When all minimal k-vertex separators induce complete or edgeless subgraphs

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Abstract

Define $\mathcal{D}_k$ to be the class of graphs such that, for every independent set $\{v_1, \ldots, v_h\}$ of vertices with $2 \leq h \leq k$, if S is an inclusion-minimal set of vertices whose deletion would put $v_1, \ldots, v_h$ into h distinct connected components, then S induces a complete subgraph; also, let $\mathcal{D} = \bigcap_{k \geq 2} \mathcal{D}_k$. Similarly, define $\mathcal{D}'_k$ and $\mathcal{D}'$ with "complete" replaced by "edgeless," and define $\mathcal{D}^*_k$ and $\mathcal{D}^*$ with "complete" replaced by "complete or edgeless." The class $\mathcal{D}_2$ is the class of chordal graphs, and the classes $\mathcal{D}$, $\mathcal{D}'_2$, and $\mathcal{D}^*_2$ have also been characterized recently. The present paper gives unified characterizations of all of the classes $\mathcal{D}_k$, $\mathcal{D}'_k$, $\mathcal{D}^*_k$, $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}^*$.

1 Introduction: minimal k-vertex separators

When G is a graph with an independent set $\mathcal{I}$ of two or more vertices, call $S \subset V(G)$ an *$\mathcal{I}$-separator* of G if the vertices of $\mathcal{I}$ are in $|\mathcal{I}|$ separate connected components of the subgraph of G that is induced by $V(G)-S$; a *minimal $\mathcal{I}$-separator* is an inclusion-minimal $\mathcal{I}$-separator. When $|\mathcal{I}| = k$, these will also be called *(minimal) k-vertex separators.* Minimal 2-vertex separators have long been studied as "minimal separators" (sometimes called "minimal vertex separators" or "minseps"); see [1, 3, 9].

The present paper will characterize the graphs for which every minimal k-vertex separator induces a complete subgraph, an edgeless subgraph, or a complete or edgeless subgraph. Each of these characterizations will imply the existence of a forbidden induced subgraph characterization of the graph class, and so the existence of a polynomial-time recognition algorithm.

Let $v \sim w$ [and $v \not\sim w$] denote that vertices v and w are adjacent [respectively, nonadjacent]. The *components* of a graph are simply its connected components. For any $S \subset V(G)$, let $G - S$ denote the subgraph of

G that is induced by $V(G) - S$. If S is a minimal $\mathcal{I}$-separator of G, then the minimality of S implies that each $x \in S$ has neighbors x' and x'' in, respectively, components G' and G'' of $G - S$ such that G' and G'' contain different elements of $\mathcal{I}$. Observe that vertices in different components of $G - S$ cannot be adjacent in G.

2 Complete minimal k-vertex separators

Let $\mathcal{D}_k$ denote the class of graphs such that every minimal $\mathcal{I}$-separator induces a complete subgraph whenever $2 \leq |\mathcal{I}| \leq k$, and call such graphs *complete minimal k-vertex separator graphs*. Let $\mathcal{D} = \bigcap_{k\geq 2} \mathcal{D}_k$, and note that $\mathcal{D} \subseteq \cdots \subseteq \mathcal{D}_4 \subseteq \mathcal{D}_3 \subseteq \mathcal{D}_2$.

Proposition 1, from Dirac's seminal paper [2], shows that $\mathcal{D}_2$ is the familiar class of *chordal graphs*; see [1, 9].

Proposition 1 *A graph is in $\mathcal{D}_2$ if and only if no induced subgraph is a cycle of length* 4 *or more.*

Figure 1 shows a graph that is in $\mathcal{D}_3$ (and $\mathcal{D}_2$), but is not in $\mathcal{D}_4$. Each minimal 3-vertex separator is an complete set (of cardinality 2), but $\{s_1, s_2, s_3, s_4\}$ is a minimal $\{r_1, r_2, r_3, r_4\}$-separator that does not induce an complete subgraph.

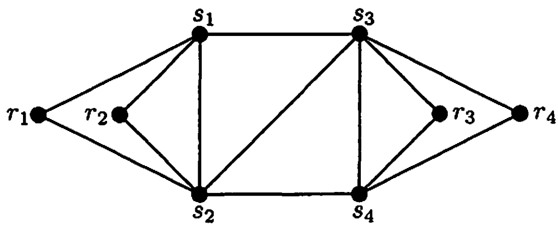

Figure 1: A graph in $\mathcal{D}_3$ but not $\mathcal{D}_4$.

Proposition 2, from [7], characterizes $\mathcal{D}$, where C_n [and P_n] denote the cycle [respectively, path] on n vertices (so C_n has length n and P_n has length $n-1$), and $2P_3$ denotes the graph that consists of two components, each a copy of P_3.

Proposition 2 *A graph is in $\mathcal{D}$ if and only if no induced subgraph is isomorphic to C_4, C_5, P_5, or $2P_3$.*

The graph G in Figure 1 is not in $\mathcal{D}$, since deleting the vertices s_2 and s_3 would leave an induced subgraph isomorphic to $2P_3$. Inserting an additional edge s_1s_4 into G would produce a graph that is in $\mathcal{D}$.

Define a *chord of a set* $\{\pi_1, \ldots, \pi_s\}$ *of paths* (or a *chord of a path* π_1 if $s = 1$) to be an edge xy where $x, y \in V(\pi_1) \cup \cdots \cup V(\pi_s)$ but $xy \notin E(\pi_1) \cup \cdots \cup E(\pi_s)$. Define a *$\pi_i$-to-$\pi_j$ chord* to be a chord $x_i x_j$ of $\{\pi_i, \pi_j\}$ where $x_i \in V(\pi_i) - V(\pi_j)$ and $x_j \in V(\pi_j) - V(\pi_i)$. Let π_i° denote the set of vertices of the path π_i that are not endpoints of π_i, and call paths π_i and π_j *internally disjoint* when $\pi_i^\circ \cap \pi_j^\circ = \emptyset$. Motivated by the definition of "chordless cycle" in [11], define a *chordless path* to be a path of length at least 2 that has no chord—thus, chordless paths are induced paths that are long enough for a chord to have been possible.

Theorem 3 will generalize (and prove again) both Propositions 1 and 2.

Theorem 3 *A graph is in $\mathcal{D}_k$ if and only if no two internally disjoint chordless paths π_1 and π_2 have a total of at most* $\min\{k, 4\}$ *distinct endpoints with no π_1-to-π_2 chord. Consequently, $\mathcal{D}_k = \mathcal{D}_4$ whenever $k \geq 4$ and $\mathcal{D} = \mathcal{D}_4$.*

Proof. First suppose two internally disjoint chordless paths π_1 and π_2 of G have a total of at most $\min\{k, 4\}$ distinct endpoints with no π_1-to-π_2 chord (arguing by contraposition). Let $\mathcal{I}$ be the set of endpoints of π_1 and π_2 (so $\mathcal{I}$ is independent with $2 \leq |\mathcal{I}| \leq 4$), and let $S = \{x_1, x_2\}$ where $x_1 \not\sim x_2$ and each $x_i \in \pi_i^\circ$. But then S is a minimal $\mathcal{I}$-separator of the subgraph of G induced by $V(\pi_1) \cup V(\pi_2)$, and so S is contained in a minimal $\mathcal{I}$-separator of G where the subgraph induced by S is not complete (since $x_1 \not\sim x_2$) and $2 \leq |\mathcal{I}| \leq \min\{k, 4\} \leq k$. Therefore, $G \notin \mathcal{D}_k$.

Conversely, suppose $G \notin \mathcal{D}_k$, say with S a minimal $\mathcal{I}$-separator of G that does not induce a complete subgraph and has $2 \leq |\mathcal{I}| \leq k$. To be specific, suppose $x_1, x_2 \in S$ where $x_1 \not\sim x_2$. By the minimality of S, each x_i has neighbors x_i' and x_i'' in, respectively, components G_i' and G_i'' of $G - S$.

Case $1'$: $G_1' \neq G_2'$. Define paths τ_1' and τ_2' to consist of, respectively, the single edge $x_1' x_1$ and the single edge $x_2' x_2$.

Case $2'$: $G_1' = G_2'$. Let σ' denote a chordless x_1-to-x_2 path in G with $\sigma'^\circ \subset V(G_1')$, and choose $v' \in \sigma'^\circ$. Define paths τ_1' and τ_2' to consist of, respectively, the v'-to-x_1 and v'-to-x_2 subpaths of σ'.

Now consider similar *cases* $1''$ *and* $2''$ for the components G_1'' and G_2'' of $G - S$ by replacing each superscript $'$ with $''$ in the corresponding case above so as to define paths τ_1'' and τ_2''. Finally, for each $i \in \{1, 2\}$, define π_i to be the chordless path $\tau_i' \cup \tau_i''$ in G (noting that each $x_i \in \pi_i^\circ$). In every combination of cases $1', 2'$ and $1'', 2''$, the two paths π_1, π_2 will be internally disjoint and chordless with a total of at most $\min\{k, 4\}$ distinct endpoints, with no π_1-to-π_2 chord.

Consequently, if $k \geq 4$, then $\min\{k, 4\} = 4$ and the preceding two paragraphs show $\mathcal{D}_k = \mathcal{D}_4$, and so $\mathcal{D} = \mathcal{D}_4$. □

To illustrate Theorem 3 in Figure 1 with $k = 4$, $\mathcal{I} = \{r_1, r_2, r_3, r_4\}$, and $S = \{s_1, s_2, s_3, s_4\}$, observe that the paths $\pi_1 = r_1, s_1, r_2$ and $\pi_2 = r_3, s_4, r_4$ have four distinct endpoints with no π_1-to-π_2 chord.

Theorem 3 has Propositions 1 and 2 as consequences; in the latter's characterization of $\mathcal{D} = \mathcal{D}_4$, the subgraphs C_4 and C_5 arise when π_1 and π_2 share the same two endpoints, and the subgraphs P_5 and $2P_3$ arise when π_1 and π_2 have, respectively, a total of three or four endpoints. Similarly, $\mathcal{D}_3$ can be characterized by no induced subgraph being isomorphic to C_4, C_5, or P_5.

3 Edgeless minimal k-vertex separators

Let $\mathcal{D}'_k$ denote the class of graphs such that every minimal $\mathcal{I}$-separator induces an edgeless subgraph whenever $2 \leq |\mathcal{I}| \leq k$, and call such graphs *edgeless minimal k-vertex separator graphs.* Let $\mathcal{D}' = \bigcap_{k \geq 2} \mathcal{D}'_k$, and note that $\mathcal{D}' \subseteq \cdots \subseteq \mathcal{D}'_4 \subseteq \mathcal{D}'_3 \subseteq \mathcal{D}'_2$.

Proposition 4, from [6], shows that $\mathcal{D}'_2$ is also the graph class characterized in [10], whose members are called *unichord-free graphs* in [4, 5].

Proposition 4 *A graph is in $\mathcal{D}'_2$ if and only if no cycle has a unique chord.*

Figure 2 shows a graph that is in $\mathcal{D}'_3$ (and $\mathcal{D}'_2$), but is not in $\mathcal{D}'_4$. Each minimal 3-vertex separator is an independent set (of cardinality 3 or 4), but $\{s_1, s_2, s_3, s_4\}$ is a minimal $\{r_1, r_2, r_3, r_4\}$-separator that does not induce an edgeless subgraph.

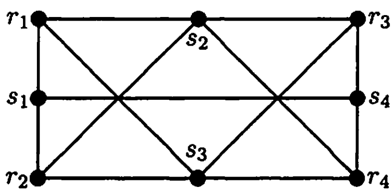

Figure 2: A graph in $\mathcal{D}'_3$ but not $\mathcal{D}'_4$.

Theorem 5 will generalize (and reprove) Proposition 4.

Theorem 5 *A graph is in $\mathcal{D}'_k$ if and only if no two internally disjoint chordless paths π_1 and π_2 have a total of at most* $\min\{k, 4\}$ *distinct endpoints with a unique π_1-to-π_2 chord x_1x_2 and each $x_i \in \pi_i^\circ$. Consequently, $\mathcal{D}'_k = \mathcal{D}'_4$ whenever $k \geq 4$ and $\mathcal{D}' = \mathcal{D}'_4$.*

Proof. The proof given above for Theorem 3 will also prove Theorem 5 after making the following four substitutions: replace each $\mathcal{D}$ (subscripted or not) with $\mathcal{D}'$; replace "no π_1-to-π_2 chord" with "a unique π_1-to-π_2 chord x_1x_2 and each $x_i \in \pi_i^\circ$"; replace "$x_1 \not\sim x_2$" with "$x_1 \sim x_2$"; and replace "complete" with "edgeless." □

To illustrate Theorem 5 in Figure 2 with $k = 4$, $\mathcal{I} = \{r_1, r_2, r_3, r_4\}$, and $S = \{s_1, s_2, s_3, s_4\}$, observe that the paths $\pi_1 = r_1, s_1, r_2$ and $\pi_2 = r_3, s_4, r_4$ have four distinct endpoints with the unique π_1-to-π_2 chord s_1s_4 and $s_1 \in \pi_1^\circ$ and $s_4 \in \pi_2^\circ$.

Theorem 5 has Proposition 4 as a consequence. Moreover, a graph is in $\mathcal{D}' = \mathcal{D}'_4$ if and only if it contains none of the graphs shown in Figure 3 as an induced subgraph; in each of these six graphs, the "hollow vertices"

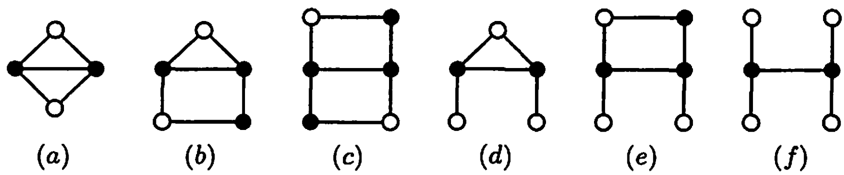

Figure 3: The six forbidden induced subgraphs for $\mathcal{D}' = \mathcal{D}'_4$.

show possible endpoints for the two internally disjoint paths π_1 and π_2, with the "horizontal edge" halfway up being their unique π_1-to-π_2 chord. In subgraphs (a), (b), and (c), the two paths share the same two endpoints; in (d) and (e), they share a total of three endpoints; in (f), they have a total of four endpoints. (Similarly, $\mathcal{D}'_3$ can be characterized by no induced subgraph being a cycle with pendant edges attached at two consecutive vertices or being one of the subgraphs (a), (b), or (c) in Figure 3.)

The graph G in Figure 2 is not in $\mathcal{D}'$, since deleting vertices s_2 and s_3 would leave an induced subgraph isomorphic to subgraph (f) in Figure 3. Deleting the edge s_1s_4 from G would produce a graph that is in $\mathcal{D}'$.

4 Extreme minimal k-vertex separators

Let $\mathcal{D}_k^*$ denote the class of graphs such that every minimal $\mathcal{I}$-separator induces a complete or edgeless subgraph whenever $2 \le |\mathcal{I}| \le k$, and (motivated by [8]) call such graphs *extreme minimal k-vertex separator graphs.* Let $\mathcal{D}^* = \bigcap_{k\ge2} \mathcal{D}_k^*$, and note that $\mathcal{D}^* \subseteq \cdots \subseteq \mathcal{D}_4^* \subseteq \mathcal{D}_3^* \subseteq \mathcal{D}_2^*$. Of course, $\mathcal{D}_k \cup \mathcal{D}'_k \subseteq \mathcal{D}_k^*$ and $\mathcal{D} \cup \mathcal{D}' \subseteq \mathcal{D}^*$.

Proposition 6, from [8], characterizes $\mathcal{D}_2^*$, where a *block* is an inclusion-maximal 2-connected subgraph and a graph G is the *edge sum* of graphs G_1 and G_2 if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ where $V(G_1) \cap V(G_2) = \{x, y\}$ and $E(G_1) \cap E(G_2) = \{xy\}$.

Proposition 6 *A graph is in $\mathcal{D}_2^*$ if and only if every block is the edge sum of graphs in $\mathcal{D}_2 \cup \mathcal{D}_2'$.*

Both Figure 1 and 2 show graphs that are in $\mathcal{D}_3^*$ (and $\mathcal{D}_2^*$), but are not in $\mathcal{D}_4^*$. Each minimal 3-vertex separator is an independent set (of cardinality 3 or 4), but $\{s_1, s_2, s_3, s_4\}$ is a minimal $\{r_1, r_2, r_3, r_4\}$-separator that induces neither a complete nor an edgeless subgraph.

Although there does not seem to be a direct generalization of Proposition 6 to $\mathcal{D}_k^*$ when $k \geq 3$, Theorem 7 will characterize each class $\mathcal{D}_k^*$ (along with $\mathcal{D}^*$) while generalizing the results and proofs of sections 1 and 2.

Theorem 7 *A graph is in $\mathcal{D}_k^*$ if and only if no three internally disjoint chordless paths π_1, π_2, and π_3 have a total of at most $\min\{k, 6\}$ distinct endpoints with no π_1-to-π_2 chord and a unique π_2-to-π_3 chord x_2x_3 and $x_2 \in \pi_2^\circ$ and $x_3 \in \pi_3^\circ$. Consequently, $\mathcal{D}_k^* = \mathcal{D}_6^*$ whenever $k \geq 6$ and $\mathcal{D}^* = \mathcal{D}_6^*$.*

Proof. First suppose three internally disjoint chordless paths π_1, π_2, and π_3 of G have a total of at most $\min\{k, 6\}$ distinct nonadjacent endpoints with no π_1-to-π_2 chord and a unique π_2-to-π_3 chord x_2x_3 and $x_2 \in \pi_2^\circ$ and $x_3 \in \pi_3^\circ$ (arguing by contraposition). Let $\mathcal{I}$ be the set of endpoints of π_1, π_2, and π_3 (so $\mathcal{I}$ is independent with $2 \leq |\mathcal{I}| \leq 6$), and let $S = \{x_1, x_2, x_3\}$ where $x_1 \not\sim x_2 \sim x_3$ and each $x_i \in \pi_i^\circ$. But then S is a minimal $\mathcal{I}$-separator of the subgraph of G induced by $V(\pi_1) \cup V(\pi_2) \cup V(\pi_3)$, and so S is contained in a minimal $\mathcal{I}$-separator of G where the subgraph induced by S is neither complete (because $x_1 \not\sim x_2$) nor edgeless (since $x_2 \sim x_3$) and $2 \leq |\mathcal{I}| \leq \min\{k, 6\} \leq k$. Therefore, $G \notin \mathcal{D}_k^*$.

Conversely, suppose $G \notin \mathcal{D}_k^*$, say with S a minimal $\mathcal{I}$-separator of G that does not induce a complete or edgeless subgraph and has $2 \leq |\mathcal{I}| \leq k$. To be specific, suppose $x_1, x_2, x_3 \in S$ where $x_1 \not\sim x_2 \sim x_3$ (x_1 might or might not be adjacent to x_3). By the minimality of S, each x_i has neighbors x_i' and x_i'' in, respectively, components G_i' and G_i'' of $G - S$. Assume in addition that S and x_1, x_2, x_3 have been chosen to make $|V(G'')|$ a minimum.

Case 1′: no two of G_1', G_2', G_3' are equal. For each $i \in \{1, 2, 3\}$, define a path τ_i' to consist of the single edge $x_i'x_i$.

Case 2′: exactly two of G_1', G_2', G_3' are equal—say $G_i' = G_j' \neq G_h'$ where $\{h, i, j\} = \{1, 2, 3\}$. Let σ' denote a chordless x_i-to-x_j path in G with $\sigma'^\circ \subset V(G_i')$, and choose $v' \in \sigma'^\circ$. Define paths τ_i' and τ_j' to consist of, respectively, the v'-to-x_i and v'-to-x_j subpaths of σ'. Define τ_h' to consist of the single edge $x_h'x_h$.

Case 3′: $G_1' = G_2' = G_3'$. Let H' be the subgraph of G induced by $V(G_1') \cup \{x_1, x_2, x_3\}$ except without the edge x_2x_3 (and without x_1x_3 if $x_1 \sim x_3$). Let T' be a subtree of H' with leaves x_1, x_2, x_3 and with $|E(T')|$ a minimum; specifically, say T' consists of a vertex v' and three v'-to-x_i

paths (each either chordless or the single edge $v'x_i$). Let G' be the subgraph of H' induced by $V(T')$, and note that $\{x_1, x_2, x_3\}$ is independent in G'. Each edge $E(G') - E(T')$ must join two neighbors of v' in T' (otherwise, that edge could be inserted into T' and two or more edges deleted so as to contradict the minimality of $|E(T')|$). Similarly, there can be at most two edges in $E(G') - E(T')$ that join neighbors of v' in T' (if there were three, then two of them could be inserted into T' and the three edges incident with v' deleted so as to contradict the minimality of $|E(T')|$). Thus, one of the following three subcases occurs:

$3'_0$: The subgraph $G' = T'$ has no triangles and so consists of the three internally disjoint v'-to-x_i paths; take $a' = v'$.

$3'_1$: The subgraph of G' induced by $V(T')$ consists of one triangle $a'b'c'$ and three vertex disjoint paths from $\{a', b', c'\}$ to $\{x_1, x_2, x_3\}$ where at most one of a', b', c' is in the independent set $\{x_1, x_2, x_3\}$; without loss of generality, suppose $a' \notin \{x_1, x_2, x_3\}$.

$3'_2$: The subgraph of G' induced by $V(T')$ consists of two triangles with vertices in $\{a', b', c', d'\}$ and three vertex disjoint paths from $\{a', b', c', d'\}$ to $\{x_1, x_2, x_3\}$ where either $\{a', b', c', d'\} \cap \{x_1, x_2, x_3\} = \emptyset$ or exactly one of the two vertices of $\{a', b', c', d'\}$ that are in both triangles (and neither of the other two vertices) is in the independent set $\{x_1, x_2, x_3\}$; without loss of generality, suppose $a' \notin \{x_1, x_2, x_3\}$.

In each subcase, define internally disjoint paths $\tau'_1, \tau'_2, \tau'_3$ such that each τ_i is the a'-to-x_i path in G' (and so is chordless or a single edge).

Now consider similar *cases* $1''$, $2''$, *and* $3''$ (with three subcases $3''_i$) for the components G''_1, G''_2, G''_3 of $G - S$ by replacing each superscript $'$ with $''$ in the corresponding case above so as to define paths τ''_1, τ''_2, and τ''_3. Subcases $3''_1$ and $3''_2$ cannot occur (otherwise, one or two of x_1, x_2, x_2 could have been replaced by one or two vertices of G'', contradicting the assumed minimality of $|V(G'')|$). Thus, case $3''$ will always have $G'' = T''$. Finally, for each $i \in \{1, 2, 3\}$, define π_i to be the chordless path $\tau'_i \cup \tau''_i$ in G (noting that each $x_i \in \pi_i^\circ$).

If neither subcase $3'_1$ nor $3'_2$ occurs, then no chord will exist between τ'_i and τ'_j in G' and the paths π_1, π_2, π_3 will be as described in the theorem.

On the other hand, if subcase $3'_1$ or $3'_2$ does occur, then there were at least two choices for a', and so a' can be chosen so as either to have $\{i, j\} = \{1, 3\}$ for every chord between τ'_i and τ'_j in G' or to have $\{i, j\} = \{1, 2\}$ for every such chord. (Either way, there will only be one or two such chords.) If $\{i, j\} = \{1, 3\}$ always holds, then the three paths π_1, π_2, π_3 will be as described in the theorem. If $\{i, j\} = \{1, 2\}$ always holds, then interchanging π_1 and π_2 (do this if $x_1 \sim x_3$) or interchanging π_2 and π_3 (do this if $x_1 \not\sim x_3$) will make the edges of $E(G') - E(T')$ all into π_1-to-π_3 chords, and the new π_1, π_2, π_3 will be as described in the theorem.

Therefore, in every combination of cases $1', 2', 3'$ and $1'', 2'', 3''$, the three paths π_1, π_2, π_3 will end up being internally disjoint and chordless with a total of at most $\min\{k, 6\}$ distinct endpoints, with no π_1-to-π_2 chord and with a unique π_2-to-π_3 chord x_2x_3 and $x_2 \in \pi_2^\circ$ and $x_3 \in \pi_3^\circ$.

Consequently, if $k \geq 6$, then $\min\{k, 6\} = 6$ and the preceding paragraphs show $\mathcal{D}_k^* = \mathcal{D}_6^*$, and so $\mathcal{D}^* = \mathcal{D}_6^*$. □

To illustrate Theorem 7 in Figure 1 with $k = 4$, $\mathcal{I} = \{r_1, r_2, r_3, r_4\}$, and $S = \{s_1, s_2, s_3, s_4\}$, observe that the paths $\pi_1 = r_1, s_1, r_2$ and $\pi_2 = r_3, s_4, r_4$ and $\pi_3 = r_1, s_2, r_2$ share four distinct endpoints with no π_1°-to-π_2° chord and the unique π_2°-to-π_3° chord s_2s_4. Similarly in Figure 2, the paths $\pi_1 = r_1, s_2, r_2$ and $\pi_2 = r_1, s_1, r_2$ and $\pi_3 = r_3, s_4, r_4$ share four distinct endpoints with no π_1°-to-π_2° chord and the unique π_2°-to-π_3° chord s_1s_4.

Theorem 7 would also lead to forbidden induced subgraph characterizations, much as Theorems 3 and 5 did; for instance, characterizing $\mathcal{D}^* = \mathcal{D}_6^*$ would require 20 forbidden subgraphs. By Proposition 2 and Theorems 5 and 7, the "house graph"—shown as (*b*) in Figure 3—is the smallest graph that is in $\mathcal{D}^*$ but is not in $\mathcal{D}$ or $\mathcal{D}'$.

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