DISTRIBUTIONS FOR TRANSFORMATIONS OF DERANGEMENTS

AMY MIHNEA AND FREDERICK HOFFMAN

ABSTRACT. In this paper we present some patterns related to derangements. We find the distribution of the δ'-transformation applied to all unicyclic derangements of order n, and the distribution of the δ'-transformation applied to all derangements of order n, considered in one-line notation. We introduce the notion of matrix of forbidden pairs that helps us in solving our problems. We also give and prove a theorem related to derangements.

1. INTRODUCTION

Let d_n denote the number of derangements of order n; i.e., permutations π of the set $\{1, 2, \ldots, n\}$, such that $\pi(i) \neq i$ for all i. It is well known that:

$$d_n = \sum_{i=0}^{n}(-1)^i \binom{n}{i}(n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!},$$

$$d_n = nd_{n-1} + (-1)^n \text{ for } n \geq 2,$$

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \text{ for } n > 2.$$

The problem of counting derangements is also known as *The Hatcheck Problem*: How many ways can a hatcheck girl hand back the n hats of n gentlemen, 1 to each gentleman, with no man getting his own hat? We can tell from the language of The Hatcheck Problem that this is a very old problem, but it is still interesting [3].

In this paper we find the distribution of the δ'-transformation applied to all unicyclic derangements of order n, and the distribution of the δ'-transformation applied to all derangements of order n, considered in one-line notation. By a unicyclic derangement of order n is meant a derangement of order n with a single cycle of length n. In using the transformation mentioned here, we were inspired by the use of the δ-transformation in data compression [1]. We also observed that the unicyclic derangements of order n are the unique outputs of the Burrows-Wheeler Transform [2] for all permutations of order n given as inputs.

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2. The δ'-Transformation for All Unicyclic Derangements of Order n

Given a string $Y = (Y[1], Y[2], Y[3], \dots, Y[n])$, we define its δ-transformation as the difference string $\delta Y = (Y[1], Y[2]-Y[1], Y[3]-Y[2], \dots, Y[n]-Y[n-1])$ and its δ'-transformation as the difference string $\delta' Y = (Y[2]-Y[1], Y[3]-Y[2], \dots, Y[n]-Y[n-1], Y[1]-Y[n])$. We consider a collection C of pairs of elements from the set $\{1, 2, \dots, n\}$, which can contain the same pair more than one time. We define *the matrix of forbidden pairs* for C, which is an n x n matrix P where $P[i,j]$ represents the frequency of the pair (i,j) in the collection C, i and j from 1 to n.

First we present some theoretical aspects, some of which will help us in finding the distribution of the δ'-transformation for all unicyclic derangements of order n. We denote this distribution by D'. We consider derangements represented in one-line notation (not in cycle notation).

Let $d = (d_1, d_2, \dots, d_n)$ be a unicyclic derangement of order n. Next, when we talk about (d_i, d_{i+1}), we consider i and $i+1$ modulo n (e.g., $d_{n+1} = d_1$), to include the pair (d_n, d_1). For each pair (d_i, d_{i+1}), we can't have $d_i = i$ or $d_{i+1} = i+1$. Also, we can't have $(d_i, d_{i+1}) = (i+1, i)$ because this derangement would no longer have a single cycle of order n.

Let $s = (s_1, s_2, \dots, s_n)$ be the difference string resulting from applying the δ'-transformation to d. We want to find the distribution of all $s_i = d_{i+1} - d_i$, i taking values from 1 to n, for all unicyclic derangements d of order n.

Let M be a matrix whose rows are the $(n-1)!$ unicyclic derangements of order n, written in one-line notation. This matrix can be generated, for example, by taking as rows the unique outputs of the Burrows-Wheeler Transform, presented in [1] or [2], for all permutations of order n given as inputs. From the cycle representation we can easily see that there are $(n-1)!$ such derangements. We consider that all the cyclic representations start with 1. The order of the rows in M does not matter, so M can be generated in different ways.

The words *is followed* here have a cyclic meaning (the last element of the cycle is followed by the first element). In the cycle representation of all such derangements, 1 is always in the first position and each value $k \neq 1$ appears equally often in the second position. So 1 is followed by a fixed $k \neq 1$ $(n-2)!$ times. The elements that follow 1 in this representation will be in the first column/entry of the matrix M.

For any fixed $k \neq 1$ there are $(n-2)!$ rows of M with first entry k. This is because each value of k appears equally often. The statement is true for all positions c from 1 to n; i.e., all the values $k \neq c$ appear an equal number of times, $(n-2)!$, in the c position (in column c), in the matrix M.

To prove this for a fixed $c \neq 1$, we consider the pair (c, k), $c \neq k$ in all the cyclic representations (starting with 1) of our derangements. If $k = 1$, we have c in the last position, n, in these cyclic representations, and the number of times this can happen is $(n-2)!$. If $k \neq 1$, this pair can start at positions $2, 3, \ldots, n-1$. But for each of these $n-2$ positions, there are $(n-3)!$ derangements that contain the fixed pair (c, k) at that fixed position, so (c, k) appears $(n-3)!(n-2) = (n-2)!$ times in the cyclic representations (starting with 1) of all unicyclic derangements of order n. This means that $k \neq c$ appears an equal number of times, $(n-2)!$, in the c position (column c), in the matrix M.

We consider each pair of adjacent columns (m_i, m_{i+1}) of M, $1 \leq i \leq n$ where subscripts are as above taken modulo n. For each of the n pairs of adjacent columns, consider the n x n matrix P_i of forbidden pairs for the pairs in (m_i, m_{i+1}). When we construct P_i we use the word *clear* to mean, "set the entry equal to zero." In this matrix we clear the diagonal elements, $\{(i, i)\}$. Then we clear row i, column $i+1$ and the element at position $(i+1, i)$. We set the remaining elements $P_i[row, col]$ equal to the frequency of (row, col) in the corresponding pair of adjacent columns (m_i, m_{i+1}) of M.

For each i' and j' from 1 to n, $P_i[i', j'] = 0$ means that the pair (i', j') cannot appear in the pair of adjacent columns (m_i, m_{i+1}) of M and $P_i[i', j'] \neq 0$ means that we can have $m_i = i'$ and $m_{i+1} = j'$. If $P_i[i', j'] \neq 0$, we call (i', j') a *legal* pair, or a pair that *qualifies.*

Our problem is reduced to finding, for all n matrices P_i, how many times each possible $j' - i'$ appears. Each pair (i', j') that qualifies appears in (m_i, m_{i+1}) equally often; i.e., $(n-1)!/(n^2 - n - (n-1) - (n-2) - 1) = (n-1)!/(n^2 - 3n + 2) = (n-1)!/((n-2)(n-1)) = (n-3)!$ times. A generalization of this fact is given in the next section as a theorem. In the previous expression, the denominator was obtained by considering the total number of elements of the n x n matrix P_i and subtracting the number of elements resulted from clearing the main diagonal, row i, column $i+1$ and the element at position $(i+1, i)$.

First, we will find the distribution vector of $j' - i'$ for the unique pairs (i', j') that appear in (m_i, m_{i+1}), i from 1 to n, denoted by D'_0, and then we will multiply this by $(n-3)!$, to get $D' = D'_0$ x $(n-3)!$. So in this section, from now on, the matrices of forbidden pairs P_i will have $P_i[i', j'] = 1$ instead of $P_i[i', j'] = (n-3)!$, for each pair (i', j') that can appear in (m_i, m_{i+1}).

To simplify the problem of finding D'_0, we consider n steps, each step i corresponding to an adjacent pair of columns (m_i, m_{i+1}), i from 1 to n. In each step i, to construct P_i, we start with an initial matrix of all possible pairs from which we eliminate the pairs that do not qualify. For this initial matrix, the distribution vector of all possible $j' - i'$ (i.e. $col - row$) is:

s_i	-(n-1)	-(n-2)	...	-2	-1	1	2	...	n-2	n-1
Frequency	1	2	...	n-2	n-1	n-1	n-2	...	2	1

We denote this basic distribution vector by Basic_DV. In other words, we found the distribution of all possible values $j' - i'$ when i' and j' take values from 1 to n, neglecting the zero values (when $i' = j'$). For an n x n matrix, in this distribution vector, each s_i appears $n - |s_i|$ times at each of the n steps. To obtain our initial distribution vector, because we have n steps, we multiply each number in the second row by n and denote the result by n x Basic_DV.

We consider that each matrix P_i has the main diagonal cleared. In this paragraph, the word *delete* refers to deleting from D'_0, which is the distribution vector of $j' - i'$ for the unique pairs (i', j') that appear in (m_i, m_{i+1}). We have to delete the elements that can't be in D'_0. When we clear row i from matrix P_i, it means that we delete the differences $(col - row)$: $1 - i, 2 - i, \ldots, (i-1) - i, (i+1) - i, \ldots, n - i$. When we clear column $i+1$ from matrix P_i, it means that, in addition, we delete the differences $(col - row)$: $(i+1) - 1, (i+1) - 2, \ldots, (i+1) - (i-1), (i+1) - (i+2), \ldots, (i+1) - n$. These are equal to $i, i-1, \ldots, 2, -1, -2, \ldots, i+1-n$. Then, when we clear the element at $(i+1, i)$ from matrix P_i, we delete the difference $i - (i+1) = -1$. So these are the elements that we delete at step i:

(1) $1 - i, 2 - i, \ldots, -1, 1, \ldots, n - i$
(2) $i, i-1, \ldots, 2, -1, -2, \ldots, i+1-n$
(3) -1

To find the distribution of the δ'-transformation for all unicyclic derangements of order n, we can use the following method. The first steps are a summarized version of the details that we gave before, for a clearer understanding of the method.

We denote by D' the distribution vector of the δ'-transformation of all unicyclic derangements of order n.

First, we will find the distribution vector of $j' - i'$ for the unique pairs (i', j') that appear in the pair of columns (m_i, m_{i+1}), i from 1 to n, denoted by D'_0, and then we will multiply this by $(n-3)!$. Our final distribution vector will be $D' = D'_0$ x $(n-3)!$. The goal is now to find D'_0. We start from the basic distribution vector Basic_DV multiplied by n, or n x Basic_DV, where Basic_DV is presented in the next table.

s_i	-(n-1)	-(n-2)	...	-2	-1	1	2	...	n-2	n-1
Frequency	1	2	...	n-2	n-1	n-1	n-2	...	2	1

Each pair of adjacent columns in M will have an associated matrix of forbidden pairs, and in total, we have n distinct pairs of adjacent columns.

The matrix of forbidden pairs for (m_1, m_2) will have the main diagonal, row 1, column 2 and the element at (2, 1) cleared.

...

The matrix of forbidden pairs for (m_i, m_{i+1}) will have the main diagonal, row i, column $i+1$ and the element at $(i+1, i)$ cleared.

...

The matrix of forbidden pairs for (m_n, m_1) will have the main diagonal, row n, column 1 and the element at $(1, n)$ cleared.

When we talk about a matrix of forbidden pairs, we totally neglect the main diagonal because we can't have pairs of equal elements in (m_i, m_{i+1}). Summarizing the results, we observe that from n x Basic_DV we have to subtract the distribution vector of the differences corresponding to two matrices (2 x Basic_DV) because our cleared elements span a matrix twice, by rows and by columns respectively. But when we cleared all rows i and columns $i+1$ by subtracting 2 x Basic_DV, we cleared their intersection (the element at $(i, i+1)$) twice. Therefore, we have to add back to D'_0 one copy of each of the differences corresponding to the elements at $(1,2), (2,3), \ldots, (n-1, n), (n, 1)$, i.e. $n-1$ of 1's and 1 of $-(n-1)$.

We also have to take into account the following cleared elements: (2,1), (3,2), ..., $(i+1, i), \ldots, (n, n-1), (1, n)$, which correspond to $n-1$ of (-1)'s and 1 of $n-1$ to be subtracted from D'_0. We get n x Basic_DV - 2 x Basic_DV = $(n-2)$ x Basic_DV, to which we add $n-1$ of 1's and 1 of $-(n-1)$ and from which we subtract $n-1$ of (-1)'s and 1 of $n-1$.

The sum of the elements in Basic_DV is $2(1+2+\ldots+(n-1)) = n(n-1)$. The sum of the elements in $(n-2)$ x Basic_DV is $n(n-1)(n-2)$. If we add n elements and subtract n elements, we still get $n(n-1)(n-2)$ elements, so the sum of the elements in the distribution vector D'_0 is $n(n-1)(n-2)$ and the sum of the elements in the distribution vector $D' = D'_0$ x $(n-3)!$ is $n(n-1)(n-2)((n-3)!) = n! = (n-1)!n$. So our distribution vector D' will have a sum of $(n-1)!n$, which we expect, taking into consideration that we have $(n-1)!$ unicyclic derangements of order n, and each such derangement has n elements.

Our final distribution vector D' is $(n-2)$ x Basic_DV x $(n-3)! = (n-2)!$ x Basic_DV, to which we add $(n-3)!(n-1)$ of 1's and $(n-3)!$ of $-(n-1)$'s and from which we subtract $(n-3)!(n-1)$ of (-1)'s and $(n-3)!$ of $(n-1)$'s.

The next table presents the distribution of the δ'-transformation for all unicyclic derangements of order n.

TABLE 1. The distribution of the δ'-transformation for all unicyclic derangements of order n

s_i	(n-2)x Basic_DV	Adjustments	(n-2)xBasic_DV + Adjustments	Our Final Distribution [(n-2)xBasic_DV + Adjustments] x (n-3)!
-(n-1)	1(n-2)	+1	1(n-2) + 1	(n-1)·(n-3)!
-(n-2)	2(n-2)		2(n-2)	[2(n-2)]·(n-3)!
...	...	...	...	...
-2	(n-2)(n-2)		(n-2)(n-2)	[(n-2)(n-2)]·(n-3)!
-1	(n-1)(n-2)	-(n-1)	(n-1)(n-2) - (n-1)	[(n-1)(n-3)]·(n-3)!
1	(n-1)(n-2)	+(n-1)	(n-1)(n-2) + (n-1)	[(n-1)(n-1)]·(n-3)!
2	(n-2)(n-2)		(n-2)(n-2)	[(n-2)(n-2)]·(n-3)!
...	...	...	...	...
n-2	2(n-2)		2(n-2)	[2(n-2)]·(n-3)!
n-1	1(n-2)	-1	1(n-2) - 1	(n-3)·(n-3)!

3. A THEOREM RELATED TO ALL UNICYCLIC DERANGEMENTS OF ORDER n

In the next theorem we prove that any pair (a, b), a and b taking values from 1 to n, with $a \neq b$, $a \neq i$, $b \neq j$ and $(a, b) \neq (j, i)$ appears in the pair of columns (m_i, m_j) of M (the matrix of all unicyclic derangements of order n, represented in one-line notation) with equal probability; that is $(n-3)!$ times.

We consider a pair of columns (m_i, m_j) with fixed i, j taking values from 1 to n, $i < j$ and fixed a, b taking values from 1 to n, with $a \neq b$, $a \neq i$, $b \neq j$ and $(a, b) \neq (j, i)$. In the proof of the theorem, when we refer to cycle representations we mean the cycle representations for all unicyclic derangements of order n. We consider that every cycle notation starts with 1. For an easier understanding we separate the proof into two cases: 1) $i = 1$ and 2) $i > 1$. We suppose $n > 2$.

Theorem. Any pair (a, b), a and b taking values from 1 to n $(n > 2)$, with $a \neq b$, $a \neq i$, $b \neq j$ and $(a, b) \neq (j, i)$ appears in the pair of columns (m_i, m_j) of the matrix M of all unicyclic derangements of order n, represented in one-line notation, with equal probability; that is, $(n-3)!$ times. There are $(n-1)!/(n-3)! = (n-1)(n-2)$ such distinct pairs (a, b), for each fixed i, j (i, j taking values from 1 to n, with $i \neq j$).

Proof.

Case 1: $i = 1$

In cycle representation, we have $(n-2)!$ possibilities to start with $(1, a)$.

i) If $a = j$, then a must be followed by b. This happens $(n-3)!$ times, so we proved that (a, b) appears in the pair of columns (m_1, m_j) $(n-3)!$ times.

ii) If $a \neq j$ and $b \neq 1$, we have to count how many pairs (j, b) we have in the cycle representations that start with $(1, a)$. (j, b) can start at positions $3, 4, \ldots, n-1$, so it appears $(n-4)!(n-3) = (n-3)!$ times.

iii) If $a \neq j$ and $b = 1$, we have to count how many times j appears in the last position, n, of the cycle representations that start with $(1, a)$. This happens $(n-3)!$ times.

Case 2: $i > 1$

Case 2.1: $a = 1$

In this case, i appears in the last position, n, of the cycle representations. This happens $(n-2)!$ times.

i) If $b = i$, then (j, b) starts at position $n-1$, and there are $(n-3)!$ ways to fill in the remaining $n-3$ positions.

ii) If $b \neq i$, we try to place (j, b) to start at one of positions $2, 3, \ldots, n-2$, so we have $(n-4)!(n-3) = (n-3)!$ possibilities.

Case 2.2: $a \neq 1$

In cycle notation, the pair (i, a) can start at positions $2, 3, \ldots, n-1$, so we have $(n-3)!(n-2) = (n-2)!$ possibilities that contain (i, a).

i) If $a = j$, then a must be followed by b. If $b \neq 1$ this happens $(n-4)!(n-3) = (n-3)!$ times, because the triplet (i, a, b) can start at positions $2, 3, \ldots, n-2$. If $b = 1$, the pair (i, a) has to start at position $n-1$, so we have $(n-3)!$ possibilities.

In this case, we must have $b \neq i$; otherwise, the cycle representation would contain the cycle $(a\ b)$, and this is not possible.

ii) If $a \neq j$ and $b \neq 1$, we consider two cases: $b = i$ and $b \neq i$.

If $b = i$, the triplet (j, b, a) can start at positions $2, 3, \ldots, n-2$, so we have $(n-4)!(n-3) = (n-3)!$ possibilities.

If $b \neq i$, we have to count how many cycle representations contain the non-overlapping pairs (i, a) and (j, b), where (i, a) can start at positions $2, \ldots, n-1$, and (j, b) can start at positions $2, \ldots, n-1$. We know that $i > 1$; so (i, a) can't start at position 1, and also $j > i > 1$, so $j \neq 1$; therefore, (j, b) can't start at position 1 (from our assumptions, position 1 of the cycle representations is filled with 1). In this case, we have $a \neq 1$ and $b \neq 1$; so, i and j can't be in the last position, n, of the cycle representations.

If (i, a) starts at position 2, there are $n-4$ possibilities to place (j, b).

If (i, a) starts at position 3, there are $n-5$ possibilities to place (j, b). Position 2 is blocked for (j, b), because position 3 is taken.

If (i, a) starts at position 4, there are $1 + (n-6) = n-5$ possibilities to place (j, b). Position 2 becomes available for (j, b).

If (i, a) starts at position 5, there are $2 + (n-7) = n-5$ possibilities to place (j, b).

...

If (i, a) starts at position $n-3$, there are $(n-6)+1 = n-5$ possibilities to place (j, b). Position $n-1$ is available for (j, b).

If (i, a) starts at position $n-2$, there are $n-5$ possibilities to place (j, b). Position $n-1$ is blocked for (j, b) because it is taken.

If (i, a) starts at position $n-1$, there are $n-4$ possibilities to place (j, b).

Summarizing the results, we have in total $(n-5)!(2(n-4)+(n-4)(n-5)) = (n-5)!(n-4)(n-3) = (n-3)!$ cycle representations that contain the non-overlapping pairs (i, a) and (j, b), where (i, a) can start at positions $2, \ldots, n-1$ and (j, b) can start at positions $2, \ldots, n-1$.

Thus, subcase $(b \neq i)$ is solved.

iii) If $a \neq j$ and $b = 1$, we have j in the last position, and (i, a) can appear in positions $2, 3, \ldots, n-2$; so, we have $(n-4)!(n-3) = (n-3)!$ possibilities.

Here we can't have $b = i$, because we are in case 2, for which $i > 1$.

The first statement of the theorem is now proved and it follows that the second statement is also true.

4. The δ'-Transformation for All Derangements of Order n

We generalize the results from the preceding sections by applying the δ'-transformation to all derangements of order n and finding patterns. Regarding the matrices of forbidden pairs, for an easier understanding of their construction, we will sometimes use the terms *black* for zero (cleared) elements and *white* for the elements that were not cleared yet.

We use the basic distribution vector for a matrix of forbidden pairs, Basic_DV. This corresponds to a matrix of forbidden pairs from which just the main diagonal is cleared, and for which the non-cleared cells are filled with 1. Basic_DV gives the distribution of all possible differences $col - row$ $(col \neq row)$ in an n x n matrix.

s_i	-(n-1)	-(n-2)	...	-2	-1	1	2	...	n-2	n-1
Frequency	1	2	...	n-2	n-1	n-1	n-2	...	2	1

We denote the number of derangements of order n by $d(n)$. We construct the n x n matrix of forbidden pairs corresponding to position (column) $k = 1$ in a matrix of all derangements of order n, of dimension $d(n)$ x n. The order of the rows in the matrix of all derangements does not matter. In the matrix of forbidden pairs, we clear the main diagonal, row 1 and column 2; the number of white cells we are left with is: $n^2 - n - (n-1) - (n-2) = n^2 - 3n + 3$.

Position (2,1) should be filled with $d(n-2)$, and we are left with $n^2 - 3n + 3 - 1 = n^2 - 3n + 2 = (n-2)(n-1)$ white cells. We fill the remaining $n-2$ white cells from row 2 and the remaining $n-2$ white cells from column 1 with $d(n-3) + d(n-2)$. We are left with $(n-2)(n-1) - 2(n-2) =$

$(n-3)(n-2)$ white cells, which we fill with $d(n-4)+2d(n-3)+d(n-2)$. These results can be generalized for any position k.

In the next table, it can be seen how a number from the third and the fourth column (for $n > 3$) is obtained by adding the number to its left and the one above this number. Next, when we refer to k or $k+1$ we refer to these numbers modulo n. We do this so that we take into account the case $k = n, k+1 = 1$.

n	Number in cell $(k+1, k)$	Number in the remaining white cells from row $k+1$ and from column k	Number in the remaining white cells
3	$d(1)=0$	1	
4	$d(2)=1$	$d(1)+d(2)=1$	2
5	$d(3)=2$	$d(2)+d(3)=3$	$d(1)+2d(2)+d(3)=4$
6	$d(4)=9$	$d(3)+d(4)=11$	$d(2)+2d(3)+d(4)=14$
7	$d(5)=44$	$d(4)+d(5)=53$	$d(3)+2d(4)+d(5)=64$
8	$d(6)=265$	$d(5)+d(6)=309$	$d(4)+2d(5)+d(6)=362$

The sum of the frequencies for the matrix of forbidden pairs corresponding to position k should equal $d(n)$.

$[1]d(n-2)+[2(n-2)][d(n-3)+d(n-2)]+[(n-3)(n-2)][d(n-4)+2d(n-3)+d(n-2)]$
$= d(n-2)+2(n-2)d(n-3)+2(n-2)d(n-2)+(n-3)(n-2)d(n-4)+2(n-3)(n-2)d(n-3)+(n-3)(n-2)d(n-2)$
$= [d(n-2)+2(n-2)d(n-2)+(n-3)(n-2)d(n-2)]+[2(n-2)d(n-3)+2(n-3)(n-2)d(n-3)]+(n-3)(n-2)d(n-4)$
$= [1+(n-2)(n-1)]d(n-2)+2(n-2)^2d(n-3)+(n-3)(n-2)d(n-4)$
$= d(n-2)+(n-2)(n-1)d(n-2)+2(n-2)^2d(n-3)+(n-3)(n-2)d(n-4)$
$= (n-3)[d(n-3)+d(n-4)]+(n-2)(n-1)d(n-2)+2(n-2)^2d(n-3)+(n-3)(n-2)d(n-4)$
$= (n-2)(n-1)d(n-2)+(n-1)(2n-5)d(n-3)+(n-3)(n-1)d(n-4)$

We know that

$d(n) = (n-1)[d(n-1)+d(n-2)]$
$= (n-1)d(n-1)+(n-1)d(n-2)$
$= (n-1)[(n-2)(d(n-2)+d(n-3))]+(n-1)[(n-3)(d(n-3)+d(n-4))]$
$= (n-2)(n-1)d(n-2)+(n-2)(n-1)d(n-3)+(n-3)(n-1)d(n-3)+(n-3)(n-1)d(n-4)$
$= (n-2)(n-1)d(n-2)+(n-1)(2n-5)d(n-3)+(n-3)(n-1)d(n-4)$

We observe that the sum of our frequencies is indeed $d(n)$.

For fixed n and different positions k's, the matrix of forbidden pairs contains the same numbers in the white cells, the same number of times, but they are placed differently in the matrix, according to the previous table. When k goes from 1 to n, the cell $(k+1, k)$ filled with $d(n-2)$ moves like this: $(2,1), (3,2), (4,3), \ldots, (n, n-1), (1, n)$. These give us $(n-1)d(n-2)$ of (-1)'s and $d(n-2)$ of $(n-1)$'s.

Also, for k from 1 to n, row $k+1$ and column k sweep an n x n matrix twice (except for the cells $(k+1, k)$ considered before and the forbidden pairs), so we start from $2[d(n-3)+d(n-2)]$ x Basic_DV, from which we subtract the differences corresponding to the elements $(k+1, k)$ twice: $2(n-1)[d(n-3)+d(n-2)]$ of (-1)'s and $2[d(n-3)+d(n-2)]$ of $(n-1)$'s. We subtract them twice, because when we consider the sweep by rows and the sweep by columns, the elements $(k+1, k)$ appear twice, being the intersection of row $k+1$ and column k. The strips for forbidden pairs (corresponding to row k, column $k+1$) intersect row $k+1$ and column k on the main diagonal, at positions (k, k) and $(k+1, k+1)$, which are cleared by default, so we neglect them. So, in this case, we subtract $2n[d(n-3)+d(n-2)]$ elements: $2(n-1)[d(n-3)+d(n-2)]$ of (-1)'s and $2[d(n-3)+d(n-2)]$ of $(n-1)$'s.

The cells that are left are filled with $d(n-4)+2d(n-3)+d(n-2)$. Suppose these cells are filled with 1. We could get the differences corresponding to them by considering n x Basic_DV from which we subtract 2 x Basic_DV (corresponding to the cleared row k and column $k+1$, giving us 2 sweeps of the matrix), and we subtract again 2 x Basic_DV (corresponding to row $k+1$ and column k considered previously, giving us other 2 sweeps of the matrix). When we did this we subtracted the differences corresponding to the intersection cells $(k, k+1)$ and $(k+1, k)$ twice, so we add back one copy of them; i.e., we add $n-1$ of (-1)'s and 1's and 1 of $-(n-1)$ and $(n-1)$. We get: $(n-4)$ x Basic_DV, to which we add $n-1$ of (-1)'s and 1's and 1 of $-(n-1)$ and $(n-1)$. Taking into account the numbers that fill the white cells we consider here, we get: $[d(n-4)+2d(n-3)+d(n-2)](n-4)$ x Basic_DV, to which we add $(n-1)[d(n-4)+2d(n-3)+d(n-2)]$ of (-1)'s and 1's and $[d(n-4)+2d(n-3)+d(n-2)]$ of $-(n-1)$'s and $(n-1)$'s.

Our resulting distribution will be: $2[d(n-3)+d(n-2)]$ x Basic_DV + $[d(n-4)+2d(n-3)+d(n-2)](n-4)$ x Basic_DV

(1) to which we add $n[d(n-2)]$ elements: $(n-1)d(n-2)$ of (-1)'s and $d(n-2)$ of $(n-1)$'s;
(2) from which we subtract $2n[d(n-3)+d(n-2)]$ elements: $2(n-1)[d(n-3)+d(n-2)]$ of (-1)'s and $2[d(n-3)+d(n-2)]$ of $(n-1)$'s;
(3) to which we add $2n[d(n-4)+2d(n-3)+d(n-2)]$ elements: $(n-1)[d(n-4)+2d(n-3)+d(n-2)]$ of (-1)'s and 1's and $d(n-4)+2d(n-3)+d(n-2)$ of $-(n-1)$'s and $(n-1)$'s.

We want to check if this distribution sums up to $n[d(n)]$.
The sum of the elements in Basic_DV is $2[1+2+\ldots+(n-1)] = n(n-1)$.
The sum of the elements in $2[d(n-3)+d(n-2)]$ x Basic_DV + $[d(n-4)+2d(n-3)+d(n-2)](n-4)$ x Basic_DV is $2n(n-1)[d(n-3)+d(n-2)]+n(n-1)(n-4)[d(n-4)+2d(n-3)+d(n-2)]$

(1) to which we have to add $n[d(n-2)]$ elements;

(2) from which we subtract $2n[d(n-3)+d(n-2)]$ elements;

(3) to which we have to add $2n[d(n-4)+2d(n-3)+d(n-2)]$ elements.

We get:

$2n(n-1)[d(n-3)+d(n-2)]+n(n-1)(n-4)[d(n-4)+2d(n-3)+d(n-2)]+n[d(n-2)]-2n[d(n-3)+d(n-2)]+2n[d(n-4)+2d(n-3)+d(n-2)]$

$=n\{2(n-1)[d(n-3)+d(n-2)]+(n-1)(n-4)[d(n-4)+2d(n-3)+d(n-2)]+d(n-2)-2[d(n-3)+d(n-2)]+2[d(n-4)+2d(n-3)+d(n-2)]\}$

$=n\{2(n-2)[d(n-3)+d(n-2)]+(n-1)(n-4)d(n-4)+(n-1)(n-4)2d(n-3)+(n-1)(n-4)d(n-2)+d(n-2)+2d(n-4)+4d(n-3)+2d(n-2)\}$

$=n\{2d(n-1)+[(n-1)(n-4)d(n-4)+2d(n-4)]+[2(n-1)(n-4)d(n-3)+4d(n-3)]+(n-1)(n-4)d(n-2)+3d(n-2)\}$

$=n\{2d(n-1)+[(n-3)(n-2)d(n-4)+(n-3)(n-2)d(n-3)]+(n-3)(n-2)d(n-3)+(n-1)(n-4)d(n-2)+3d(n-2)\}$

$=n\{2d(n-1)+(n-2)[(n-3)(d(n-3)+d(n-4))]+(n-3)(n-2)d(n-3)+(n-1)(n-4)d(n-2)+3d(n-2)\}$

$=n[2d(n-1)+(n-2)d(n-2)+(n-3)(n-2)d(n-3)+(n-1)(n-4)d(n-2)+3d(n-2)]$

$=n[2d(n-1)+(n-2)d(n-2)+(n-2)d(n-3)-(n-2)d(n-3)+(n-3)(n-2)d(n-3)+(n-1)(n-4)d(n-2)+3d(n-2)]$

$=n[2d(n-1)+d(n-1)+(n-4)(n-2)d(n-3)+(n-1)(n-4)d(n-2)+3d(n-2)]$

$=n[3d(n-1)+3d(n-2)+(n-4)(n-2)d(n-3)+(n-1)(n-4)d(n-2)]$

$=n[3d(n-1)+3d(n-2)+(n-4)(n-2)d(n-2)+(n-4)(n-2)d(n-3)-(n-4)(n-2)d(n-2)+(n-1)(n-4)d(n-2)]$

$=n[3d(n-1)+3d(n-2)+(n-4)d(n-1)-(n-4)(n-2)d(n-2)+(n-1)(n-4)d(n-2)]$

$=n[(n-1)d(n-1)+3d(n-2)-(n-4)(n-2)d(n-2)+(n-1)(n-4)d(n-2)]$

$=n[(n-1)d(n-1)+3d(n-2)+(n-4)d(n-2)]$

$=n[(n-1)d(n-1)+(n-1)d(n-2)]$

$=n[d(n)]$

We obtained the expected result.

The next table contains the distribution of the δ'-transformation applied to all derangements of order n. As we previously showed, the elements in this table add up to $n[d(n)]$.

For clarity, we rewrote the elements $2[d(n-3)+d(n-2)]$ x Basic_DV + $[d(n-4)+2d(n-3)+d(n-2)](n-4)$ x Basic_DV as $\{2[d(n-3)+d(n-2)]+[d(n-4)+2d(n-3)+d(n-2)](n-4)\}$ x Basic_DV = $[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]$ x Basic_DV.

TABLE 2. The distribution of the δ'-transformation for all derangements of order n

s,	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]$ x Basic_DV			
-(n-1)	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]1$			$+d(n-4)+2d(n-3)+d(n-2)$
-(n-2)	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]2$			
-(n-3)	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]3$			
...	...	...	...	...
-2	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)](n-2)$			
-1	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)](n-1)$	$+(n-1)d(n-2)$	$-2(n-1)[d(n-3)+d(n-2)]$	$+(n-1)[d(n-4)+2d(n-3)+d(n-2)]$
1	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)](n-1)$			$+(n-1)[d(n-4)+2d(n-3)+d(n-2)]$
2	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)](n-2)$			
...	...	...	...	...
n-3	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]3$			
n-2	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]2$			
n-1	$[(n-4)d(n-4)+2(n-3)d(n-3)+(n-2)d(n-2)]1$	$+d(n-2)$	$-2[d(n-3)+d(n-2)]$	$+d(n-4)+2d(n-3)+d(n-2)$

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