A Fundamental Theorem of Multigraph Decomposition of a $\lambda K_{m,n}$

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ABSTRACT. Much research has been done on the edge decomposition of λ copies of the complete graph G with respect to some specified subgraph H of G. This is equivalent to the investigation of (G, H)-designs of index λ. In this paper we present a fundamental theorem on the decomposition of λ copies of a complete bipartite graph. As an application of this result we show that necessary conditions are sufficient for the decomposition of λ copies of a complete bipartite graph into several multi-subgraphs H with number of vertices less than or equal to 4 and the number of edges less than or equal to 4, with some exceptions where decompositions do not exist. These decomposition problems are interesting to study as various decompositions do not exist even when necessary conditions are satisfied.

1. Introduction

The decomposition problem of a graph into subgraphs all of which belong to a specific class of graphs has been well studied where the subgraphs are simple (see [1], [2], and references therein). We consider connected graphs G with vertex set V of size/order n and edge set E of size e, and we allow the edges to occur with a frequency greater or equal to 1. By λ copies of a simple graph G, denoted by λG, we mean the graph with the same vertex set of G with each edge of G having multiplicity λ. For example, a λK_n is a λ-fold complete multigraph of order n and a $\lambda K_{m,n}$ is a λ-fold complete bipartite graph with V partitioned into two subsets V_1 and V_2 such that the size of V_1 equals m and the size of V_2 equals n. The decomposition of copies of a complete graph or a complete bipartite graph into proper multigraphs has not received much attention yet, see [**3, 4, 6, 7, 8**].

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DEFINITION 1. *Given a graph G and a subgraph H, a decomposition of G into t subgraphs in $\Delta = \{G_1, G_2, \ldots, G_t\}$ such that any edge of G is an edge of exactly one of the G_i's, and the G_i's are isomorphic to the graph H is called an H-decomposition of G. The elements of Δ are also called H-blocks. An H-decomposition of a λG is also called a $(\lambda G, H)$-design with index λ.*

Following well known observations are very useful for our purpose.

LEMMA 1. *If an H-decomposition of λ copies of G exists, then an H-decomposition of $n\lambda$ copies of G exists for any integer $n \geq 2$.*

LEMMA 2. *If an H-decomposition of λ_1 copies of G and λ_2 copies of G exists, then an H-decomposition of $\lambda_1 + \lambda_2$ copies of G exists.*

In this paper we present a fundamental theorem on the decomposition of a $\lambda K_{m,n}$ into isomorphic subgraphs and use it to settle graph decomposition problem for several subgraphs. All graphs considered in this paper are connected graphs. The fundamental theorem reduces proving the necessary conditions are sufficient to mostly finding examples of decompositions for certain small bipartite graphs, but the problem is still very interesting. First reason is that these small complete bipartite graphs are not always of the same size and more interesting reason is that there are several examples of non-existence: for example, a $3K_{m,n}$, a $4K_{m,n}$, and a $6K_{m,n}$ can be decomposed into copies of a specific subgraph called EL graph (defined in Section 5), but a $5K_{m,n}$ can not be for any m and n.

We begin with a simple problem of the decomposition of a $\lambda K_{m,n}$ into a small graph to motivate the Fundamental theorem and its application.

2. Decomposition of $\lambda K_{m,n}$ into LO graphs

DEFINITION 2. *Let $V = \{a, b, c\}$. An LO graph $< a, b, c >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{b, c\}$ are 1 and 2, respectively. We write abc to denote an LO graph $< a, b, c >$ when there is no confusion.*

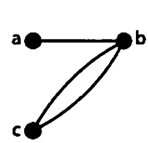

FIGURE 1. An LO graph $< a, b, c >$

2.1. Decomposition of $3K_{m,n}$.

THEOREM 1. *Necessary conditions are sufficient for an LO-decomposition of a* $3K_{m,n}$.

Proof: An LO Graph needs three vertices, hence $m+n$ must be greater than or equal to 3. If $m = 1$, and $n = 2$, with $V_1 = \{a_1\}$ and $V_2 = \{b_1, b_2\}$, we have the unique LO decomposition $\{b_1a_1b_2, b_2a_1b_1\}$. For $m = 1$ and $n = 3$, and $V_1 = \{a_1\}$ and $V_2 = \{b_1, b_2, b_3\}$, the LO decomposition is $\{b_1a_1b_2, b_2a_1b_3, b_3a_1b_1\}$. We can use these two decompositions to obtain an LO decomposition for any $3K_{m,n}$ as follows: First consider $m = 1$ and any n, with $V_1 = \{a_1\}$ and $V_2 = \{b_1, b_2, \ldots, b_n\}$. If n is even, say $n = 2t$, we can use the set union of $\{b_{2i-1}a_1b_{2i}, b_{2i}a_1b_{2i-1}\}$ over $i = 1, \ldots, t$ to obtain the required LO decomposition. For n odd, say $n = 2t - 1$, the decomposition is obtained with the LO graphs $b_1a_1b_2, b_2a_1b_3$, and $b_3a_1b_1$ on the first three vertices along with the LO graphs $b_{2i}a_1b_{2i+1}$, and $b_{2i+1}a_1b_{2i}$ for $i = 2, \ldots, t-1$. For general, a $3K_{m,n}$ with $V_1 = \{a_1, a_2, \ldots, a_m\}$, repeat the same decomposition for a_i $(i = 1, 2, \ldots, m)$ as we have done for a_1 for the $3K_{m,n}$. □

The fundamental theorem for the decomposition of λ copies of a complete bipartite graph is a generalization of this method and simplifies our proofs for other subgraphs, but first we state a natural necessary condition for a decomposition to exist.

LEMMA 3. *$\lambda mn \equiv 0(mod\ t)$ is a necessary condition for an H- decomposition of a $\lambda K_{m,n}$ if the order of H is t.*

This follows since there are λmn edges in the multi-bipartite graph and graph H has t edges.

2.2. LO decomposition of $\lambda K_{m,n}$ for any index λ. From the previous lemma, $\lambda mn \equiv 0(\text{mod } 3)$ is a necessary condition for the existence of an LO decomposition of a $\lambda K_{m,n}$.

LEMMA 4. *An LO decomposition of a $3t\lambda K_{m,n}$ and a $2t\lambda K_{m,n}$ exists.*

Proof: Theorem 1 gives an LO decomposition of a $3K_{m,n}$ for all m and n. Taking multiple copies, we will have a decomposition of $\lambda = 3t$ by Lemma 1. For $\lambda \neq 3t$, we have either $m \equiv 0(\text{mod } 3)$ or $n \equiv 0(\text{mod } 3)$. Without loss of generality we take $m = 3t$ and n any integer.

Case $\lambda = 2$: Consider the graph $2K_{3,1}$, on $V_1 = \{1, 2, 3\}$ and $V_2 = \{4\}$. Its LO decomposition is $\{142, 143\}$. Hence a $2K_{3,n}$ can always be decomposed. Assume $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5, \ldots, n\}$, then the LO graphs $\{1i2, 1i3\}$ $(i = 4, 5, \ldots, n)$ provide the necessary LO decomposition.

Hence, if $m = 3t$, let $V_1 = \{1, 2, 3, 4, 5, 6, \ldots, 3t-2, 3t-1, 3t\}$ and $V_2 = \{a_1, a_2, \ldots, a_n\}$, the required decomposition consists of the LO graphs $ji(j+1)$, and $ji(j+2)$, $j = 1, 4, \ldots, 3t-2$; $i = 1, 2, \ldots, n$ (subscript i is used in place of a_i). □

Thus from Lemma 1, 3 and 4, we have that the necessary conditions are sufficient for an LO decomposition of a $\lambda K_{m,n}$. A formal statement and proof for the ease of reference is given in Section 4 as Theorem 3.

In the next section we introduce a very applicable result on the decomposition of a complete bipartite graphs into isomorphic subgraphs.

3. A fundamental theorem for the decomposition of a $\lambda K_{m,n}$

We will refer to the following theorem and its corollaries and remarks as the FT in the rest of the paper.

THEOREM 2. *(The fundamental theorem for the decomposition of a $\lambda K_{m,n}$): If an H-decomposition of a $\lambda K_{t,s}$ exists, then an H-decomposition of a $\lambda\mu K_{pt,qs}$ for any positive integers μ, p and q exists.*

Proof: Suppose we have a decomposition of an $F_1 = \lambda K_{t,s}$ with graphs G_1, G_2, ..., and G_r where each edge of the F_1 is in exactly one of the G_i's. Let the F_1 have partite set A_1 of cardinality t and B_1 with cardinality s. Take $p-1$ copies A_2, A_3, ..., A_p of A_1 and place on the same side as A_1. Superimpose the edges of the F_1 onto the partite sets $A_i \cup B$ ($i = 1, 2, \ldots, p$) so that we have a decomposition of a $\lambda K_{pt,s}$. Similarly, we can extend the decomposition to a $\lambda K_{pt,qs}$, and take μ copies of the decomposition to complete the proof of the theorem. □

COROLLARY 1. *If the decomposition of a $\lambda K_{a,n}$ and the decomposition of a $\lambda K_{b,n}$ are known where $a, b = 1, 2$ or $2, 3$, then we know the decomposition of a $\lambda K_{m,n}$ for any positive integer m.*

COROLLARY 2. *If the decomposition of a $\lambda K_{a,c}$ and the decomposition of a $\lambda K_{b,d}$ are known where $a, b = 1, 2$ or $2, 3$, same for c and d, then we know the decomposition of a $\lambda K_{m,n}$ for any positive integers m and n.*

REMARK 1. *Essentially a $\lambda K_{mt,ns}$ is $t \times s$ copies of disjoint $\lambda K_{m,n}$. Therefore, once the decomposition of a $\lambda K_{m,n}$ is known, we know the decomposition of a $\lambda K_{mt,ns}$. Similarly, if the decompositions of a $\lambda K_{m,n}$ and a $\mu K_{m,n}$ are known, then we know the decomposition of a $(a\lambda + b\mu)K_{m,n}$.*

4. An application of the FT for an LO graph decomposition

As mentioned earlier, here is a formal statement and a proof based on the FT.

THEOREM 3. *An LO decomposition of a $\lambda K_{m,n}$ exists for $\lambda > 1$, except when $\lambda \neq 3t$ and neither m, n are congruent to 0(mod 3), i.e. the necessary conditions are sufficient.*

Proof: By Lemma 3 a necessary condition for an LO decomposition for a $\lambda K_{m,n}$ is $\lambda mn \equiv 0 \pmod 3$. Hence, $\lambda \neq 1$ and for $\lambda = 3t$ (any positive integer t), there is no condition on m and n. For $\lambda \neq 3t$, i.e. $\lambda \equiv 1, 2 \pmod 3$, m or n has to be $\equiv 0 \pmod 3$. Earlier we have constructed LO graph decompositions for a $2K_{3,1}$, a $3K_{1,2}$ and a $3K_{1,3}$, hence by the FT and using Lemmas 1 and 2, the necessary conditions are sufficient for the existence of an LO decomposition of complete bipartite graphs. □

5. Multigraphs on 3 vertices and at most 4 edges

There are infinitely many multi-subgraphs on 3 vertices of a multi-bipartite graphs. We restrict ourselves to the following four graphs as shown in Figure 2: a P_3, an LO graph, a $2P_3$, and an EL graph, respectively.

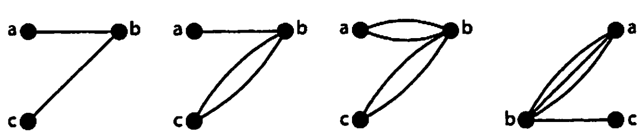

FIGURE 2. All connected graphs on 3 vertices and at most 4 edges

Recall a P_k is a path on k vertices. Hence, there is nothing to do for the graphs with two edges (a path) and its multiple a $2P_3$ with four edges except to note the following result on a P_3 decomposition (Theorem 4) and to note that a $2P_3$ decomposition of $\lambda K_{m,n}$ exists iff a P_3 decomposition of a $(\frac{\lambda}{2})K_{m,n}$ exists.

THEOREM 4. *(Heinrich [5]) Necessary conditions that λmn is even and $mn \geq 2$ are sufficient for a P_3 decomposition of a $\lambda K_{m,n}$.*

We have already given the final result on the subgraph with three edges in Theorem 3. Now we deal with the remaining subgraph with four edges.

DEFINITION 3. *Let $V = \{a, b, c\}$. An EL graph $< a, b, c >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{b, c\}$ are 3 and 1, respectively (see the last graph in Figure 2 for an example). We write abc to denote an EL graph $< a, b, c >$ when there is no confusion.*

OBSERVATION 1. *For $\lambda = 2t + 1$, where t is any nonnegative integer and $m = n = 2$, a $\lambda K_{2,2}$ does not have a EL decomposition.*

Of all the EL graphs used in the decomposition, let p be those edges of the multi-bipartite graph that have 1 edge contribution only, h those edges that have a combination of 1 and 3 edges contributed: s of 1 and q of 3, and r edges that have a 3 edge contribution. Counting the total number of edges, we have $p + s + 3q + 3r = 4(2t + 1) = T$, but $p + s = q + r =$ total number of EL graphs so that $p + s = 2t + 1 = q + r$. But the multiplicity of each edge is $2t + 1$ so that if $\{u, v\}$ is an edge of type p, then we must have a triple corresponding to this edge. Thus the total number of edges are $2t + 1 + 3(2t + 1) = T$. This means that every edge must be incident with u or v which is impossible. Thus $p = 0$ and $s = 2t + 1$. Each edge of s must be matched with a triple from q so that all the edges are of the combined type. But then the multiplicity of each edge is a multiple of 4, impossible.

The necessary condition $\lambda mn \equiv 0 \pmod 4$ for an EL decomposition of a $\lambda K_{m,n}$ yields: (A): $\lambda = 4t$, no condition on m and n. (B): $\lambda = 4t + 2$, m is even or n is even. (C): λ is odd, $mn \equiv 0 \pmod 4$: (C-I): $m \equiv 0 \pmod 4$ or $n \equiv 0 \pmod 4$; (C-II): $m, n \equiv 2 \pmod 4$.

We consider each case below.

In case (A), $\lambda = 4t$. An EL decomposition for a $\lambda K_{2,1} = 4tK_{2,1}$ with $V_1 = \{a, b\}$ and $V_2 = \{c\}$ is obtained by taking EL graphs acb and bca t times. Similarly taking EL graphs adb, bdc and cda t times, we get an EL decomposition for a $\lambda K_{3,1} = 4tK_{3,1}$ with $V_1 = \{a, b, c\}$ and $V_2 = \{d\}$. Now invoking the FT gives us the result that the necessary conditions are sufficient for this case.

In case (B), $\lambda = 4t + 2$, m or n is even. First we make an interesting observation: $(4t+2)K_{1,2}$ can not be decomposed into EL graphs. Suppose $V_1 = \{1\}$ and $V_2 = \{2, 3\}$. If the edge $\{1, 2\}$ occurs with multiplicity 3 in, say, x number of EL graphs and with multiplicity in y EL graphs, then the edge $\{1, 3\}$ will occur with multiplicity 1 in x and with multiplicity 3 in y EL graphs. Hence $3x + y = 4t + 2 = x + 3y$. There is no integer solution for this system of linear equations in two variables.

Next, we can decompose a $6K_{2,2}$ and a $6K_{2,3}$ into EL graphs as follows. A decomposition of a $6K_{2,2} = \{132, 423, 423, 231, 413, 413\}$ on $V_1 = \{1, 2\}$ and $V_2 = \{3, 4\}$. A decomposition of a $6K_{2,3} = \{413, 513, 231, 314, 514, 241, 324, 524, 524\}$ on $V_1 = \{1, 2\}$ and $V_2 = \{3, 4, 5\}$.

Hence using the FT, we can decompose a $6K_{2,2b}$ and also a $6K_{2a,2b}$ for any positive integers a and b, and also a $6K_{2a,2(b-1)+3} = 6K_{2a,2b+1}$. Since a

$4tK_{m,n}$ decomposition exists for all values of m and n, we can say that the necessary conditions are sufficient for the existence of an EL decomposition in this case as well for any $\lambda = 4t+2$, m or n even, $t > 0$.

In case (C), λ is odd. (I) Suppose $m \equiv 0 \pmod 4$ or $n \equiv 0 \pmod 4$. A decomposition of a $\lambda K_{4,1} = 3tK_{4,1}$ on $V_1 = \{a,b,c,d\}$ and $V_2 = \{e\}$ is obtained with t copies of EL graphs *aed, bed* and *ced*. Using the FT, a $\lambda K_{4t,1}$ has a EL decomposition and so a $\lambda K_{4t,n}$ has the required decomposition if $\lambda \equiv 3 \pmod 4$. We notice that 9 is $1 \pmod 4$ and though a $5K_{4,1}$ does not have required decomposition (see Theorem 5 below), a $9K_{4,1}$ has the decomposition as 9 is a multiple of 3. Therefore, if $\lambda = 4t+1$ and greater than 5, then as $4t+1 = 4(t-2)+9$, we have a required decomposition for a $\lambda K_{m,n}$ for all odd λ when m or n is $0 \pmod 4$.

(II) Suppose $m = 4t+2$ and $n = 4s+2$. Recall that $(2t+1)K_{2,2}$ does not have a EL decomposition by Observation 1. Even though it is immediate from this, one can check that it is not possible to decompose $5K_{2,2}$ in a different way as well: the decomposition needs $\frac{5mn}{4} = 5$ EL graphs, but as each of the four edges need to occur single twice, it is impossible. What helps us is that $3K_{2,6}$ has the required EL decomposition. Let $V_1 = \{1,2\}$ and $V_2 = \{3,4,5,6,7,8\}$. The EL graphs in the decomposition are $\{231, 241, 513, 613, 714, 814, 625, 725, 825\}$.

Therefore a $(4t+i)K_{2,6}$ decomposition for $i = 1,3$ exists except for a $5K_{2,6}$, and repeated application of the FT gives the decompositions in all other cases except that a $5K_{m,n}$ cannot be EL decomposed for any m and n. The reason is that we need $\frac{5mn}{4}$ EL graphs, hence $\frac{5mn}{4}$ edges which can come with multiplicity 3, but there are only mn distinct edges in $5K_{m,n}$ and $\frac{5mn}{4}$ is bigger than mn. We summarize the results obtained in this section in the following theorem.

THEOREM 5. *The necessary conditions are sufficient for a decomposition of $\lambda K_{m,n}$ into graphs of 3 vertices and at most 4 edges, except for decomposing a $(2t+1)K_{2,2}$, a $(4t+2)K_{1,2}$ or a $5K_{m,n}$ into EL graphs (where a decomposition does not exist).*

6. Multigraphs on 4 vertices and at most 4 edges

We discuss the decomposition of a $\lambda K_{m,n}$ into multigraphs of 4 vertices and 3 edges, and of 4 vertices and 4 edges in each of the following subsections, respectively. Recall that we consider only the connected graphs.

6.1. Decompositions of a $\lambda K_{m,n}$ into graphs on 4 vertices and 3 edges. Figure 3 includes all graphs on 4 vertices and 3 edges: a $K_{1,3}$

and a P_4 and the decomposition problem for these graphs has long been solved in a most general setting as follows.

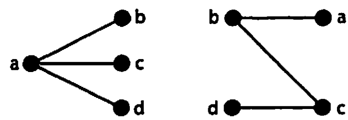

FIGURE 3. All connected graphs on 4 vertices and 3 edges: a $K_{1,3}$ and a P_4

THEOREM 6. *(Heinrich [5]) If $m \geq n$ and either λ is even or m and n are even, there is a $(\lambda K_{m,n}, P_k)$-design if and only if $\lambda mn \equiv 0(mod\ k-1)$, $m \geq \lceil \frac{k}{2} \rceil$ and $n \geq \lceil \frac{k-1}{2} \rceil$.*

Since the necessary conditions for a decomposition of a $\lambda K_{m,n}$ into P_4s are $m = n \geq 2$ and even and $\lambda mn \equiv 0(\text{mod } 3)$, by Theorem 6, the necessary conditions are sufficient for the decomposition.

Here is an argument using the FT. The necessary condition for a decomposition of a $\lambda K_{m,n}$ into $K_{1,3}$ is $\lambda mn \equiv 0(\text{mod } 3)$. That is, if $\lambda \not\equiv 0(\text{mod } 3)$, then m or $n \equiv 0(\text{mod } 3)$. If $\lambda \equiv 0(\text{mod } 3)$, then m or $n \geq 3$ (without loss of generality, assume $n \geq 3$ here). In the first case ($\lambda \not\equiv 0(\text{mod } 3)$), since a $K_{1,3}$ can be decomposed into a $K_{1,3}$ (itself), by the FT, a $K_{m,n=3t}$ can be decomposed into $K_{1,3}$s and so does a $\lambda K_{m,n=3t}$. In the second case ($\lambda \equiv 0(\text{mod } 3)$), if m or $n \equiv 0(\text{mod } 3)$, a $\lambda K_{m,n}$ can be decomposed into $K_{1,3}$s using the arguments in the first case. Now assume $m \not\equiv 0(\text{mod } 3)$ and $n \not\equiv 0(\text{mod } 3)$ ($n \geq 3$). A $3K_{1,4}$ on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4\}$ can be decomposed into four $K_{1,3}$s: $\{a123, a124, a134, a234\}$. Since a $3K_{1,3}$ can also be decomposed into $K_{1,3}$s, by the FT, a $3K_{1,3t+4s}$ can be decomposed into $K_{1,3}$s. Also, a $3K_{1,5}$ on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3, 4, 5\}$ can be decomposed into five $K_{1,3}$s: $\{a123, a145, a124, a235, a345\}$. Thus, a $3K_{m,n}$ and a $3tK_{m,n}$ can be decomposed into $K_{1,3}$s by the FT (notice that for any $n > 5$, n can be written as $3t + 4s$). We have that the necessary condition for the decomposition of a $\lambda K_{m,n}$ into $K_{1,3}$ is sufficient. We conclude the results in the section in the following theorem.

THEOREM 7. *The necessary conditions are sufficient for the decomposition of a $\lambda K_{m,n}$ into graphs of 4 vertices and 3 edges.*

6.2. Decompositions of a $\lambda K_{m,n}$ into graphs on 4 vertices and 4 edges. Figure 4 includes all graphs on 4 vertices and 4 edges and the decomposition problem for one of these graphs C_4 has long been settled, see for example,

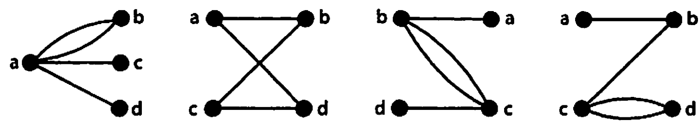

FIGURE 4. All connected graphs on 4 vertices and 4 edges

THEOREM 8. *(Sotteau [9]) A bipartite graph $K_{m,n}$ can be decomposed into cycles of length $2k$ if and only if m and n are even, $m \geq k$, $n \geq k$, and $2k$ divides mn.*

The second graph in Figure 4 is a C_4 graph. The necessary conditions for a decomposition of a $\lambda K_{m,n}$ into C_4s are that m and n are even (which implies $\lambda mn \equiv 0 \pmod 4$) and $m \geq 2$ and $n \geq 2$, by Theorem 8, the necessary conditions are sufficient for the decomposition.

6.2.1. *$OLL_{1,3}$ graph decomposition.* The first graph in Figure 4 is an example of an $OLL_{1,3}$ graph which is defined as follows.

DEFINITION 4. *Let $V = \{a, b, c, d\}$. An $OLL_{1,3}$ graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$ and $\{a, c\}$ and $\{a, d\}$ are 2, 1 and 1, respectively. We write abcd to denote an $OLL_{1,3}$ graph $< a, b, c, d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into $OLL_{1,3}$ graphs are $m \geq 3$ or $n \geq 3$ and λmn is $0 \pmod 4$, and if m (or n) is less than 3, then λn (or λm) is $0 \pmod 4$.

If $\lambda = 2$, then $2mn \equiv 0 \pmod 4$ implies m is even or n is even. Notice that both a $2K_{1,3}$ and a $2K_{2,3}$ can not be decomposed into $OLL_{1,3}$ graphs due to the necessary conditions. Without loss of generality, we only need to show the case when n is even. A $2K_{1,4}$ on $V_1 = \{1\}$ and $V_2 = \{a, b, c\}$ can be decomposed into two $OLL_{1,3}$ graphs $\{1abc, 1dbc\}$. Also, a $2K_{1,6}$ on $V_1 = \{1\}$ and $V_2 = \{a, b, c, d, e, f\}$ can be decomposed into three $OLL_{1,3}$ graphs $\{1abc, 1dbe, 1fce\}$. By the FT, a decomposition of a $2K_{m,n}$ for n even and $n \geq 4$ into $OLL_{1,3}$ graphs exists. Notice that if $n = 2$, then $m \geq 3$ and $2m \equiv 0 \pmod 4$ by the necessary conditions, which implies $m \geq 4$ and m is even. Thus, a decomposition of a $2K_{m,n}$ for n even into $OLL_{1,3}$ graphs exists when the necessary conditions are satisfied.

If $\lambda = 4$, the necessary conditions are $m \geq 3$ or $n \geq 3$. A $4K_{1,3}$ on $V_1 = \{1\}$ and $V_2 = \{a, b, c\}$ can be decomposed into three $OLL_{1,3}$ graphs $\{1abc, 1bac, 1cab\}$, and hence a $4K_{t,3s}$ by the FT. Also, A $4K_{1,4}$ on $V_1 = \{1\}$ and $V_2 = \{a, b, c, d\}$ can be decomposed into four $OLL_{1,3}$ graphs $\{1abc, 1abc, 1dbc, 1dbc\}$, and hence a $4K_{t,4s}$ by the FT. A $4K_{1,5}$ on

$V_1 = \{1\}$ and $V_2 = \{a, b, c, d, e\}$ can be decomposed into five $OLL_{1,3}$ graphs $\{1abc, 1bcd, 1cde, 1dae, 1eab\}$, and hence a $4K_{t,5s}$ by the FT. Since any number greater than or equal 3 can be written as a linear combination of 3, 4 or 5, a decomposition of $4K_{m,n}$ into $OLL_{1,3}$ graphs exists, and hence a $4tK_{m,n}$.

If λ is even, then we can write $\lambda = 4t$ or $\lambda = 4t + 2$ and a decomposition of $\lambda K_{m,n}$ into $OLL_{1,3}$ graphs exists by Remark 1. Thus, the necessary conditions for any even λ are sufficient.

If $\lambda = 3$, then $3mn \equiv 0 \pmod 4$ implies either m or n is $0 \pmod 4$ or $m, n \equiv 0 \pmod 2$. A $3K_{1,4}$ on $V_1 = \{a\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into three $OLL_{1,3}$ graphs $\{a123, a412, a324\}$. By the FT, a decomposition of a $3K_{m,4t}$ exists. Notice that for the case when $m, n \equiv 0 \pmod 2$, we only need to show $m, n \equiv 2 \pmod 4$. The decomposition of a $3K_{2,2}$ or a $3K_{2,6}$ into $OLL_{1,3}$ graphs does not exist by the necessary conditions. A $3K_{6,6}$ on $V_1 = \{1, 2, 3, 4, 5, 6\}$ and $V_2 = \{a, b, c, d, e, f\}$ can be decomposed into 27 $OLL_{1,3}$ graphs as follows. Decompose each of the nine $3K_{1,4}$ into three $OLL_{1,3}$ graphs: $\{1\}$ and $\{a, b, c, d\}$, $\{2\}$ and $\{c, d, e, f\}$, $\{3, 4, 5, 6\}$ and $\{c\}$, $\{3, 4, 5, 6\}$ and $\{d\}$, $\{2, 3, 4, 5\}$ and $\{a\}$, $\{2, 3, 4, 5\}$ and $\{b\}$, $\{3, 4, 5, 1\}$ and $\{e\}$, $\{3, 4, 5, 1\}$ and $\{f\}$, $\{6\}$ and $\{a, b, e, f\}$. By the FT, a decomposition of a $3K_{4t+6,4s+6}$ into $OLL_{1,3}$ exists. Thus, a decomposition of a $3K_{m,n}$ into $OLL_{1,3}$ exists when the necessary conditions are satisfied.

Notice that for any $\lambda > 1$ that is odd, the necessary conditions are the same as the ones for $\lambda = 3$, and these conditions are also included in the necessary conditions for $\lambda = 2$. Since we can write $\lambda = 2t + 3$, a decomposition of $\lambda K_{m,n}$ into $OLL_{1,3}$ graphs exists by Remark 1. Thus, the necessary conditions for any odd λ are sufficient. Combining the case for λ even, we have the following theorem.

THEOREM 9. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into $OLL_{1,3}$ graphs are sufficient.*

6.2.2. *LOL graph decomposition.* The third graph in Figure 4 is an example of an LOL graph which is defined as follows.

DEFINITION 5. *Let $V = \{a, b, c, d\}$. An LOL graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ are 1, 2 and 1, respectively. We write abcd to denote an LOL graph $< a, b, c, d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into LOL graphs are $m \geq 2$ and $n \geq 2$ and λmn is $0 \pmod 4$. If $\lambda = 2$, then $2mn \equiv 0 \pmod 4$ implies m is even or n is even. A $2K_{2,2}$ on $V_1 = \{a, b\}$

and $V_2 = \{c, d\}$ can be decomposed into two LOL graphs $\{acbd, cadb\}$. By the FT, a decomposition of a $2K_{2t,2s}$ exists.

There does not exist an LOL decomposition of a $2K_{2,2t+1}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, \ldots, 2t+1\}$. If a decomposition exists, the number of LOL graphs in the decomposition is $2t+1$ which is an odd number. However, if an edge $\{a, i\}$ appears singly, then it has to appear once more as a single edge in another LOL graph, which implies that the number of LOL graphs in the decomposition must be even, a contradiction. Similarly, a $(4s+2)K_{2,2t+1}$ cannot be decomposed into LOL graphs.

A $2K_{4,3}$ on $V_1 = \{1, 2, 3, 4\}$ and $V_2 = \{a, b, c\}$ can be decomposed into six LOL graphs $\{4a2b, 4a3c, a1b3, a1c3, b2c4, 3b4c\}$. By the FT, a decomposition of a $2K_{4t,3s}$ exists. Similarly, A $2K_{6,3}$ on $V_1 = \{a, \ldots, f\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into nine LOL graphs $\{b3a1, a1d2, a2c1, a2e3, b3d2, f3c1, b2f3, f1b2, f1e3\}$. By the FT, a decomposition of a $2K_{6t,3s}$ exists. Combing with the decomposition of a $2K_{2t,2s}$, a decomposition of a $2K_{2t,n}$ into LOL graphs exists, i.e., the necessary conditions are sufficient for $\lambda = 2$.

For $\lambda = 3$, the necessary condition $3mn \equiv 0 (\text{mod } 4)$ implies that either m or $n \equiv 0 (\text{mod } 4)$, or m and n are both $\equiv 0 (\text{mod } 2)$. Both a $3K_{2,4}$ (on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3, 4\}$) and a $3K_{3,4}$ (on $V_1 = \{a, b, c\}$ and $V_2 = \{1, 2, 3, 4\}$) can be decomposed into six LOL graphs and nine LOL graphs, respectively, as follows: $\{b1a4, b2a4, b3a1, a2b4, a3b1, a4b1\}$ and $\{b1a4, 1a2c, 2a3b, c2b4, 2b3c, 1b4a, a3c2, a4c1, b1c4\}$. By the FT, a decomposition of a $3K_{m,4s}$ exists.

A $3K_{2,4t+2}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, \ldots, 4t+2\}$ can not be decomposed into LOL graphs. Notice that the degree of point a (as well as b) is $3(4t+2)$ which is an even number. If a decomposition exists, there are $3(2t+1)$ LOL graphs in the decomposition, that is, the number of LOL graphs is an odd number. Since the point a (as well as b) is in each of the $3(2t+1)$ LOL graphs and each point in an LOL graphs has an odd degree, the total degree of a (as well as b) in those $3(2t+1)$ LOL graphs must be an odd number, which is a contradiction. Similarly, a $\lambda K_{2,4t+2}$ for λ odd cannot be decomposed into LOL graphs. A $3K_{6,6}$ on $V_1 = \{a, \ldots, f\}$ and $V_2 = \{1, \ldots, 6\}$ can be decomposed into 27 LOL graphs $\{2a1f, 2b1f, 2c1f, 3d2f, 2e3c, 3a2f, 3b2f, 6c2e, 1d3f, 5e4d, 4a3f, 4b3f, 4c3e, 1d4e, 6e5f, 5a4f, 5b4f, 6c4f, 6d5c, 1e6c, 6a5f, 6b5f, 1c5d, 2d6f, 2e1d, 1a6f, 1b6f\}$. By the FT, a decomposition of a $3K_{6,4t+2}$ into LOL graphs exists (since a decomposition of a $3K_{6,4(t-1)}$ exists and a decomposition of a $3K_{6,6}$ exists). Also, by the FT, a decomposition of a $3K_{4s+2,4t+2}$ ($s \geq 1$ and $t \geq 1$)

into LOL graphs exists (since a decomposition of a $3K_{6,4t+2}$ exists and a decomposition of a $3K_{4s,4t+2}$ exists). Except for a $3K_{2,4t+2}$ which cannot be decomposed into LOL graphs, we have shown that the necessary conditions for decomposing a $3K_{m,n}$ into LOL graphs are sufficient for $\lambda = 3$.

For any odd λ, the necessary conditions ($m \geq 2$ and $n \geq 2$ and $mn \equiv 0 (\text{mod } 4)$) are the same as the necessary conditions for $\lambda = 3$. Notices that these conditions are also included in the necessary conditions for $\lambda = 2$. Except for a decomposition of a $\lambda K_{2,4t+2}$ (for λ odd) into *LOL* graphs which does not exists, by the FT, a decomposition of a $\lambda K_{m,n}$ into *LOL* graphs exists (since any odd number can be written as a linear combination of 2 and 3), i.e., the necessary conditions for decomposing a $\lambda K_{m,n}$ for λ odd into *LOL* graphs are sufficient.

If $\lambda = 4$, the necessary conditions are $m \geq 2$ and $n \geq 2$. By the FT, a decomposition of a $4K_{2,2}$ into *LOL* graphs exists since the existence of a decomposition of a $2K_{2,2}$. A $4K_{2,3}$ on $V_1 = \{a, b\}$ and $V_2 = \{1, 2, 3\}$ can be decomposed into six *LOL* graphs $\{b1a2, b2a3, b3a1, a1b2, a2b3, a3b1\}$. By the FT, a decomposition of a $4K_{m,n}$ into *LOL* graphs exists, and hence a $4tK_{m,n}$.

For any even λ (i.e., $\lambda = 4t$ or $4t+2$), we know that a decomposition of a $4tK_{m,n}$ into *LOL* graphs exists. For $\lambda = 4t+2$, the necessary conditions ($m \geq 2$ and $n \geq 2$and $mn \equiv 0 (\text{mod } 2)$) are the same as the necessary conditions for $\lambda = 2$. Except for a decomposition of a $(4t+2)K_{2,2s+1}$ into *LOL* graphs which does not exists, by the FT, a decomposition of a $(4t+2)K_{m,n}$ into *LOL* graphs exists. That is, the necessary conditions for decomposing a $\lambda K_{m,n}$ for λ even into *LOL* graphs are sufficient. The following theorem concludes the results obtained in this section.

THEOREM 10. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into LOL graphs are sufficient, except for a $(2s+1)K_{2,4t+2}$ or a $(4t+2)K_{2,2s+1}$ (where a decomposition into LOL graphs does not exist).*

6.2.3. *LLO graph decomposition.* The last graph in Figure 4 is an example of an *LLO* graph which is defined as follows.

DEFINITION 6. *Let $V = \{a, b, c, d\}$. An LLO graph $< a, b, c, d >$ on V is a graph where the frequency of edges $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ are 1, 1 and 2, respectively. We write abcd to denote an LLO graph $< a, b, c, d >$ when there is no confusion.*

The necessary conditions for the decomposition of a $\lambda K_{m,n}$ into *LLO* graphs are $m \geq 2$ and $n \geq 2$ and λmn is $0 (\text{mod } 4)$. If $\lambda = 2$, then $2mn \equiv 0 (\text{mod } 4)$ implies m is even or n is even. A $2K_{2,2}$ on $V_1 = \{a, b\}$

and $V_2 = \{1,2\}$ can be decomposed into two LLO graphs $\{a1b2, b1a2\}$. A $2K_{2,3}$ on $V_1 = \{a,b\}$ and $V_2 = \{1,2,3\}$ can be decomposed into three LLO graphs $\{2a1b, 1a3b, 3a2b\}$. By the FT, a decomposition of a $2K_{2t,n}$ into LLO graphs exists, i.e., necessary conditions for the decomposition are sufficient for $\lambda = 2$.

If $\lambda = 4$, the necessary conditions are $m \geq 2$ and $n \geq 2$. A $4K_{2,2}$ on $V_1 = \{a,b\}$ and $V_2 = \{1,2\}$ can be decomposed into four LLO graphs $\{a1b2, 1b2a, b2a1, 2a1b\}$. A $4K_{2,3}$ on $V_1 = \{a,b\}$ and $V_2 = \{1,2,3\}$ can be decomposed into six LLO graphs $\{a1b2, a2b3, a3b1, b1a2, b2a3, b3a1\}$. By the FT, a decomposition of a $4K_{m,n}$ into LLO graphs exists, and hence that of a $4tK_{m,n}$.

For any even λ (i.e., $\lambda = 4t$ or $4t+2$), we know that a decomposition of a $4tK_{m,n}$ into LLO graphs exists. For $\lambda = 4t+2$, the necessary conditions ($m \geq 2$ and $n \geq 2$and $mn \equiv 0 (\text{mod } 2)$) are the same as the necessary conditions for $\lambda = 2$. By the FT, a decomposition of a $(4t+2)K_{m,n}$ into LLO graphs exists. That is, the necessary conditions for decomposing a $\lambda K_{m,n}$ for λ even into LLO graphs are sufficient.

For $\lambda = 3$, the necessary condition $3mn \equiv 0 (\text{mod } 4)$ implies that either m or $n \equiv 0 (\text{mod } 4)$, or m and n are both $\equiv 0 (\text{mod } 2)$. A $3K_{2,2}$ (on $V_1 = \{a,b\}$ and $V_2 = \{1,2\}$) cannot be decomposed into LLO graphs. This is also true for a $\lambda K_{2,2}$ where λ is odd. There are four edges in a $(2k+1)K_{2,2}$, and the degree of each vertex is even $(2(2k+1))$. Also, there are $2k+1$ LLO graphs in a decomposition if it exists. In an LLO graph in a decomposition, either both a and b occur with degree 2 or both 1 and 2 occur with degree 2. Also, if a and b occur with degree 2, then both 1 and 2 occur with odd degrees, and vice versa. If there are s LLO graphs in the decomposition where a and b have even degrees (1 and 2 have odd degrees) and t LLO graphs where a and b have odd degrees (1 and 2 have even degrees), then $s+t = 2k+1$ which is odd. Since the degree of a in the $(2k+1)K_{2,2}$ is even, t must be even. Similarly, s must be even (since the degree of 1 in the $(2k+1)K_{2,2}$ is even). We have $s+t$ is even, a contradiction to that $s+t$ is odd.

A $3K_{2,3}$ cannot be decomposed into LLO graphs because necessary conditions are not satisfied. A $3K_{2,4}$ (on $V_1 = \{a,b\}$ and $V_2 = \{1,2,3,4\}$) can be decomposed into six LLO graphs $\{b4a1, b4a2, b1a3, a2b1, a3b2, a4b3\}$. A $3K_{3,4}$ on $V_1 = \{a,b,c\}$ and $V_2 = \{1,2,3,4\}$ can be decomposed into nine LLO graphs $\{3b1a, 1c2a, c4a3, 3c2b, a1b3, 2a4b, b2c3, 3a4c, 4b1c\}$. By the FT, a decomposition of a $3K_{m,4t}$ into LLO graphs exists. A $3K_{2,6}$ (on $V_1 = \{a,b\}$ and $V_2 = \{1,\ldots,6\}$) can be decomposed into nine LLO

graphs $\{6b1a, 1b2a, 1b3a, b6a4, b6a5, 1a2b, 6a3b, a5b4, a4b5\}$. By the FT, a decomposition of a $3K_{2,4t+2}$ into LLO graphs exists (since a decomposition of a $3K_{2,4(t-1)}$ exists and a decomposition of a $3K_{2,6}$ exists). Also, by the FT, a decomposition of a $3K_{4s+2,4t+2}$ ($s \geq 1$ and $t \geq 1$) into LLO graphs exists (since a decomposition of a $3K_{2,4t+2}$ exists and a decomposition of a $3K_{4s,4t+2}$ exists). We have shown that the necessary conditions for decomposing a $3K_{m,n}$ into LLO graphs are sufficient for $\lambda = 3$.

For any odd λ, the necessary conditions ($m \geq 2$ and $n \geq 2$ and $mn \equiv 0 (\text{mod } 4)$) are the same as the necessary conditions for $\lambda = 3$. Notices that these conditions are also included in the necessary conditions for $\lambda = 2$. Except for a decomposition of a $(2k+1)K_{2,2}$ into LLO graphs which does not exists, by the FT, a decomposition of a $\lambda K_{m,n}$ into LLO graphs exists, i.e., the necessary conditions for decomposing a $\lambda K_{m,n}$ for λ odd into LLO graphs are sufficient. The following theorem concludes the results obtained in this section.

THEOREM 11. *The necessary conditions of decomposing a $\lambda K_{m,n}$ into LLO graphs are sufficient, except for a $(2k+1)K_{2,2}$ (where a decomposition into LLO graphs does not exist).*

COROLLARY 3. *The necessary conditions are sufficient for the decomposition of a $\lambda K_{m,n}$ into graphs of 4 vertices and 4 edges, except for decompositions of a $(2s+1)K_{2,4t+2}$ and a $(4t+2)K_{2,2s+1}$ into LOL graphs which do not exist, and a $(2k+1)K_{2,2}$ where the decomposition into LLO graphs does not exist.*

7. Energy of a graph

The Huckel Molecular Orbital theory provided the motivation for the idea of the energy of a graph: the sum of the absolute values of the eigenvalues associated with the graph (see [2]). In this section we show that the sum of the energies of the decomposed LO subgraphs is greater than the energy of the original graph.

DEFINITION 7. *Define $\alpha_f(G, \zeta) = \min_P \sum_i f(G_i)$, where $P = \{G_1, \dots G_t\}$ ranges over all ζ-decompositions of G and $f(G_i)$ is some non-negative value (such as weight or energy, see below) assigned to G_i, or its cost function.*

DEFINITION 8. *The sum of the absolute values of the eigenvalues of adjacency matrix of graph G is called the energy of the graph.*

In other words, the energy of a graph G is $E = \Sigma_1^n |\lambda_i|$; λ_i are eigenvalues of the adjacency matrix of G ([2]).

EXAMPLE 1. *Consider a $3K_{1,2}$ with $V_1 = \{0\}$ and $V_2 = \{1, 2\}$ and a decomposition of the $3K_{1,2}$ into two LO graphs:*

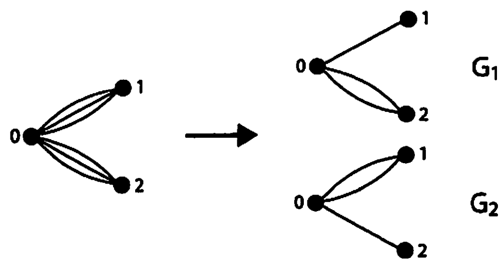

FIGURE 5. Decomposition of a $3K_{1,2}$ into 2 LO graphs

The adjacency matrix of graph G_1 is:

$$M(G_1) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

The adjacency matrix of graph G_2 is:

$$M(G_2) = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The adjacency matrix of the $3K_{1,2}$ is the sum of the first two

$$\begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Note that the non-zero entries occur on the edges of the square of the matrix. Energy of G_1 = energy of $G_2 = 2 * \sqrt{5}$ while energy of $3K_{1,2} = E(3K_{1,2}) = 6*\sqrt{2}$. Thus, if $f(G_i) = E(G_i)$, we have $\alpha_f(3K_{1,2}, \zeta) = 4*\sqrt{5} > 6*\sqrt{2}$ where ζ is an LO-decomposition of $3K_{1,2}$.

THEOREM 12. *An LO-decomposition of a* $\lambda K_{m,n}$ *yields* $\alpha_f(\lambda K_{m,n}, LO) > 2\lambda * \sqrt{mn}$ *where* $f(G_i) = E(G_i) = 2\sqrt{5}$.

Proof: Each LO graph has 3 edges and energy $2\sqrt{5}$. Also the total number of edges of a $\lambda K_{m,n}$ is λmn. Thus,

$$\alpha_f(\lambda K_{m,n}, LO) = \frac{\lambda mn}{3} * 2\sqrt{5} \qquad (1)$$

On the other hand since the non-zero eigenvalues of a $K_{m,n}$ are $\pm\sqrt{mn}$, the energy of a $\lambda K_{m,n}$ is:

$$\lambda 2\sqrt{mn} \qquad (2)$$

Since $2\lambda \neq 0$ is common to (1) and (2), we have to compare $\frac{mn\sqrt{5}}{3}$ and $\sqrt{mn}$. We see that since one of m, n must be greater than 1 (LO graphs have 3 vertices), we must have $mn \geq 2 \Rightarrow \sqrt{mn} \geq \sqrt{2}$. Also, $\sqrt{10} > 3$. Thus,

$\frac{mn*\sqrt{5}}{3} = \frac{\sqrt{mn}*\sqrt{mn}*\sqrt{5}}{3} \geq \frac{\sqrt{2}*\sqrt{5}}{3} *\sqrt{mn} = \frac{\sqrt{10}}{3} *\sqrt{mn} > \sqrt{mn}$ yielding our result that (1) > (2) for all possible LO decompositions of a $\lambda K_{m,n}$. □

8. Summary

In this paper a fundamental theorem on the decomposition of a $\lambda K_{m,n}$ is presented and applied to obtain decompositions of a $\lambda K_{m,n}$ into subgraphs having four or less vertices and edges. Applying the fundamental theorem to prove that certain necessary conditions are sufficient reduces the proofs to find examples of decompositions for certain small bipartite graphs. We also showed in several cases where decompositions do not exist even when the necessary conditions are satisfied. It is evident that the fundamental theorem will be beneficial to others since the same idea can be applied to find the decompositions of a $\lambda K_{m,n}$ into subgraphs with vertices or edges more than four.

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