Note on the Independent Roman Domination Number of a Graph

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Abstract

A Roman dominating function (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number, $\gamma_R(G)$, of G is the minimum weight of a Roman dominating function on G. An RDF f is called an independent Roman dominating function if the set of vertices assigned non-zero values is independent. The independent Roman domination number, $i_R(G)$, of G is the minimum weight of an independent RDF on G. In this paper, we improve previous bounds on the independent Roman domination number of a graph.

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1 Introduction

For a graph $G = (V(G), E(G))$, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of $V(G)$. So we will write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer to f). A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function*, or just RDF, if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the value $f(V(G)) = \sum_{u \in V} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0, V_1, V_2)$ is called a γ_R-function if it is an RDF and $f(V(G)) = \gamma_R(G)$, (see for example [5, 9, 10]).

For a subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. A subset S of vertices of a graph G is an *independent set* if $G[S]$ has no edge. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set of G.

Cockayne et al. in [5] introduced the concept of independent Roman domination in graphs. An RDF $f = (V_0, V_1, V_2)$ is called an *independent Roman dominating function* (IRDF) if the set $V_1 \cup V_2$ is an independent set. The *independent Roman domination number*, $i_R(G)$, is the minimum weight of an IRDF on G. We call an IRDF $f = (V_0, V_1, V_2)$ in a graph G an i_R-function if $f(V(G)) = i_R(G)$. The concept of independent Roman domination in graphs is studied in [1, 3, 4, 6, 8]. If v is a vertex of G then the *degree* of v is denoted by $\deg(v) = \deg_G(v)$, and the *maximum degree* of G is denoted by $\Delta(G)$. In [1] the authors proved that if $\Delta(G) \leq 3$ then $i_R(G) = \gamma_R(G)$, and presented the following upper bound for the independent Roman domination number in terms of the Roman domination number and maximum degree $\Delta(G)$.

Theorem 1 ([1]) *For any graph G with $\Delta(G) \geq 3$,*

$$i_R(G) \leq \gamma_R(G) + ((\gamma_R(G) - 2)/2)(\Delta(G) - 3).$$

In this paper we improve Theorem 1 for all graphs, and characterize graphs with maximum degree at most six which achieve equality. We make use of the following.

Lemma 2 ([1]) *Let $f = (V_0^f, V_1^f, V_2^f)$ be an RDF for a graph G. If V_2^f is independent, then there is an independent RDF g for G such that $w(g) \leq w(f)$.*

Theorem 3 ([2] and [11]) *For any graph G,*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}.$$

By $S(a, b)$ we mean a double-star in which the two central vertices have degrees a and b, respectively, and by $K_{a,b}$ we mean a complete bipartite graph in which the partite sets have cardinalities a and b, respectively. Let $f = (V_0^f, V_1^f, V_2^f)$ be an RDF in a graph G and $v \in V_2^f$. A vertex $u \in V_0^f$ is a *private neighbor* of $v \in V_2^f$ if $N(u) \cap V_2^f = \{v\}$.

2 Main result

Let $k \geq 4$. Let H be a bipartite graph with partite sets A and B each of cardinality $k-1$ such that $i_R(H) \geq k$ with the condition that if $i_R(H) = k$ then for every $i_R(H)$-function f, either $f(A) = 0$ or $f(B) = 0$. Let G_k be the graph obtained from H by adding two new vertices x and y, and adding edges xy, xu for all $u \in A$, and yv for all $v \in B$.

Theorem 4 *For any connected graph G with $4 \leq \Delta(G) \leq 6$, $i_R(G) \leq \frac{\Delta(G)+1}{4}\gamma_R(G)$, with equality if and only if $G = S(k,k)$ or $G = G_k$, where $k = \Delta(G)$.*

Proof. Let G be a graph with $4 \leq \Delta(G) \leq 6$. Our goal is to find an RDF $g = (V_0^g, V_1^g, V_2^g)$ with the desired weight such that V_2^g is independent, and then apply Lemma 2. Let $f = (V_0^f, V_1^f, V_2^f)$ be a $\gamma_R(G)$-function such that $G[V_2^f]$ has minimum number of edges. Observe that each vertex of V_2^f has at least one private neighbor in V_0^f. Now if there is a vertex $x \in V_2^f$ having exactly one private neighbor in V_0^f, say y, then we replace $f(x)$ and $f(y)$ with 1. Hence we may assume that every vertex of V_2^f has at least two private neighbors in V_0^f. Let A be the set of isolated vertices of $G[V_2^f]$. Let B the set of all vertices of $G[V_2^f]$ that belong to a K_2-component of $G[V_2^f]$, and $C = V_2^f - (A \cup B)$. If $B \cup C = \emptyset$, then clearly $i_R(G) \leq \gamma_R(G) < \frac{\Delta(G)+1}{4}\gamma_R(G)$. Thus we may assume that $B \cup C \neq \emptyset$. Note that any vertex of B is of degree one in $G[V_2^f]$, and any vertex of C belongs to a component of $G[V_2^f]$ of order at least three. If a vertex x of $B \cup C$ has precisely two private neighbors x_1, x_2 in V_0^f, then $((V_0^f - \{x_1, x_2\}) \cup \{x\}, V_1^f \cup \{x_1, x_2\}, V_2^f - \{x\})$ is a $\gamma_R(G)$-function such that $|E(G[V_2^f - \{x\}])| < |E(G[V_2^f])|$, a contradiction. Thus any vertex of $B \cup C$ has at least three private neighbors in V_0^f. In particular, $\Delta(G[C]) \leq \Delta(G) - 3$. Let X be a maximum independent set in $G[C]$ containing all vertices of degree one in $G[C]$. By Theorem 3,

$$|X| \geq |C|/(\Delta(G[C]) + 1) \geq |C|/(\Delta(G) - 2).$$

Also, since every vertex of $C - X$ has at least two neighbors in X, every vertex of $C - X$ dominates at most $\Delta(G) - 2$ vertices of $V(G) - V_0^f$. Let

Y be a subset of B containing one end vertex of every K_2-component of $G[B]$. Thus $|Y| = \frac{|B|}{2}$. Let Z be the set of all vertices of V_0^f that are not dominated by $A \cup Y \cup X$. Then any vertex of Z is dominated by some vertex of $(B - Y) \cup (C - X)$. Furthermore,

$$\begin{aligned}|Z| &\le (\Delta(G) - 1)(|B| - |Y|) + (\Delta(G) - 2)(|C| - |X|)\\ &= (\Delta(G) - 1)|B|/2 + (\Delta(G) - 2)(|C| - |X|).\end{aligned}$$

Now $g = ((V_0^f - Z) \cup (B - Y) \cup (C - X), V_1^f \cup Z, A \cup Y \cup X)$ is an RDF for G such that $A \cup Y \cup X$ is independent. Let $s = 2|A| + 2|Y| + 2|X| + |V_1^f|$, $s_1 = \frac{\Delta(G)-3}{2}|B|$ and $s_2 = (\Delta(G) - 4)|C|$. By Lemma 2,

$$\begin{aligned}
i_R(G) &\le w(g) && (1)\\
&= s + |Z| && (2)\\
&\le s + (\Delta(G) - 1)|B|/2 + (\Delta(G) - 2)(|C| - |X|) && (3)\\
&= s + (\Delta(G) - 1)|B|/2 + (\Delta(G) - 2)(|C| - |X|) && (4)\\
&= \gamma_R(G) + s_1 + s_2 - (\Delta(G) - 4)(|X| && (5)\\
&\le \gamma_R(G) + s_1 + s_2 - (\Delta(G) - 4)(\frac{|C|}{\Delta(G) - 2}) && (6)\\
&= \gamma_R(G) + s_1 + \frac{s_2(\Delta(G) - 3)}{\Delta(G) - 2} && (7)\\
&\le \gamma_R(G) + \frac{\Delta(G) - 3}{2}(|B| + |C|) && (8)\\
&\le \gamma_R(G) + \frac{\Delta(G) - 3}{2}.\frac{\gamma_R(G)}{2} && (9)\\
&= \frac{\Delta(G) + 1}{4}\gamma_R(G). && (10)
\end{aligned}$$

Assume now that the equality holds, that is

$$i_R(G) = \frac{\Delta(G) + 1}{4}\gamma_R(G).$$

Then all of the inequalities (1), (3), (6), (8) and (9) become equality. From (8) and (9) we obtain that $|A| = |V_1^f| = 0$. We show that $|C| = 0$. If $k = \Delta(G) < 6$ then from (7) and (8) we have $|C| = 0$. Thus assume that $k = 6$. From (2) and (3) we obtain that $|X| = |C|/4$. Furthermore, Z is an independent set, and any vertex of $C - X$ dominates four vertices of Z. Further, as we assumed earlier, for any vertex $x \in C - X$, $\deg_{G[C]}(x) = 2$.

Then Theorem 3 implies that $|X| > |C|/4$, a contradiction. Thus $|C| = 0$. We show that $|B| = 2$. Suppose that $|B| > 2$. Then $|B|$ is even. Let $P_1 : x_1y_1$, $P_2 : x_2y_2$,...,$P_{|B|/2} : x_{|B|/2}y_{|B|/2}$ be the components of $G[B]$. Clearly $N(B - Y)$ is an independent set, and any vertex of $B - Y$ has precisely $\Delta(G) - 1$ private neighbors in V_0^f. Thus for $i \neq j$, no vertex of $N(\{x_i, y_i\} - Y)$ is adjacent to a vertex of $N(\{x_j, y_j\} - Y)$. Moreover, for each $i = 1, 2, ..., |B|/2$, replacing the vertex of $Y \cap \{x_i, y_i\}$ with the vertex of $\{x_i, y_i\} - Y$ produces a set that plays the same role of Y. We conclude that for each $i = 1, 2, ..., |B|/2$, if a vertex of $N(x_i)$ has degree more than one then it is adjacent only to some vertex in $N(y_i)$. Consequently $G[N[x_i] \cup N[y_i]]$ is a component of G, and G is disconnected, a contradiction. Hence, $|B| = 2$, and thus $\gamma_R(G) = 4$, and $i_R(G) = 1 + \Delta(G)$. Let $B = \{x, y\}$. Then $\deg(x) = \deg(y) = \Delta(G)$ and both $N(x) - \{y\}$ and $N(y) - \{x\}$ are independent. Let $X_1 = N(x) - \{y\}$, and $Y_1 = N(y) - \{x\}$. If there is no edge between X_1 and Y_1 then $G[X_1 \cup Y_1] = S(\Delta(G), \Delta(G))$. Thus assume that there is some edge between X_1 and Y_1. Let $g = (V_0^g, V_1^g, V_2^g)$ be an $i_R(G[X_1 \cup Y_1])$-function. Suppose that $w(g) \leq \Delta(G) - 1$. If $V_2^g \cap X_1 \neq \emptyset$ and $V_2^g \cap Y_1 \neq \emptyset$, then we extend g to an IRDF for G of weight less than $i_R(G)$ by assigning 0 to both x and y, a contradiction. Thus assume without loss of generality that $V_2^g \cap X_1 = \emptyset$. Since $w(g) \leq \Delta(G) - 1$, we find that $V_2^g \cap Y_1 \neq \emptyset$. Then we extend g to an IRDF for G of weight less than $i_R(G)$ by assigning 1 to x, and 0 to y, a contradiction. We deduce that $w(g) \geq \Delta(G)$. Assume that $w(g) = \Delta(G)$. If $V_2^g \cap X_1 \neq \emptyset$ and $V_2^g \cap Y_1 \neq \emptyset$, then we extend g to an IRDF for G of weight less than $i_R(G)$ by assigning 0 to both x and y, a contradiction. Thus assume without loss of generality that $V_2^g \cap X_1 = \emptyset$. Then $g(u) \neq 0$ for each $u \in Y_1$. Since Y_1 is an independent set, and $w(g) = \Delta(G)$, we find that $|Y_1 \cap V_2^g| = 1$, $|Y_1 \cap V_1^g| = \Delta(G) - 2$, and $f(X_1) = 0$. Consequently, $G = G_{\Delta(G)}$.

Conversely, if $G = S(k, k)$ then clearly $i_R(G) = k + 1$ and $\gamma_R(G) = 4$. Let $G = G_k$. It is clear that $\gamma_R(G) = 4$ and $(V(G) - \{x, y\}, \emptyset, \{x, y\})$ is a $\gamma_R(G)$-function. By assumption $A = N(x) - \{y\}$ and $B = N(y) - \{x\}$. Let $h_1 = (V(G) - (\{x\} \cup B), B, \{x\})$. By Lemma 2, h_1 is an IRDF for G and thus $i_R(G) \leq k + 1$. Let $h_2 = (V_0^{h_2}, V_1^{h_2}, V_2^{h_2})$ be an $i_R(G)$-function. If $x \in V_2^{h_2}$, then $A \cup \{y\} \subseteq V_0^{h_2}$, and thus $V_1^{h_2} = B$. So $w(h_2) \geq k + 1$. Similarly if $y \in V_2^{h_2}$, then $w(h_2) \geq k + 1$. Thus we may assume that $\{x, y\} \cap V_2^{h_2} = \emptyset$. Clearly $V_2^{h_2} \neq \emptyset$. Assume that $V_2^{h_2} \cap A \neq \emptyset$. If $V_2^{h_2} \cap B \neq \emptyset$ then $h_2(x) = h_2(y) = 0$, and the restriction of h_2 on $G[A \cup B]$ (say $h_2|_{G[A \cup B]}$) is an IRDF for $G[A \cup B]$, and so $w(h_2) \geq w(h_2|_{G[A \cup B]}) \geq i_R(G[A \cup B])$, and by assumption $w(h_2) \geq k + 1$, since $h_2(A) \neq 0$ and $h_2(B) \neq 0$. Thus assume that $V_2^{h_2} \cap B = \emptyset$. Since A is an independent set, and $h_2(y) \neq 0$, we find that $w(h_2) \geq k + 1$. We conclude that $i_R(G) = k + 1$. ■

Corollary 5 *If* $4 \leq k \leq 6$, *then for a k-regular graph* G, $i_R(G) = \frac{\Delta(G)+1}{4}\gamma_R(G)$ *if and only if* $G = K_{k,k}$.

Theorem 6 *For any graph* G *with* $\Delta(G) \geq 7$, $i_R(G) \leq \lceil(\Delta(G) - \frac{18}{5})\gamma_R(G)\rceil - 1$.

Proof. We follow the proof of Theorem 4. Since $\Delta(G) \geq 7$, we find that $\frac{\Delta(G)-3}{2} < \frac{(\Delta(G)-4)(\Delta(G)-3)}{\Delta(G)-2}$ and thus we deduce from (7) that

$$\begin{aligned}
i_R(G) &\leq \gamma_R(G) + \frac{\Delta(G)-3}{2}|B| + \frac{(\Delta(G)-4)(\Delta(G)-3)}{\Delta(G)-2}|C| \\
&< \gamma_R(G) + \frac{(\Delta(G)-4)(\Delta(G)-3)}{\Delta(G)-2}(|B|+|C|) \\
&\leq \gamma_R(G) + \frac{(\Delta(G)-4)(\Delta(G)-3)}{\Delta(G)-2}\frac{\gamma_R(G)}{2} \\
&\leq \frac{\Delta(G)^2 - 6\Delta(G) + 10}{\Delta(G)-2}\gamma_R(G) \\
&= (\Delta(G) - 4 + \frac{2}{\Delta(G)-2})\gamma_R(G) \\
&\leq (\Delta(G) - \frac{18}{5})\gamma_R(G).
\end{aligned}$$

■

Corollary 7 *If* G *is a graph in which the vertices of degree at least* 4 *form an independent set, then* $i_R(G) = \gamma_R(G)$.

Proof. Let f, A, B and C be as defined in the proof of Theorem 4. If $B \cup C \neq \emptyset$, then as it was seen, any vertex of $B \cup C$ has at least three private neighbors in V_0^f, and thus has degree at least four, a contradiction. Thus $B \cup C = \emptyset$, and so $V_2^f = A$. By Lemma 2, $i_R(G) \leq \gamma_R(G)$, and thus $i_R(G) = \gamma_R(G)$. ■

Recall that a graph is *claw-free* if it does not contain a $K_{1,3}$ as an induced subgraph.

Corollary 8 ([8]) *If* G *is a claw-free graph, then* $i_R(G) = \gamma_R(G)$.

Proof. Let f, A, B and C be as defined in the proof of Theorem 4. Assume that $B \cup C \neq \emptyset$. It was seen that any vertex of $B \cup C$ has at least three private neighbors in V_0^f. Since G is claw-free, the private neighbors of any vertex of $B \cup C$ in V_0^f form a clique. But then replacing a vertex of $B \cup C$ with one of its private neighbors in V_0^f forms a $\gamma_R(G)$-function g such that $G[V_2^g]$ has fewer edges than $G[V_2^f]$, a contradiction. Thus $B \cup C = \emptyset$, and so $V_2^f = A$. By Lemma 2, $i_R(G) \leq \gamma_R(G)$, and thus $i_R(G) = \gamma_R(G)$. ■

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