Edge Domination in Grids

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Abstract

It has been conjectured that the edge domination number of the $m \times n$ grid graph, denoted by $\gamma'(P_m \Box P_n)$, is $\lceil mn/3 \rceil$, when $m, n \geq 2$. Our main result gives support for this conjecture by proving that $\lceil mn/3 \rceil \leq \gamma'(P_m \Box P_n) \leq mn/3 + n/12 + 1$, when $m, n \geq 2$. We furthermore show that the conjecture holds when mn is a multiple of three and also when $m \leq 13$. Despite this support for the conjecture, our proofs lead us to believe that the conjecture may be false when m and n are large enough and mn is not a multiple of three. We state a new conjecture for the values of $\gamma'(P_m \Box P_n)$.

Keywords: edge dominating set, total dominating set, vertex cover.
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1 Introduction

Let $G = (V, E)$ be a graph with n vertices. In this paper, we will study edge dominating sets in grid graphs. Edge domination is the natural analog of vertex domination and has been studied in a number of papers, see for example [1, 2, 3, 4, 7, 9, 10, 11]. For an edge uv, we call u and v the *endvertices* of uv. An edge *dominates* itself and the edges adjacent to it (edges incident to its endvertices). Two edges are *independent* if they have no endvertices in common.

An *edge dominating set* of a graph is a set of edges $E' \subseteq E$ such that each edge in E is either in E' or shares an endvertex with some edge in E'. The *edge domination number* of a graph is the number of edges of a smallest edge dominating set, which we denote as $\gamma'(G)$. It is shown in [1] that if E' is a minimum edge dominating set of G, there is an independent edge dominating set of G of cardinality $|E'|$. In fact, that the edge domination number of a graph is equal to its independent edge domination number seems to have first been proved in [6].

Yannakakis and Gavril proved that the decision problem associated with edge domination is NP-complete, even in planar graphs or bipartite graphs of maximum degree three. Polynomial-time algorithms for finding a minimum edge dominating set have been found for bipartite permutation graphs and cotriangulated graphs [10], trees [8], claw-free chordal graphs, locally connected claw-free graphs, the line graphs of total graphs, and the line graphs of chordal graphs [7]. Approximation algorithms for edge domination and weighted edge domination have been considered in a number of papers, including [11] and [5]. For example, a 2-approximation algorithm for edge domination is given in [11]. Structural results were obtained in [1], in which the graphs G with $\gamma'(G) = n/2$ and $\gamma'(G) = n - \Delta'$ were characterized, where Δ' denotes the maximum degree of any edge in G. In addition, trees and unicyclic graph with $\gamma'(G) = \lfloor n/2 \rfloor$ were characterized in [1].

The $m \times n$ grid graph is the Cartesian product of P_m and P_n, denoted $P_m \Box P_n$. Previously, edge domination in Cartesian products of graph was explored by Cutler and Halsey [3]. In that paper, they consider the edge domination numbers of $G \Box K_n$.

Prior to this paper, the edge domination number of $P_m \Box P_n$ was unknown except when (i) $m = 1$, in which case $\gamma'(P_n) = \lceil (n-1)/3 \rceil$; (ii) $m \in \{2, 3\}$, in which case $\gamma'(P_m \Box P_n) = \lceil nm/3 \rceil$, see [4, 9]; and (iii) $m = 4$ and $n \not\equiv 1 (\text{mod } 3)$, in which case $\gamma'(P_m \Box P_n) = \lceil 4n/3 \rceil$, see [9]. It was further stated in [9] that $\gamma'(P_m \Box P_n) \leq \lceil mn/3 \rceil$, when $m \in \{5, 7\}$, but no proof is given (however, these two cases are not difficult to verify, which we do below).

The following is conjectured in [9].

Conjecture 1 *[9]* $\gamma'(P_m \Box P_n) = \lceil mn/3 \rceil$, *when* $m, n \geq 2$.

Motivated by this conjecture, in this paper we further study edge dominating sets in grids. Our main result is that $\gamma'(P_m \Box P_n) \geq \lceil mn/3 \rceil$, for all $m, n \geq 2$. This is proved in Section 2. In Section 2 we furthermore prove that if nm is a multiple of three then Conjecture 1 holds and every minimum edge dominating set in $P_m \Box P_n$ is independent. In Section 3, we

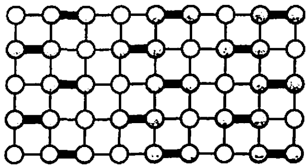


Figure 1: A minimal edge dominating set in $P_5\Box P_9$.

show that Conjecture 1 holds when $2 \leq n \leq 10$ (or $2 \leq m \leq 10$). In Section 4, we prove an upper bound on $\gamma'(P_m\Box P_n)$, show that Conjecture 1 holds in some additional cases, and conclude the paper with a new conjecture.

2 Proof of the Main Result

We begin by illustrating that $\lceil mn/3 \rceil$ is an upper bound on $\gamma'(P_m\Box P_n)$ when $n \equiv 0(\bmod 3)$. See Figure 1 for a solution to $P_5\Box P_9$. This result also appears in [9], but we include a proof for completeness, as it will be used in the proof of the lower bound.

Theorem 1 *[9] $\gamma'(P_m\Box P_n) \leq \lceil mn/3 \rceil$ when $n \equiv 0(\bmod 3)$.*

Proof: Select the second edge on the first row and every third edge thereafter (so the fifth, eighth and so on); select the first edge on the second row and every third edge thereafter; and then alternate that pattern row by row. It is not difficult to see that this produces an edge dominating set of size $nm/3$. □

In Theorem 2 below we give support for Conjecture 1, by proving that $\lceil nm/3 \rceil$ is a lower bound for $\gamma'(P_m\Box P_n)$ when $n, m \geq 2$. We prove this by introducing a specific edge-weighting that easily gives us a lower bound of $\lceil (nm-1)/3 \rceil$. However some extra work will be required to achieve the lower bound of $\lceil nm/3 \rceil$.

Theorem 2 *Let $m, n \geq 2$. Then $\gamma'(P_m\Box P_n) \geq \lceil nm/3 \rceil$.*

Furthermore, if nm is a multiple of three, then every minimum edge dominating set is independent.

Proof: Let $m, n \geq 2$ and let $G = P_m\Box P_n$. Let the vertex set and edge set of G be defined as follows.

$$\begin{aligned} V(G) &= \{v_{i,j} \mid i = 1,2,\dots,m \;\; j = 1,2,\dots,n\} \\ E(G) &= \{v_{i,j}v_{i,j+1} \mid i = 1,2,\dots,m \;\; j = 1,2,\dots,n-1\} \cup \\ &\quad\; \{v_{i,j}v_{i+1,j} \mid i = 1,2,\dots,m-1 \;\; j = 1,2,\dots,n\} \end{aligned}$$

In other words, $v_{i,j}$ is the vertex in row i and column j if we think of G as being laid out in m rows and n columns. Now assign a weight of one to all edges, except for the edges in the first row, last row, first column and last column which each get weight 3/2. Let this weight be denoted by the function w_e, which implies that the following holds.

$$w_e(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } 1 < i < m \\ \frac{3}{2} & \text{if } i = 1 \text{ or } i = m \end{cases}$$

$$w_e(v_{i,j}v_{i+1,j}) = \begin{cases} 1 & \text{if } 1 < j < n \\ \frac{3}{2} & \text{if } j = 1 \text{ or } j = n \end{cases}$$

Let D be a minimum edge dominating set in G. We will now define a weight $w_v(v_{i,j})$ for each vertex $v_{i,j} \in V(D)$ as follows.

$$w_v(v_{i,j}) = \left(\sum_{u \in N(v_{i,j}) \setminus V(D)} w_e(v_{i,j}u) \right) + \left(\sum_{u \in N(v_{i,j}) \cap V(D)} w_e(v_{i,j}u)/2 \right)$$

In other words, for every edge $xy \in E(G)$ we assign $w_e(xy)/2$ to each of x and y if $x, y \in V(D)$ and otherwise we assign $w_e(xy)$ to the vertex in $V(D) \cap \{x, y\}$.

For every edge $uv \in E(D)$, let $w^*(uv) = w_v(u) + w_v(v)$. We will now prove the following claims.

Claim A: $\sum_{v \in V(D)} w_v(v) = 2mn - 2$.

Proof of Claim A: Claim A follows from the calculation below. Note that the first equality below holds by double counting and the next two equalities follow from the definition of w.

$$\begin{aligned} \textstyle\sum_{v \in V(D)} w_v(v) &= \textstyle\sum_{xy \in E(G)} w(xy) \\ &= (m-2)(n-1) + (n-2)(m-1) + \\ &\quad\; \frac{3 \times 2(m-1)}{2} + \frac{3 \times 2(n-1)}{2} \\ &= 2mn - 2 \end{aligned}$$

($\square$)

Claim B: $w^*(e) \leq 6$ **for all** $e \in D$.

Proof of Claim B: The proof is by case analysis. For the first case, consider an edge $e = v_{i,j}v_{i,j+1} \in D$. If $i > 1$, then at least one of the vertices $v_{i-1,j}$ and $v_{i-1,j+1}$ belongs to $V(D)$, as the edge $v_{i-1,j}v_{i-1,j+1}$ needs to be dominated. Analogously, if $i < m$, then one of the vertices $v_{i+1,j}$ and $v_{i+1,j+1}$ belongs to $V(D)$. For the remainder of the proof, consider the following cases, which exhaust all other possibilities.

$i = 1$ **and** $1 < j < n-1$: In this case, the edges $v_{i,j}v_{i+1,j}$ and $v_{i,j+1}v_{i+1,j+1}$ add a total of at most 3/2 to $w^*(e)$. Furthermore, the edges $v_{i,j-1}v_{i,j}$, $v_{i,j}v_{i,j+1}$ and $v_{i,j+1}v_{i,j+2}$ add at most 3/2 each to $w^*(e)$. Therefore, $w_D^*(e) \leq 6$.

$i = 1$ **and** $j = 1$: In this case, the edges $v_{1,1}v_{2,1}$ and $v_{1,2}v_{2,2}$ add a total of at most 2 to $w^*(e)$. Furthermore, the edges $v_{1,1}v_{1,2}$ and $v_{1,2}v_{1,3}$ add at most 3/2 each to $w^*(e)$. Therefore, $w_D^*(e) \leq 5$.

$i = 1$ **and** $j = n$: Analogously to when $i = j = 1$ we get $w_D^*(e) \leq 5$.

$i = m$ **and** $1 \leq j \leq n$: This case is proved analogously to when $i = 1$ considering the three cases for j in turn.

$1 < i < m$ **and** $j = 1$ **or** $j = n$: This is proved analogously to when $i = 1$ and $1 < j < n$.

$1 < i < m$ **and** $1 < j < n$: In this case, the edges $v_{i,j}v_{i+1,j}$ and $v_{i,j+1}v_{i+1,j+1}$ together add a total of at most 3/2 to $w^*(e)$. Analogously, the edges $v_{i,j}v_{i-1,j}$ and $v_{i,j+1}v_{i-1,j+1}$ add a total of at most 3/2 to $w^*(e)$. Finally, the edges $v_{i-1,j}v_{i,j}$, $v_{i,j}v_{i,j+1}$ and $v_{i,j+1}v_{i,j+2}$ add at most 1 each to $w^*(e)$. Therefore, $w_D^*(e) \leq 6$.

This completes the proof of Claim B when $e = v_{i,j}v_{i,j+1} \in D$. The case when $e = v_{i,j}v_{i+1,j} \in D$ can be proved analogously. (□)

Claim C: If $xy, yz \in D$, **then** $w_v(x) + w_v(y) + w_v(z) < 10$.

Proof of Claim C: By Claim B, we note that $w_v(x) + w_v(y) \leq 6$. Let $z = v_{i,j}$ and consider the following cases. If $1 < i < m$ and $1 < j < n$, then $w_v(z) < 4$ as the edge yz adds 1/2 to $w_v(z)$, and each of the other three edges incident with z add at most one to $w_v(z)$. This implies that $w_v(x) + w_v(y) + w_v(z) \leq 9.5$ in this case. If z lies in one of the corners of our grid, then $w_v(z) \leq 3$ and we are done. So we may assume that $i = 1$ or $i = m$ and $1 < j < n$, or alternatively $j = 1$ or $j = n$ and $1 < i < m$. However, in this case, the three edges incident with z have weights 3/2, 3/2 and 1, respectively, and the edge yz only contributes with $w_e(yz)/2$ to $w_v(z)$, which implies that $w_v(z) \leq 3.5$. So again, $w_v(x) + w_v(y) + w_v(z) \leq 9.5$. This completes the proof of Claim C. (It is, in fact, possible to show that $w_v(x) + w_v(y) + w_v(z) \leq 9$, but this is not needed for our proof). (□)

Claim D: We may assume that D is an independent edge dominating set, as otherwise the theorem holds.

Proof of Claim D: Assume that D is not independent and let $xy, yz \in D$. By Claim C, we note that $w_v(x)+w_v(y)+w_v(z) < 10$. By Claim A and B, we note that the following holds.

$$6|D| - 2 = 10 + 6(|D| - 2) \geq 10 + \sum_{e \in (D \setminus \{xy, yz\})} w^*(e) >$$

$$\sum_{v \in V(D)} w_v(v) = 2mn - 2.$$

Therefore, $|D| > mn/3$. This proves that $|D| \geq \lceil nm/3 \rceil$. Furthermore, if either n or m is congruent to zero modulo three, then $|D| \leq mn/3$, by Theorem 1, contradicting the fact that $|D| > mn/3$. This implies that D is independent in this case. (□)

Claim E: We may assume that $n = m = 1$ or $n = m = 2$ modulo three, as otherwise the theorem holds.

Proof of Claim E: By Claim A and B, we note that

$$6|D| \geq \sum_{e \in D} w^*(e) \geq \sum_{v \in V(D)} w_v(v) = 2mn - 2.$$

Therefore, $|D| \geq (mn - 1)/3$. If $mn - 1$ is not divisible by three, then we must have $|D| \geq mn/3$ and therefore $|D| \geq \lceil mn/3 \rceil$, as $|D|$ is an integer. This would complete the proof in this case, so we may assume that $mn - 1$ is divisible by 3, which implies that $n = m = 1$ or $n = m = 2$ modulo three. (□)

Claim F: We may assume that $w^*(e) = 6$ for all $e \in D$, as otherwise the theorem holds.

Proof of Claim F: Assume that $w^*(e) < 6$, for some $e \in D$. By Claim A and B, we note that $6|D| > \sum_{e \in D} w^*(e) = 2mn - 2$, which implies that $|D| > (mn - 1)/3$. As m, n and $|D|$ are integers, this implies that $|D| \geq mn/3$, which in turn implies that $|D| \geq \lceil mn/3 \rceil$. We would therefore be done in this case. (□)

Claim G: We may assume that the following hold, as otherwise the theorem holds.

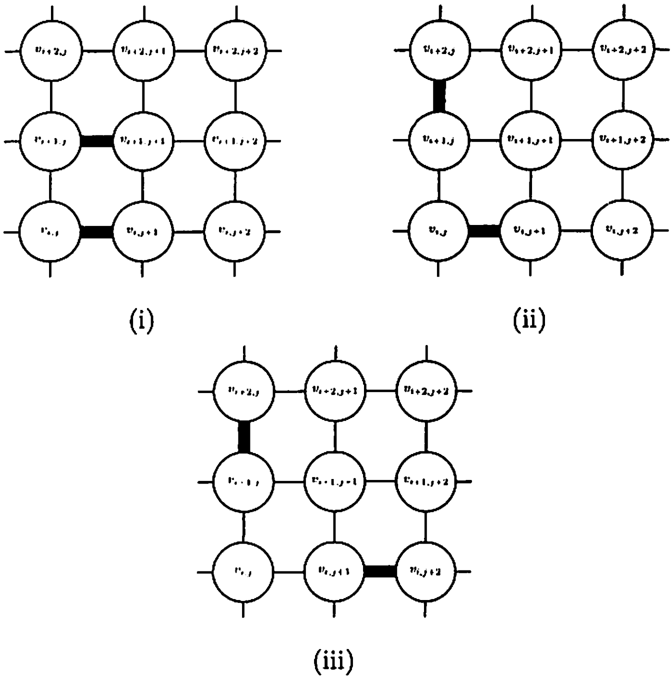


Figure 2: We show in Claim G that we may assume that the above configurations do not appear.

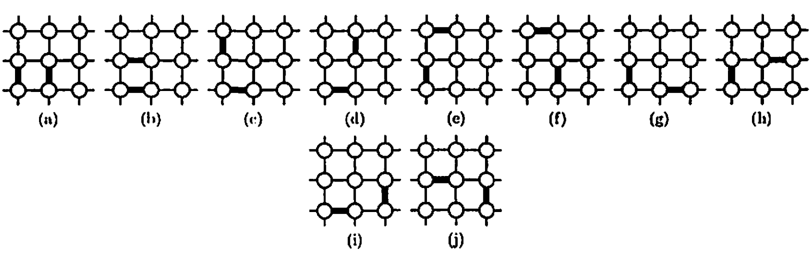


Figure 3: By symmetry of (i) and (ii) in Claim G, we may assume that the above configurations do not appear.

(i): **We do not have** $v_{i,j}v_{i,j+1}$, $v_{i+1,j}v_{i+1,j+1} \in D$ **for any** i, j **(see Figure 2 (i)).**

(ii): **We do not have** $v_{i,j}v_{i,j+1}$, $v_{i+1,j}v_{i+2,j} \in D$ **for any** i, j **(see Figure 2 (ii)).**

(iii): **We do not have** $v_{i,j+1}v_{i,j+2}$, $v_{i+1,j}v_{i+2,j} \in D$ **for any** i, j **(see Figure 2 (iii)).**

(iv): $v_{1,1}$, $v_{1,m}$, $v_{n,1}$, $v_{n,m} \notin V(D)$.

Proof of Claim G: We shall show that if any of the configurations (i), (ii), or (iii) appear, then $|D| \geq mn/3$.

First assume that $v_{i,j}v_{i,j+1}$, $v_{i+1,j}v_{i+1,j+1} \in D$ (see Figure 2 (i)). Analogously to the proof of Claim B, it is not difficult to show that $w^*(v_{i,j}v_{i,j+1}) < 6$ and $w^*(v_{i+1,j}v_{i+1,j+1}) < 6$ (as the edges $v_{i,j}v_{i+1,j}$ and $v_{i,j+1}v_{i+1,j+1}$ only contribute $w(v_{i,j}v_{i+1,j})/2$ and $w(v_{i,j+1}v_{i+1,j+1})/2$ to each of $w^*(v_{i,j}v_{i,j+1})$ and $w^*(v_{i+1,j}v_{i+1,j+1})$). By Claim F, we have now proved part (i).

Now assume that $v_{i,j}v_{i,j+1}$, $v_{i+1,j}v_{i+2,j} \in D$ (see Figure 2 (ii)). Again analogous to the proof of Claim B, we note that $w^*(v_{i+1,j}v_{i+2,j}) < 6$ (as the edge $v_{i,j}v_{i+1,j}$ only contributes $w(v_{i,j}v_{i+1,j})/2$ to $w^*(v_{i+1,j}v_{i+2,j})$). By Claim F, we have now proved part (ii).

By symmetry, we may now assume that none of the configurations in Figure 3 appear. In order to prove Claim G(iii), we will prove the following subclaim.

Subclaim G.1: **If** $v_{i',j'+1}v_{i',j'+2} \in D$ **and** $v_{i'+1,j'}v_{i'+2,j'} \in D$**, then** $v_{i'+1,j'+2}v_{i'+1,j'+3} \in D$ **and** $v_{i'+2,j'+1}v_{i'+3,j'+1} \in D$.

Proof of Subclaim G.1: We assume that $v_{i',j'+1}v_{i',j'+2}$, $v_{i'+1,j'}v_{i'+2,j'} \in D$. For the sake of contradiction, assume that $v_{i'+1,j'+1} \in V(D)$, and let $e' \in D$ be the edge containing $v_{i'+1,j'+1}$ as an endpoint. The other endpoint cannot be $v_{i'+1,j'+2}$ (see Figure 3(g)) or $v_{i'+2,j'+1}$ (see Figure 3(c)). Furthermore, it cannot be $v_{i',j'+1}$ or $v_{i'+1,j'}$ as D is an independent edge dominating set (by Claim D). This contradiction implies that $v_{i'+1,j'+1} \notin V(D)$.

As $V(D)$ is a vertex cover, we must now have $v_{i'+2,j'+1}$, $v_{i'+1,j'+2} \in V(D)$. Let $e_1, e_2 \in D$ be chosen such that $v_{i'+2,j'+1}$ is an endpoint of e_1 and $v_{i'+1,j'+2}$ is an endpoint of e_2. We note that $e_1 = v_{i'+2,j'+1}v_{i'+3,j'+1}$ (by Figure 3(h)) and $e_2 = v_{i'+1,j'+2}v_{i'+1,j'+3}$ (by Figure 3(d)). This completes the proof of Subclaim G.1. (($\square$))

We now complete the proof of Claim G(iii). Assume that $v_{i,j+1}v_{i,j+2}$, $v_{i+1,j}v_{i+2,j} \in D$ (see Figure 2 (iii)). By Subclaim G.1, we note that $v_{i+1,j+2}v_{i+1,j+3}$, $v_{i+2,j+1}v_{i+3,j+1} \in D$. Using Subclaim G.1 again gives us $v_{i+2,j+3}v_{i+2,j+4}$, $v_{i+3,j+2}v_{i+4,j+2} \in D$. Continuing this process gives us an infinite sequence of edges belonging to D, a contradiction to G being finite. This proves part (iii).

We will now prove part (iv). If $e = v_{1,1}v_{1,2} \in D$, then in the proof of Claim B we showed that $w^*(e) \leq 5$, a contradiction to Claim F. Analogously (by symmetry), if $e = v_{1,1}v_{2,1} \in D$, then $w^*(e) \leq 5$, a contradiction to Claim F. Therefore, $v_{1,1} \notin V(D)$. Analogously (by symmetry) we have $v_{1,m}$, $v_{n,1}$, $v_{n,m} \notin V(D)$. (□)

We will now complete the proof of the theorem. By symmetry, we note that we do not have any of the configurations in Figure 3. By Claim G(iv), we have $v_{1,1} \notin V(D)$. Therefore $v_{1,2}, v_{2,1} \in V(D)$. By Claim G(iii), we note that either $v_{1,2}v_{2,2} \in D$ or $v_{2,1}v_{2,2} \in D$. Assume without loss of generality that $v_{2,1}v_{2,2} \in D$ (as otherwise we could swap n and m). This implies that $v_{1,2}v_{1,3} \in D$, as D is an independent edge dominating set (by Claim D). By Claim G(iv), we must therefore have $m \geq 3$.

By Claim G(i) and (ii), we note that $v_{3,1} \notin V(D)$. By Figure 3(d), note that $v_{3,2}v_{3,3} \in D$. If $m \geq 4$, then we furthermore must have $v_{4,1} \in V(D)$ and by Claim G(iii) we have $v_{4,1}v_{4,2} \in D$. This would imply that $m \geq 5$.

Analogously to above, we would in this case have $v_{5,2}v_{5,3} \in D$. If $m \geq 6$, then analogously we would have $m \geq 7$ and $v_{6,1}v_{6,2} \in D$. Continuing this process, we see that $m \geq 3$ is odd and the following edges belong to D.

$$D^* = \{v_{1,2}v_{1,3},\ v_{2,1}v_{2,2},\ v_{3,2}v_{3,3},\ v_{4,1}v_{4,2},\ v_{5,2}v_{5,3},\ \dots,\\ v_{m-1,1}v_{m-1,2},\ v_{m,2}v_{m,3}\}$$

As $n \geq 2$ and n is not divisible by three, we note that $n \geq 4$ (as $n = 2$ is impossible due to the edge $v_{1,2}v_{1,3}$). If $v_{2i,3} \in V(D)$ for any $1 \leq i < m/2$, then $w^*_D(v_{2i,1}, v_{2i,2}) = 5.5 < 6$, a contradiction to Claim F. So there are no edges $v_{j,3}v_{j,4}$ in D for any $1 \leq j \leq m$. Now observe that $D \setminus D^*$ is an edge dominating set for $P_m \square P_{n-3}$ (by removing columns 1, 2 and 3 from G). If $n = 4$, then note that there are $(m-1)/2$ edges in $D \setminus D^*$ as the edges $v_{2,3}v_{2,4}, v_{4,3}v_{4,4}, \dots, v_{n-1,3}v_{n-1,4}$ need to be covered by D. Therefore, $|D| \geq m + (m-1)/2 \geq \lceil 4m/3 \rceil$, completing the proof in this case.

So we may assume that $n \geq 5$ and by induction we may assume that $|D \setminus D^*| \geq \lceil m(n-3)/3 \rceil$. However, this implies that $|D| \geq \lceil m(n-3)/3 \rceil + m = \lceil nm/3 \rceil$, completing the proof. □

Theorem 1 and Theorem 2 imply the following corollary.

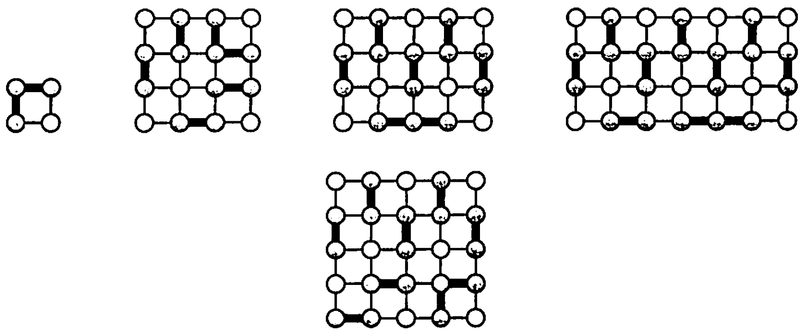


Figure 4: Examples of minimum edge dominating sets that are not independent.

Corollary 3 *If* $n \equiv 0 (\text{mod } 3)$, *then* $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil$ *and every minimum edge dominating set is independent.*

By Corollary 3, we note that every minimum edge dominating set of $P_m \square P_n$ is an independent edge dominating set when nm is a multiple of three. This is not the case for all grid graphs. In fact, it is not difficult to show that $P_2 \square P_2$, $P_4 \square P_4$, $P_4 \square P_5$, $P_5 \square P_5$, $P_4 \square P_7$, $P_7 \square P_7$, $P_4 \square P_{10}$, $P_5 \square P_{10}$ and $P_7 \square P_{10}$ have minimum edge dominating sets that are not independent. See Figure 4 for some examples.

3 Exact Solutions for small n

When $2 \leq n \leq 10$, the following theorem gives us the exact value of $\gamma'(P_m \square P_n)$.

Theorem 4 $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil$ *when* $2 \leq n \leq 10$ *and* $m > 1$.

Proof: We may assume that $m \not\equiv 0 (\text{mod } 3)$ and $n \not\equiv 0 (\text{mod } 3)$, by Corollary 3. We may also assume that $m \geq n$ as otherwise we swap n and m.

Consider the case when $n = 2$. If $m \equiv 1 (\text{mod } 3)$, then select the edge in row 1 and use the solution given in Theorem 1 on the remaining $m - 1$ rows. If $m \equiv 2 (\text{mod } 3)$, then select the edges in rows 1 and 2 and use the solution given in Theorem 1 on the remaining $m - 2$ rows.

Now consider the case when $n = 4$ and $m \not\equiv 0 (\text{mod } 3)$. See Figure 5. The middle part of the first figure can be repeated j times (on top of each

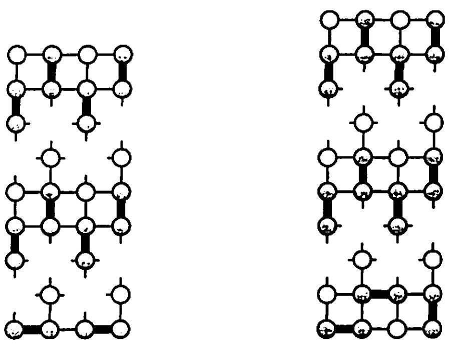


Figure 5: By taking the middle part and adding it any number of times we obtain a solution for $P_m \square P_4$, where m is congruent to 1 or 2 modulo 3 and $m \geq 4$.

other) in order to obtain a solution to $P_{4+3j} \square P_4$. Analogously, the middle part of the second figure can be repeated j times (on top of each other) in order to obtain a solution to $P_{5+3j} \square P_4$. This completes the case when $n = 4$ (as if $m \equiv 0 (\text{mod } 3)$, we were done by Corollary 3).

Now consider the case when $n = 5$ and $m \not\equiv 0 (\text{mod } 3)$. See Figure 6. Analogously to the case when $n = 4$, we obtain solutions to $P_{4+3j} \square P_5$ and $P_{5+3j} \square P_5$ $(j \geq 0)$, completing the case when $n = 5$.

The construction in Figure 7 gives us solutions to $P_{4+3j} \square P_7$ and $P_{5+3j} \square P_7$ $(j \geq 0)$, completing the case when $n = 7$.

The construction in Figure 8 gives us solutions to $P_{5+3j} \square P_8$ and $P_{7+3j} \square P_8$ $(j \geq 0)$, completing the case when $n = 8$.

The construction in Figure 9 gives us solutions to $P_{7+3j} \square P_{10}$ and $P_{8+3j} \square P_{10}$ $(j \geq 0)$, completing the case when $n = 10$. □

4 Upper Bound and Concluding Remarks

By Theorem 2, $\lceil mn/3 \rceil$ is a lower bound for $\gamma'(P_m \square P_n)$ for all $m, n \geq 2$. It was conjectured in [9] that $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil$, when $m, n \geq 2$. However, we would like to conjecture that this may be false for large values of m and n, when neither nm is not a multiple of three. In fact, we believe that the following may be true.

Conjecture 2 *There exists an $\epsilon > 0$, such that $\gamma'(P_m \square P_n) > mn/3 + \epsilon n$, when $m \geq n \geq 2$, $n \not\equiv 0 (\text{mod } 3)$, and $m \not\equiv 0 (\text{mod } 3)$.*

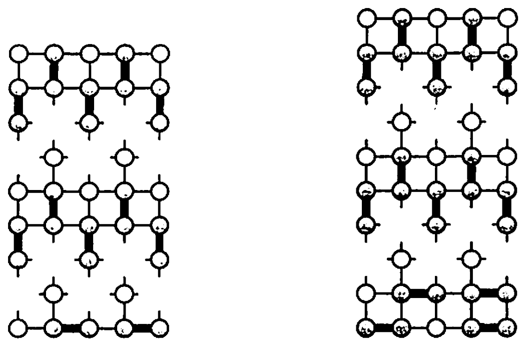


Figure 6: By taking the middle part and adding it any number of times we obtain a solution for $P_m \Box P_5$, where m is congruent to 1 or 2 modulo 3 and $m \geq 5$.

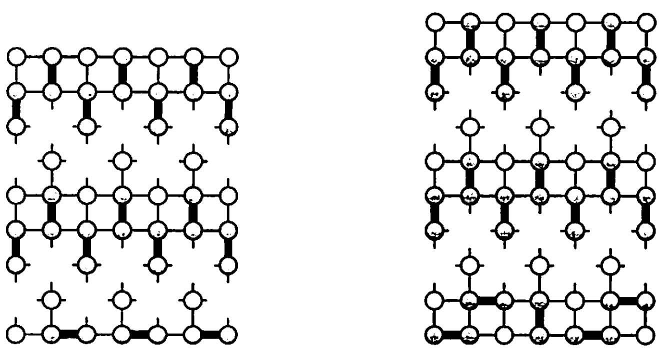


Figure 7: By taking the middle part and adding it any number of times we obtain a solution for $P_m \Box P_7$, where m is congruent to 1 or 2 modulo 3 and $m \geq 7$.

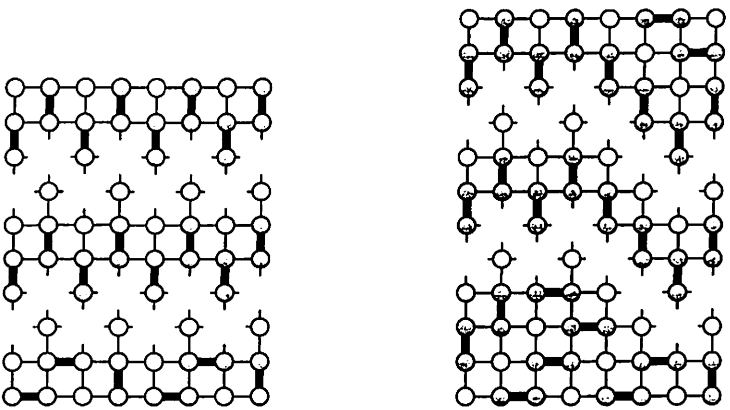


Figure 8: By taking the middle part and adding it any number of times we obtain a solution for $P_m \Box P_8$, where m is congruent to 1 or 2 modulo 3 and $m \geq 8$.

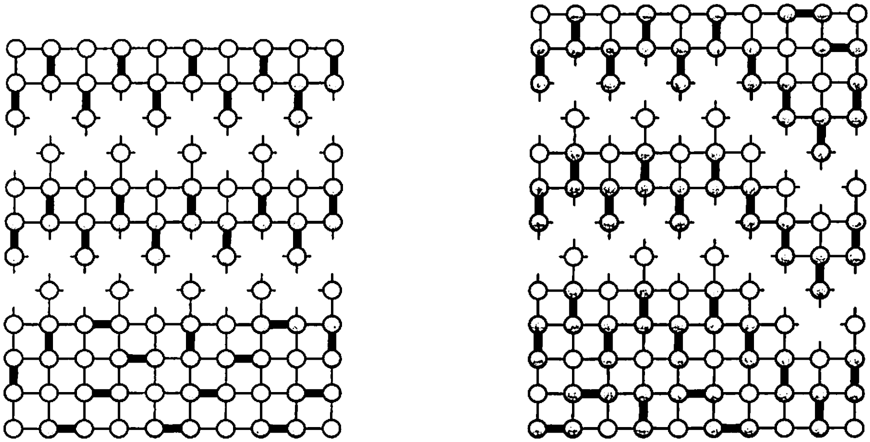


Figure 9: By taking the middle part and adding it any number of times we obtain a solution for $P_m \Box P_{10}$, where m is congruent to 1 or 2 modulo 3 and $m \geq 10$.

Note that if the above conjecture is true then we must have $\epsilon < 1/30$, as if $m \equiv 2 \pmod 3$ and $n = 10$, then Theorem 4 implies that $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil = mn/3 + 1/3$, so in this case $\epsilon n = 10\epsilon$ must be less than $1/3$.

If true, Conjecture 2 would be best possible, due to the following theorem.

Theorem 5 $\gamma'(P_m \square P_n) \leq mn/3 + n/12 + 1$, *for all* $n, m \geq 1$.

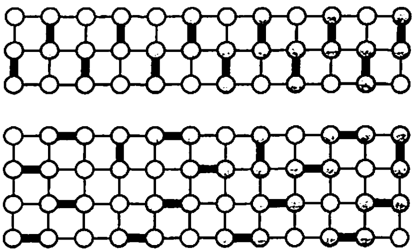
Proof: The theorem is clearly true when $n = 1$ or $m = 1$, as $\gamma'(P_1 \square P_n) = \lceil (n-1)/3 \rceil < n/3 + 1$. If $n \leq 10$ or $m \leq 10$, then $\gamma'(P_1 \square P_n) = \lceil mn/3 \rceil < mn/3 + n/12 + 1$, by Theorem 4. So assume that $n, m \geq 11$.

If $m \equiv 0 \pmod 3$ or $n \equiv 0 \pmod 3$, then we are done by Corollary 3, as $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil < mn/3 + n/12 + 1$.

Next assume that $m \equiv 1 \pmod 3$. We will now combine the solutions in Figure 10 and Figure 11 (as shown in Figure 12) to obtain a solution for $P_m \square P_n$. Assume that $n = 12i_n + r_n$ where i_n and r_n are integers such that $0 \leq r_n < 12$. Note that $r_n \not\equiv 0 \pmod 3$ as $n \not\equiv 0 \pmod 3$. Also define the integer $i > 0$, such that $m = 3i + 4$, which is possible as $m \equiv 1 \pmod 3$ and $m \geq 11$. Now consider the solution we get by placing Solution A in Figure 10 on top of each other i times and then placing this on top of Solution B from Figure 10. This gives us a solution to $P_m \square P_{12}$, which we will call Solution C. Now place the first r_n columns in Solution A on top of each other i times and then place the solution to $P_4 \square P_{r_n}$ given in Figure 11 underneath. This gives us a solution to $P_m \square P_{r_n}$, which we call Solution D. Now place i_n copies of solution C next to each other followed by solution D, which gives us an edge dominating set for $P_m \square P_n$ (See Figure 12). If we use k edges in the solution of $P_4 \square P_{r_n}$ given in Figure 11, then our solution will contain the following number of edges, which completes the part when $m \equiv 1 \pmod 3$.

$$\begin{aligned}
\tfrac{n(m-4)}{3} + 17i_n + k &= \tfrac{mn}{3} + \tfrac{n}{12} - \tfrac{17n}{12} + 17i_n + k \\
&= \tfrac{mn}{3} + \tfrac{n}{12} - \tfrac{17(12i_n + r_n)}{12} + 17i_n + k \\
&= \tfrac{mn}{3} + \tfrac{n}{12} - \tfrac{17r_n}{12} + k \\
&\leq \tfrac{mn}{3} + \tfrac{n}{12} + 1
\end{aligned}$$

So finally assume that $m \equiv 2 \pmod 3$. This case is proved analogously to the case when $m \equiv 1 \pmod 3$, except we use the partial solutions given in Figure 13 (and let $n = 4i_n + r_n$, where $0 \leq r_n \leq 3$ and $m = 3i + 2$). Again, it is not difficult to check that all solutions created this way have at most $mn/3 + n/12 + 1$ edges. $\square$



Solution A

Solution B

Figure 10: Solution A (to $P_3 \square P_{12}$) and Solution B (to $P_4 \square P_{12}$).

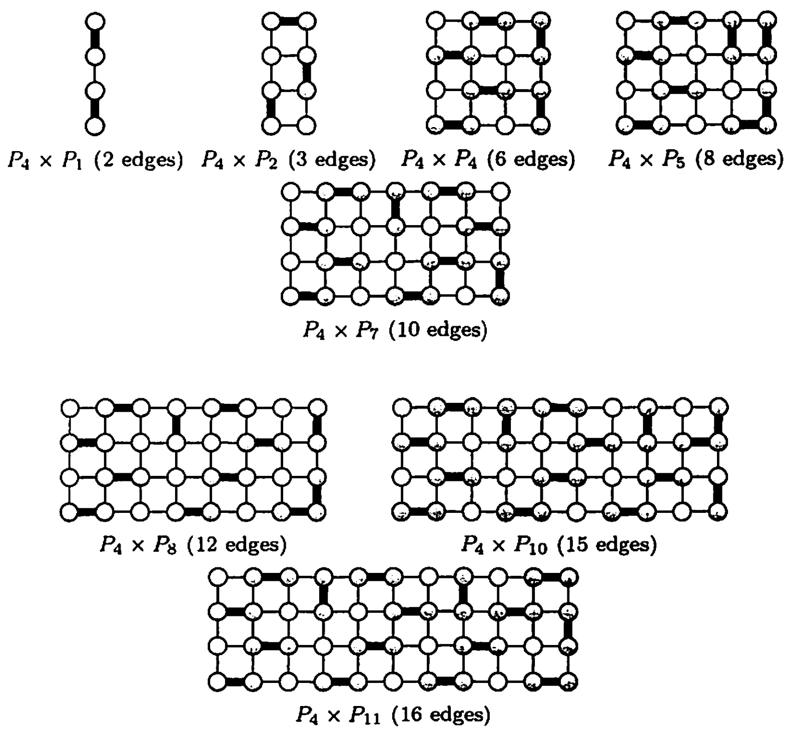


Figure 11: Solutions for $P_4 \square P_{r_n}$ (and how many edges they use).

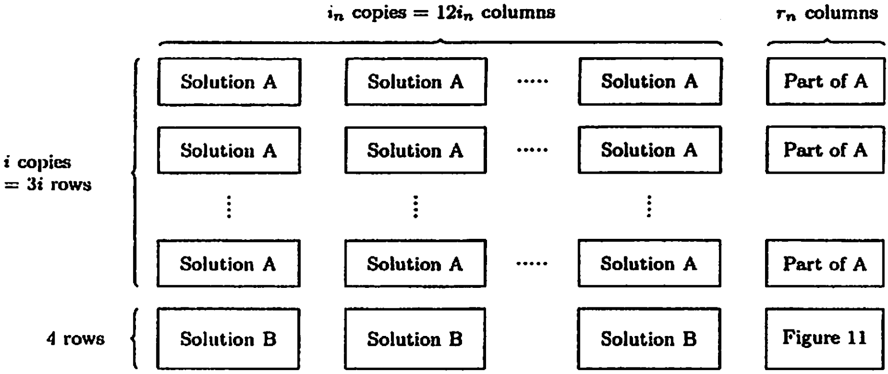


Figure 12: How to create a solution for $P_m \Box P_n$ when $m \equiv 1 (\text{mod } 3)$ and $m \geq 4$.

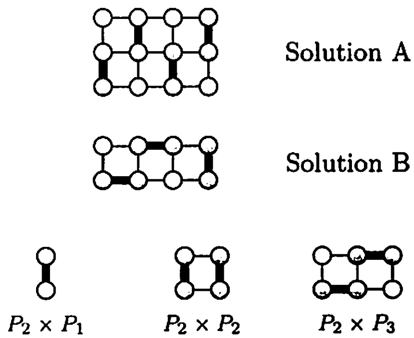


Figure 13: The partial solutions used to create a solution for $P_m \Box P_n$ when $m \equiv 2 (\text{mod } 3)$.

Note that if Conjecture 2 is true, then Conjecture 1 would be false. The reason we believe Conjecture 2 is true, despite Theorem 4 giving support for Conjecture 1, is that the number of edges, e, in an optimal solution, D, with $w^*(e) < 6$ (see the proof of Theorem 2 for a definition of w^*) seems to be proportional with n (if $m \geq n$, $n \not\equiv 0(\text{mod } 3)$, and $m \not\equiv 0(\text{mod } 3)$), which would imply that Conjecture 2 is true.

Proposition 6 *Let* $m = 14$, $n \equiv 2(\text{mod } 3)$ *and* $n \geq 11$. *Then* $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil$.

Proof: We use the usual pattern, such as in Figure 1, for the first $m-3$ rows and the leftmost $n-2$ columns. So for example, there is an edge in the edge dominating set in row 1 between the vertices in column $n-3$ and $n-2$. For the last three rows, use the usual pattern, but starting from right to left (so start with an edge in row $m-1$ between columns n and $n-1$ and edges in rows m and $m-2$ between columns $n-2$ and $n-1$) except for columns 1, 2, and 3, which will have two edges of the following form: column 1 between rows $m-2$ and $m-1$; and column 2 between rows $m-1$ and m. Next include the following two edges: in column n between rows $m-3$ and $m-4$; and in column $n-1$ between rows $m-4$ and $m-5$. Then in the last two columns, use the following pattern: every fourth edge in column n (starting between rows 1 and 2 and the last being between rows $m-8$ and $m-9$); every fourth edge between columns $n-2$ and $n-1$ (starting with row 4 and ending with row $m-6$); and every fourth edge in column $n-2$ (starting between rows 2 and 3 and the last being between rows $m-7$ and $m-8$). It is easy see that this uses $\lceil mn/3 \rceil$ edges: $\lfloor \frac{n}{3} \rfloor$ are used in each of the first $m-3$ rows and $n-3$ columns, for a total of $(m-3)(n-2)/3$. In the last three rows, $3(n-2)/3+2$ edges are used. In the last two columns and first $m-4$ rows, $3(m-6)/4+2$ edges are used. Summing, we get $mn/3 + m/12 - 1/2$, which is equal to $\lceil mn/3 \rceil$ for $m = 14$. □

A similar pattern as in Proposition 6 can be utilized when $m = 17$ and $n \equiv 2(\text{mod } 3)$, adjusting slightly where the extra two edges are located in the last two columns. However, in this case, the pattern described uses more than $\lceil mn/3 \rceil$ edges. It turns out that one can use a different pattern to show that $\gamma'(P_{17} \square P_{17}) = 97 = \lceil mn/3 \rceil$ and this pattern can be used for all $P_{17} \square P_n$, where $n \geq 17$ and $n \equiv 2(\text{mod } 3)$. This construction and a few others, including $P_{14} \square P_{14}$ and $P_{13} \square P_{13}$, can be seen at www.unf.edu/~ wkloster/edge_dom.html

As we have two contradicting conjectures, it seems interesting to determine which one (if any) is true. If Conjecture 1 is false, the smallest possible counterexample would be for $m = 16$ or one of the cases for $m = 14$ not covered by Proposition 6. This is because as the cases for $m = 11$ and $m = 13$ with $m < n$ have been checked with the aid of a computer program; that is, $\gamma'(P_m \square P_n) = \lceil mn/3 \rceil$ when $m \in \{11, 13\}$. However, we have no firm reason to believe at this time that any of these small cases will provide a counterexample to Conjecture 1 and it may be that the smallest such counterexample, if one exists at all, is for some $m > 16$.

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