof is complete. □

3 The strong product

The *strong product* $G \boxtimes H$ of graphs G and H has $G \Box H$ as a spanning subgraph. All edges of $G \Box H$ in $G \boxtimes H$ are called *Cartesian edges.* In addition $(g,h)(g',h')$ is also an edge in $G \boxtimes H$ if $gg' \in E(G)$ and $hh' \in E(H)$ and is called a *non Cartesian edge.* Fibers and projections are defined similarly as in the case of the Cartesian product. It is not hard to see that intervals of $G \boxtimes H$ do not form subproducts of intervals of the factors in contrast to the Cartesian product. Consequently, the convex sets in strong product are not subproducts. In general, not more than the distance formula

$$d_{G \boxtimes H}((g,h),(g',h')) = \max\{d_G(g,g'), d_H(h,h')\}$$

was known about metric properties of the strong product. Recently this starts to change rapidly. The hull number is discussed in [2] and [10], geodetic number and behavior of boundary sets also in [2], and intervals and convex sets have been characterized in [8]. We recall the latest result, since it is important for our discussion.

For this we need some local properties of the strong product. First we need a well known relaxation of convexity, namely 2-convexity. A subset $C \subseteq V(G)$ of a graph G is *2-convex* if $I(g,g') \subseteq C$ for any $g, g' \in C$ with $d_G(g,g') = 2$. Clearly every convex set is also 2-convex. It is clear by the distance formula that for a fixed vertex $h \in V(H)$ the set $\{(g,h) : g \in I_G(g',g'')\}$ must be in a convex set $C \subseteq G \boxtimes H$ for every pair of vertices (g',h), $(g'',h) \in C$. By comutativity of the strong product also the set $\{(g,h) : h \in I_G(h',h'')\}$, where $g \in V(G)$ is a fixed vertex, must again be in C for any pair of vertices (g,h'), $(g,h'') \in C$. We say that a *fiber condition* for $G \boxtimes H$ is fulfilled if this holds for every pair of vertices of C that belong to the same (G- or H-) fiber. Again, by the distance formula, we see that $I_{G \boxtimes H}((g,h),(g',h'))$ must be in a convex set C for every pair of vertices $(g,h),(g',h') \in C$ with additional property $d_G(g,g') = d_H(h,h')$. This is called a *diagonal property.* Finally recall that we denote with p'_G and p'_H the projection maps between graphs $G \boxtimes H$ and G and H, respectively. The following characterization is from [8].

Theorem 3.1 *Let C be a subset of $V(G \boxtimes H)$ for graphs G and H on at least two vertices. Then C is convex if and only if the following conditions hold:* *[(i)]*

(i) C is 2-convex,
(ii) the fiber condition hold,
(iii) the diagonal property is fulfilled, and
(iv) both $p'_G(C)$ and $p'_H(C)$ are convex in G and H, respectively.

Unfortunately this characterization does not help to describe the copoints of the strong product. Thus we can not derive the exact values of the copoint pre-hull number. Also it is not very helpful for deriving the convex pre-hull number with the respect to the convex pre-hull number of the factors either. Namely, if we try to have a similar construction as in the Cartesian product, we would obtain a convex set $C_G \subseteq V(G)$ in a certain G-fiber. But in general we cannot observe some box $C_G \times C_H$, for $C_H \subseteq V(H)$, since this box is usually not convex. If we insist to have C_G in G^h as a subset of some convex set $C \subseteq V(G \boxtimes H)$, we must have, by 2-convexity of C, also all vertices of the form (g', h') where h' is any neighbor of h in H and $g' \in C_G$ has two nonadjacent neighbors in C_G. On the other hand this observation is helpful for many classes of graphs since every vertex of an induced cycle C_k, $k \geq 4$, has such a property. Hence if we choose a convex set and a vertex in a right way we will have $V(C_k) \times V(H)$ as a subset of the pre-hull operator and this may yield many steps of that operator. This observation can be used to show the next theorem.

Theorem 3.2 *If G and H are connected graphs, then convex and copoint pre-hull numbers of $G \boxtimes H$ are not bounded from above by any function of convex and copoint pre-hull numbers of their factors, respectively.*

Proof. We need to construct an example of the strong product where convex and copoint pre-hull numbers have fix numbers, but their strong product has unbounded convex and copoint pre-hull number. For this we observe a product $P_n \boxtimes C_4$. Clearly, $\text{cph}(C_4) = \text{ph}(C_4) = \text{cph}(P_n) = 1$ and $\text{ph}(P_n) = 0$ for $n \geq 3$. For the convex pre-hull number let $C = \{u\}$ where u is any vertex of the C_4-fiber with respect to the end vertex of P_n and let v be the antipodal vertex of u in ${}^uC_4$. In the next scheme the positions present the vertices of $P_n \boxtimes C_4$ and the numbers at each position present the smallest number k for which that vertex is in $\ell^k(\{v\} \cup C)$:

$$\begin{array}{ccccccc} 0 & 2 & 2 & 4 & 4 & 6 & \ldots \\ 1 & 1 & 3 & 3 & 5 & 5 & \ldots \\ 0 & 2 & 2 & 4 & 4 & 6 & \ldots \\ 1 & 1 & 3 & 3 & 5 & 5 & \ldots \end{array}.$$

From the pattern it is easy to see that $n = r(v; C) \leq \text{cph}(P_n \boxtimes C_4)$.

For the copoint pre-hull number we use the same product $P_n \boxtimes C_4$, we only need to take a copoint for C. For this note that in a copoint of $P_n \boxtimes C_4$ there can not be three or more vertices of the same P_n- or C_4-fiber. Indeed, if a copoint C would have three vertices of some C_4-fiber, then this whole fiber would be in C by the fiber condition. But this yields by 2-convexity that $C = V(P_n \boxtimes C_4)$, contrary to the definition of a copoint. If some three vertices of P_n-fiber are in C, then by the fiber condition there are also three consecutive vertices of that fiber in C, say $(g,h),(g',h)$, and (g'',h). By convexity of C also (g',h') and (g',h'') are in C, where h' and h'' are neighbors of h in C_4. Thus we have again three vertices of the same C_4-fiber in C which is impossible. Thus vertices of every K_4 form a copoint of some vertex. Let C be a set of four vertices of $P_n \boxtimes C_4$ in "the corner" that induce K_4 and let v be any other vertex (out of two remaining) of the first C_4-fiber. Again we present similar scheme as above

$$\begin{array}{ccccccc} 0 & 2 & 2 & 4 & 4 & 6 & \ldots \\ 1 & 1 & 3 & 3 & 5 & 5 & \ldots \\ 0 & 0 & 2 & 4 & 4 & 6 & \ldots \\ 0 & 0 & 3 & 3 & 5 & 5 & \ldots \end{array}.$$

It is clear that $n = r(v;C) \leq \mathrm{ph}(P_n \boxtimes C_4)$ and the proof is complete.□

One could try to prove the general version of the example used in the above theorem, namely $P_n \boxtimes C_k$. However there is no pattern even for $4 \leq k \leq 10$. If we observe a vertex u in the C_k-fiber with respect to the vertex of degree 1 in P_n and take for the convex set its antipodal vertices of C_k, we obtain the following lover bounds

$$\mathrm{cph}(P_n \boxtimes C_k) \geq \left\{ \begin{array}{c|c} n & k=4,5 \\ n-1 & k=6,7 \\ \lceil \frac{n+1}{2} \rceil & k=8,9 \\ \lfloor \frac{n}{2} \rfloor & k=10 \text{ and } n>9 \end{array} \right. .$$

The proof is technical and can be done in a similar fashion as for $k=4$ in the above proof and is left to the reader.

We end the discussion with the exact result for the copoint pre-hull number for the strong product of trees.

Theorem 3.3 *If T_1 and T_2 are trees on more then two vertices, then* $\mathrm{ph}(T_1 \boxtimes T_2) = 1$.

Proof. We first describe the copoints of $T_1 \boxtimes T_2$ where T_1 and T_2 are trees each on at least two vertices. Let C be a copoint of $T_1 \boxtimes T_2$ and $(g,h) \in V(T_1 \boxtimes T_2)$ its attaching point. We denote with $T_1^1, \ldots, T_1^{k_1}$ the components of $T_1 - g$ and with $T_2^1, \ldots, T_2^{k_2}$ the components of $T_2 - h$. Note that only vertices of one of T_1^i, for $i \in \{1, \ldots, k_1\}$, can be in $C \cap T_1^h$, otherwise $(g,h) \in C$ by the fiber condition of Theorem 3.1 which is impossible. Similar, only vertices of one of T_2^i, for $i \in \{1, \ldots, k_2\}$, are in $C \cap {}^gT_2$. Say that T_1^1 and T_2^1 are these components. Let g' and h' be neighbors of g and h in T_1^1 and T_2^1, respectively. But then C consists of all vertices of $T_1^1 \times T_2^1$, as well as of every vertex (a,b) where $a \in V(T_1^1)$,

$b \in V(T_2) - V(T_2^1)$ with $d_{T_1}(a, g') = d_{T_2}(b, h)$ and of every vertex (c, d) where $c \in V(T_1) - V(T_1^1)$, $d \in V(T_2^2)$ with $d_{T_1}(c, g) = d_{T_2}(d, h')$ together with all (possible) vertices that are in $I_{T_1 \boxtimes T_2}((a, b), (a, h))$ in ${}^aT_2$ and in $I_{T_1 \boxtimes T_2}((c, d), (g, d))$ in T_1^d. In particular note that (g', h) and (g, h') are in C. It is not hard to see that the conditions of Theorem 3.1 are fulfilled for C and C is convex. (For instance the diagonal property holds since all vertices of the type (a, b) and (c, d) are included, and fiber condition is fulfilled since the interval in each fiber is added.) Suppose that there is an additional vertex (u, v) in C. Then (u, v) is a neighbor of some vertex of the type (a, b) where $b = v$ or by fiber condition there exists such a neighbor (u', v) in C or symmetric (u, v) is a neighbor of some vertex of the type (c, d) where $c = u$ or by fiber condition there exists such a neighbor (u, v') in C. By 2-convexity of C every vertex of type (a, b) and of type (c, d) has such a neighbor. Since (g', h) is of type (a, b) we see that (g, h) is also in C which is impossible.

With this it is easy to see that in $\ell(\{(g, h) \cup C\}$ for any vertex of the type $(a, b) \neq (g', h)$ a vertex $(a', b) \notin C$ where $aa' \in E(T_1)$ is added and for any vertex of the type $(c, d) \neq (g, h')$ a vertex $(c, d') \notin C$ where $dd' \in E(T_2)$ is added. This is again convex by Theorem 3.1 and, since (g, h) can be chosen so that $\deg_{T_1}(g) = 1$ and $\deg_{T_2}(h) > 1$, we have the desired equality $\mathrm{ph}(T_1 \boxtimes T_2) = 1$. □

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