# An Elementary Computation of the Conway Polynomial for $(m, 3)$ and $(m, 4)$ Torus Links

sarah-marie belcastro and Dana Rowland

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**Abstract**

Using only the skein relation and some combinatorics, we find a closed form for the Conway polynomial of $(m, 3)$ torus links and a trio of recurrence relations that define the Conway polynomial of any $(m, 4)$ torus link.

## 1 Introduction

In this paper, we use elementary methods to calculate the Conway polynomial of $(m, 3)$ and $(m, 4)$ torus links for all positive integers $m$. Fix a particular projection of an oriented link and select a particular crossing. Depending on the orientation, the crossing will either appear as in the projection of $L_+$ or $L_-$ as shown in Figure 1. Recall that the Conway polynomial $\nabla$ in one variable $z$

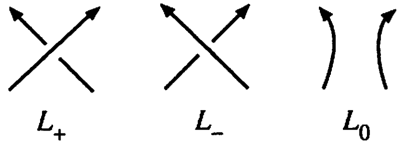



Figure 1: The projections of these three links are identical outside of the region shown.

satisfies the skein relation

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0), \tag{1}$$

where the projections of the links $L_+$ and $L_-$ are related by reversing which strand is on top for the selected crossing and a projection of the link $L_0$ (see Figure 1) is obtained from the projection of $L_+$ or $L_-$ by eliminating the particular crossing and rejoining the strands so that the orientations match.

Direct application of the skein relation yields a formula for $(m, 2)$ torus knots and links, as seen in [5, 15]. Our approach extends this method. By careful application of the skein relation, we determine recurrence relations for the Conway polynomials of $(m, 3)$ and $(m, 4)$ torus links. Using *Mathematica*, we obtain closed forms for the Conway polynomial of $(m, 2)$ and $(m, 3)$ links in Theorems 1 and 2. The recurrence relations for $(m, 4)$ torus links appear in Theorem 3.

The substitution $z = t^{1/2} - t^{-1/2}$ relates the Conway polynomial to the Alexander polynomial $\Delta$ in one variable $t$:

$$\Delta(t) = \nabla(t^{1/2} - t^{-1/2})$$

Therefore, these theorems immediately give formulas for the Alexander polynomials of $(m, 2)$, $(m, 3)$ and $(m, 4)$ torus knots and links as well. (While retrieving a Conway polynomial from a formula for the Alexander polynomial is theoretically straightforward, it is tedious and laborious in practice.)

Other formulas are known for the Alexander polynomial of $(m, n)$ torus knots, with varied methods of proof. Lickorish [9, Chapter 11] applies Fox's free differential calculus to the Wirtinger presentation of the fundamental group of the complement of the torus knot, which has two generators and one relation, to obtain the desired formula. Alternatively, Jones views the torus knots as closed braids and uses the Burau representation and the trace of Hecke algebra representations to obtain a formula for the Alexander polynomial of torus knots (see [4]). Morton gives an explicit formula for the Alexander polynomial of a torus knot, in terms of the family of Jones polynomials of its parallel links [11]. Bayram et al. calculate the Alexander polynomial of $(3, n)$-torus knots from the Alexander matrix using a Maple program [2]. Many known methods for calculating the Alexander polynomial rely on the fact that $m$ and $n$ are relatively prime. The formula for the Alexander polynomial of a torus link with any number of components can then found by making use of the fact that the $(km, kn)$ torus knot is a satellite of the $(m, n)$ torus knot; see e.g. [6]. Our approach to deriving a formula applies directly to links with any number of components; we apply the skein relation and use ambient isotopy to identify relationships among the resulting links.

The outline of this paper is as follows. In Section 2, we review some of the known invariants for torus knots and set up our notation. Sections 3 and 4 contain the proofs of Theorem 2 and Theorem 3 respectively. Finally, Section 5 describes generalization attempts.

## 2 Torus links in general

Torus links are among the most readily-recognized knots. Technically,

**Definition 1.** *A torus link $T$ is a link that can be embedded on the standard (unknotted) torus in $\mathbb{R}^3$.*

Let $m$ and $n$ be positive integers. The usual projection of the torus link $T(m, n)$ wraps $m$ times around the meridian of the torus and $n$ times around the longitude of the torus, and the projection has $m(n-1)$ positive crossings. Note that $T(m,n) = T(-m,-n)$ and $T(-m,n)$ is the mirror image of $T(m,n)$, so it suffices to restrict to the case when $m$ and $n$ are positive. In this paper, we will use a polygonalization of the standard projection, as shown in Figure 2. In this polygonal projection, there are $m$ spokes with $n-1$ crossings per spoke.

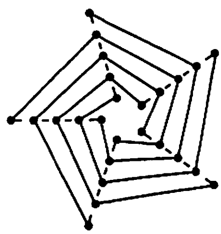


Figure 2: The torus link $T(5,4)$ wraps 5 times around the meridian of the torus and 4 times around the longitude of the torus.

Alternatively, the link $T(m,n)$ can be viewed as the closure of the braid on $n$ strings defined by $(\sigma_1\sigma_2\ldots\sigma_{n-1})^m$. The number of components of $T(m,n)$ is the greatest common divisor of $m$ and $n$. When $m$ and $n$ are relatively prime, $T(m,n)$ is a knot. The link $T(m,n)$ is equivalent to the link $T(n,m)$ by a homeomorphism of the torus.

When $m$ and $n$ are relatively prime, many of the knot invariants for $T(m,n)$ are known, including:

- The least number of crossings in any projection of $T(m,n)$ is $cr(T(m,n)) = \min(m(n-1), n(m-1))$ [4, Theorems 9.7, 15.1]. (This also holds for links; see [12, Proposition 7.5].)
- The unknotting number of $T(m,n)$ is $u(T(m,n)) = \dfrac{(m-1)(n-1)}{2}$ [7].
- The minimal genus of an orientable surface bounding $T(m,n)$ in the 4-ball is $g(T(m,n)) = \dfrac{(m-1)(n-1)}{2}$ [7].
- The Alexander polynomial of $T(m,n)$ is $\Delta_{T(m,n)}(t) = \dfrac{(1-t)(1-t^{mn})t^{-(m-1)(n-1)/2}}{(1-t^m)(1-t^n)}$. This follows from, for example, [9, Chapter 11] or [4, Theorem 9.7].
- The Jones polynomial of $T(m,n)$ is $V_{T(m,n)}(t) = \dfrac{t^{(m-1)(n-1)/2}(1-t^{m+1}-t^{n+1}+t^{m+n})}{1-t^2}$ [4, Proposition 11.9].
- The FLYPMOTH polynomial of $T(m,n)$ is

$$P_{T(m,n)}(a,z) = P_{T(m,n)}((\lambda q)^{1/2}, q^{1/2} - q^{-1/2}) =$$

$$\left(\frac{1-q}{1-q^n}\right)\frac{\lambda^{(n-1)(m-1)/2}}{1-\lambda q}\sum_{\gamma+\beta+1=n;\gamma,\beta\geq 0}(-1)^{\beta}q^{\beta m+\gamma(\gamma+1)/2}\frac{\prod_{j=-\gamma}^{\beta}(q^j-\lambda q)}{\prod_{j=1}^{\beta}1-q^j\prod_{j=1}^{\gamma}1-q^j}$$

[4, Theorem 9.7].

We contribute the following for $(m,2)$ torus links:

**Theorem 1.** *Let $m$ be a positive integer. The Conway polynomial of an $(m,2)$ torus link is*

$$\nabla(T(m,2))=\frac{(z+\sqrt{4+z^2})^m-(z-\sqrt{4+z^2})^m}{2^m\sqrt{4+z^2}}.$$

*Proof.* The torus link $T(1,2)$ is the unknot, with $\nabla(T(1,2))=1$, and the torus link $T(2,2)$ is the Hopf link, with $\nabla(T(1,2))=z$, so the theorem is valid when $m=1$ and $m=2$. For $m>2$, we can apply the skein relation from Equation 1. Let $T(m,2)=L_+$. Then $T(m-2,2)=L_-$ and $T(m-1,2)=L_0$, which produces the recurrence relation

$$\nabla(T(m,2))=\nabla(T(m-2,2))+z\nabla(T(m-1),2)).$$

See [5, 15]. The theorem follows by mathematical induction. Assume

$$\nabla(T(k,2))=\frac{(z+\sqrt{4+z^2})^k-(z-\sqrt{4+z^2})^k}{2^k\sqrt{4+z^2}}$$

for all $k<m$. Then

$$\begin{aligned}\nabla(T(m,2))&=\nabla(T(m-2,2))+z\nabla(T(m-1),2))\\&=\frac{(z+\sqrt{4+z^2})^{m-2}-(z-\sqrt{4+z^2})^{m-2}}{2^{m-2}\sqrt{4+z^2}}\\&\quad+z\frac{(z+\sqrt{4+z^2})^{m-1}-(z-\sqrt{4+z^2})^{m-1}}{2^{m-1}\sqrt{4+z^2}}\\&=\frac{(z+\sqrt{4+z^2})^{m}-(z-\sqrt{4+z^2})^{m}}{2^{m}\sqrt{4+z^2}}\end{aligned}$$

which completes the proof. □

Using the standard polygonal projection, let $c_{ij}$ denote the $j$th crossing on the $i$th spoke, where spokes are labeled clockwise from the negative $x$ axis, $j=1$ corresponds to the outermost crossing, and $j=n-1$ corresponds to the innermost crossing. Note that all the crossings in our projection of $T(m,n)$ are positive crossings. If we apply the skein relation to the above projection of $T(m,n)$ at crossing $c_{ij}$, then we have $L_+=T(m,n)$ which we denote as $T(m,n)[ij]_+$, and we then denote $L_-$ and $L_0$ by $T(m,n)[ij]_-$ and $T(m,n)[ij]_0$ respectively. The skein relation applied to $c_{ij}$ becomes:

$$\nabla(T(m,n)[ij]_+)-\nabla(T(m,n)[ij]_-)=z\nabla(T(m,n)[ij]_0)$$

By applying the skein relation at one crossing $c_{ij}$, we express the Conway polynomial of a torus link in terms of the Conway polynomials of related links that will not themselves be torus links in general. However, when we apply the skein relation at each of the $n-1$ crossings along a particular spoke, one of the links obtained is $T(m,n)[i1]_0[i2]_0\dots[i(n-1)]_0$. In this link, each of the positive crossings along spoke $i$ is eliminated, and the result is a torus link with one less spoke. Therefore, we see that

$$T(m,n)[i1]_0[i2]_0\dots[i(n-1)]_0 = T(m-1,n). \tag{2}$$

This observation motivates our strategy for finding the Conway polynomial for $(m,3)$ and $(m,4)$ torus links.

# 3 $(m,3)$ Torus links

The main result in this section is a closed form for the Conway polynomial of the $(m,3)$ torus links.

**Theorem 2.** *Let $m$ be a positive integer. The Conway polynomial of an $(m,3)$ torus link is* $\nabla(T(m,3)) =$

$$\frac{1}{3+z^2}\left(\frac{(2+z^2+z\sqrt{4+z^2})^m+(2+z^2-z\sqrt{4+z^2})^m}{2^m}-2\cos\left(\frac{2\pi m}{3}\right)\right)$$

The proof will proceed as follows. We first begin with the projection of the $T(m,3)$ link as described in Section 2 and apply two generations of skein relations in order to obtain recurrence relations for $T(m,3)$. By Equation 2 we know that $T(m,3)[11]_0[12]_0$ is ambient isotopic to $T(m-1,3)$. However, there is no reason to believe that (for example) $T(m,3)[11]_-[12]_0$ will be a recognizable knot; some secondary applications of the skein relation and manipulation of such knots by Reidemeister moves are necessary to resolve the recurrence relations (rrr). After this process results in recognizable recurrence relations (rrrr!), we manipulate these relations combinatorially. We then use *Mathematica* to solve for a closed form.

*Proof.* Since $T(1,3)$ is the unknot and $T(2,3)=T(3,2)$, we know $\nabla(T(1,3))=1$ and $\nabla(T(2,3))=1+z^2$. We computed $\nabla(T(3,3))=3z^2+z^4$ by hand, and the final necessary initial value $\nabla(T(4,3))=1+5z^2+5z^4+z^6$ was obtained using the *Mathematica* **KnotTheory** package (which can compute the Conway polynomial for torus knots, but not for links). One can verify that these polynomials match the polynomials obtained by evaluating the above formula, so the theorem holds for $m\leq 4$.

For $m>4$, we begin by applying skein relations to our standard projection of $T(m,3)=T(m,3)[11]_+$. This gives

$$\nabla(T(m,3)[11]_+) = \nabla(T(m,3)[11]_-) + z\nabla(T(m,3)[11]_0),$$

$$\nabla(T(m,3)[11]_-) = \nabla(T(m,3)[11]_-[12]_-) + z\nabla(T(m,3)[11]_-[12]_0), \text{ and}$$
$$\nabla(T(m,3)[11]_0) = \nabla(T(m,3)[11]_0[12]_-) + z\nabla(T(m,3)[11]_0[12]_0).$$

Performing the corresponding crossing changes defines three new links $K_1(m)$, $K_2(m)$, and $K_3(m)$:

$$\begin{aligned} T(m,3)[11]_-[12]_- &= K_1(m) \\ T(m,3)[11]_0[12]_- &= K_2(m) \\ T(m,3)[11]_-[12]_0 &= K_3(m) \end{aligned}$$

Next, we show that $K_2(m)$ is ambient isotopic to $K_3(m)$: Let $K^R$ indicate the projection of the link obtained by applying Reidemeister moves to $K$. Applying a couple of Reidemeister moves on each of $K_2(m)$ and $K_3(m)$, shown in Figures 3 and 4, reveals that if we rotate $K_2^R(m)$ one spoke (or one $m$-th of a

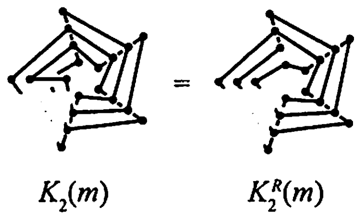



Figure 3: $K_2(m)$ is shown at left. A Reidemeister II move eliminates two crossings to produce the projection $K_2^R(m)$ shown at right.

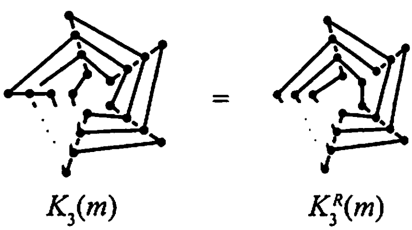


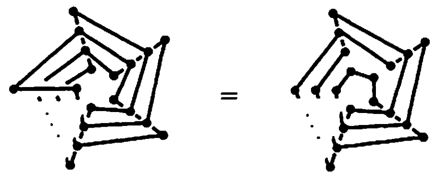

Figure 4: $K_3(m)$ is shown at left, and is ambient isotopic to the link $K_3^R(m)$ at right, which has two fewer crossings.

circle) clockwise, we obtain a link that is planar isotopic to $K_3^R(m)$.

Plugging these results into our original skein relations gives

$$\nabla(T(m,3)) = \nabla(K_1(m)) + 2z\nabla(K_2(m)) + z^2\nabla(T(m-1,3)) \qquad (3)$$

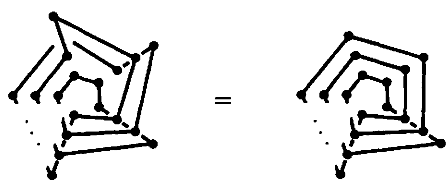
Next, we want to understand $K_1(m)$. Redrawing $K_1(m)$ as shown in Figure 5 provides some assistance.



$K_1(m)$ $\quad$ $K_1^R(m)$

Figure 5: $K_1(m)$ is shown at left, and redrawn as $K_1^R(m)$ at right to eliminate unnecessary over-over crossings.

Now the crossings on the first spoke have been eliminated. When we apply the skein relation to the projection $K_1^R(m)$ at crossing $c_{21}$, we notice that $K_1^R(m)[21]_-$ is ambient isotopic to $T(m-3,3)$ (see Figure 6).



$K_1^R(m)[21]_-$ $\quad$ $T(m\text{-}3,3)$

Figure 6: A Reidemeister II move is applied to $K_1^R(m)[21]_-$. The resulting projection is planar isotopic to the standard projection of $T(m-3,3)$.

The skein relation now becomes

$$\nabla(K_1^R(m)[21]_+) = \nabla(T(m-3,3)) + z\nabla(K_2^R(m-1)). \tag{4}$$

Similarly, applying one skein relation to the projection $K_2^R(m)$ at the crossing $c_{21}$ gives

$$\begin{aligned}\nabla(K_2^R(m)[21]_+) &= \nabla(K_3(m-1)) + z\nabla(T(m-2,3)) \\ &= \nabla(K_2(m-1)) + z\nabla(T(m-2,3)).\end{aligned} \tag{5}$$

These three recurrence relations reduce to two by plugging (4) and (5) into (3):

$$\begin{aligned}\nabla(T(m,3)) = \nabla(T(m-3,3)) &+ 3z\nabla(K_2(m-1)) \\ &+2z^2\nabla(T(m-2,3)) + z^2\nabla(T(m-1,3))\end{aligned}$$

and

$$\nabla(K_2(m)) = \nabla(K_2(m-1)) + z\nabla(T(m-2,3)).$$

We may rewrite these in a completely combinatorial manner by denoting $\nabla(T(m,3))$ by $\alpha_m(z)$ and $\nabla(K_2(m))$ by $\beta_m(z)$, to obtain

$$\alpha_m(z) = \alpha_{m-3}(z) + 2z^2\alpha_{m-2}(z) + z^2\alpha_{m-1}(z) + 3z\beta_{m-1}(z)$$

$$\text{and} \quad \beta_m(z) = \beta_{m-1}(z) + z\alpha_{m-2}(z).$$

Computing $\alpha_m(z) - \alpha_{m-1}(z)$ (noting that $\beta_{m-1}(z) - \beta_{m-2}(z) = z\alpha_{m-3}(z)$ and then simplifying) produces

$$\alpha_m(z) = (z^2+1)\alpha_{m-1} + z^2\alpha_{m-2} + (z^2+1)\alpha_{m-3} - \alpha_{m-4}.$$

From the initial values of $\nabla(T(m,3))$ we know $\alpha_1(z) = 1$, $\alpha_2(z) = 1 + z^2$, $\alpha_3(z) = 3z^2 + z^4$, and $\alpha_4(z) = 1 + 5z^2 + 5z^4 + z^6$.

We were then able to solve the recurrence relation in *Mathematica*. The command

```
RSolve[{a[m] == (z^2 + 1)a[m-1] + z^2 a[m-2]
+ (z^2 + 1) a[m-3] - a[m-4], a[1] == 1, a[2] == 1 + z^2,
a[3] == 3z^2 + z^4, a[4] == 1 + 5 z^2 + 5 z^4 + z^ 6}, a[m], m]
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returns the closed form

$$\frac{1}{3+z^2}\left(\frac{(2+z^2-z\sqrt{4+z^2})^m + (2+z^2+z\sqrt{4+z^2})^m}{2^m} - 2\cos\left(\frac{2\pi m}{3}\right)\right)$$

which can be verified via mathematical induction.

# 4 $(m,4)$ Torus links

The main result in this section is a trio of recurrences that, together with appropriate initial conditions, define the Conway polynomial of the $(m,4)$ torus links.

The standard polygonal projection of $T(m,4)$ has $m$ spokes, with three crossings per spoke. By altering the crossings along one spoke, we obtain two related projections. Let $A_m = T(m,4)[11]_0[12]_-[13]_0$ be the link obtained by eliminating the crossings at $c_{11}$ and $c_{13}$ and switching the crossing at $c_{12}$. Let $B_m = T(m,4)[11]_-[12]_0[13]_-$ be the link obtained by switching the crossings at $c_{11}$ and $c_{13}$ and eliminating the crossing at $c_{12}$. Note that the links $A_m$ and $B_m$ have fewer crossings than $T(m,4)$.

**Theorem 3.** *Let $m \geq 6$. Then the Conway polynomials of $T_m = T(m,4)$, $A_m$, and $B_m$ are related by the following recursive relationship:*

$$\begin{aligned}
\nabla(T_m) &= z^3\nabla(T_{m-1}) + 2z^3\nabla(T_{m-3}) + (z^4+4z^2+1)\nabla(T_{m-4}) + z^3\nabla(T_{m-5}) \\
&\quad +3z^2\nabla(A_m) + 2z^3\nabla(A_{m-1}) + (z^4+6z^2)\nabla(A_{m-2}) \\
&\quad +(2z^3+4z)\nabla(A_{m-3}) + z^2\nabla(A_{m-4}) \\
&\quad +z\nabla(B_m) + 2z^2\nabla(B_{m-1}) + (z^3+z)\nabla(B_{m-2}) \\
\nabla(A_m) &= z^2\nabla(T_{m-2}) + z^2\nabla(T_{m-4}) + z\nabla(T_{m-5}) \\
&\quad +2z\nabla(A_{m-1}) + z^2\nabla(A_{m-2}) + 2z\nabla(A_{m-3}) + \nabla(A_{m-4}) \\
&\quad +z\nabla(B_{m-2}) \\
\nabla(B_m) &= z\nabla(T_{m-2}) + z^2\nabla(T_{m-3}) + 2z\nabla(A_{m-2}) + \nabla(B_{m-2})
\end{aligned}$$

The proof will proceed much as the proof of Theorem 2 in Section 3. We begin with the polygonal projection of $T(m,4)$ as described in Section 2 and apply three generations of skein relations in order to obtain recurrence relations for $T(m,4)$ in terms of knots with fewer crossings. As before, some Reidemeister manipulations of links and secondary applications of the skein relation are necessary to produce reasonable recurrence relations. Because of the increased number of calculations, some straightforward calculations are noted and details left to the reader.

*Proof.* We start by applying skein relations to our standard projection of $T(m,4) = T(m,4)[11]_+[12]_+[13]_+$. As before, $T(m,4)[11]_0[12]_0[13]_0$ is ambient isotopic to $T(m-1,4)$. After performing some Reidemeister moves, we find that we can determine the Conway polynomial of $T(m,4)$ from the Conway polynomials of $T(m-1,4)$ and the four links shown in Figure 7.

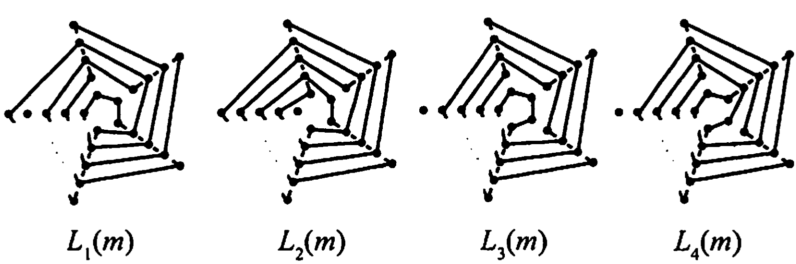


Figure 7: The Conway polynomial of $T(m,4)$ can be expressed in terms of the Conway polynomials of $T(m-1,4)$, $L_1(m)$, $L_2(m)$, $L_3(m)$, and $L_4(m)$.

Reidemeistering reveals the following set of ambient isotopies:

$$\begin{aligned}
L_1(m) &= T(m,4)[11]_0[12]_-[13]_- = T(m,4)[11]_-[12]_-[13]_0 \\
L_2(m) &= T(m,4)[11]_0[12]_-[13]_0 = T(m,4)[11]_0[12]_0[13]_- \\
&\qquad = T(m,4)[11]_-[12]_0[13]_0 = A_m \\
L_3(m) &= T(m,4)[11]_-[12]_-[13]_- \\
L_4(m) &= T(m,4)[11]_-[12]_0[13]_- = B_m
\end{aligned}$$

Applying further skein relations to $L_1(m), L_2(m), L_3(m)$, and $L_4(m)$ produces additional relationships:

$$\begin{aligned}
L_1(m)[21]_0[22]_0 &= L_2(m-1) \\
L_1(m)[21]_0[22]_-[31]_0 &= T(m-3,4) \\
L_1(m)[21]_0[22]_-[31]_- &= L_2(m-2) = L_1(m)[21]_-[22]_-[32]_0 \\
L_1(m)[21]_-[22]_0 &= L_4(m-1) \\
L_1(m)[21]_-[22]_-[32]_- &= L_1(m-1)[21]_0[22]_- \\
L_2(m)[21]_0[22]_0 &= T(m-2,4) = L_4(m)[21]_0 \\
L_2(m)[21]_0[22]_- &= L_2(m-1) = L_2(m)[21]_-[22]_0 \\
L_2(m)[21]_-[22]_- &= L_1(m-1) \\
L_3(m)[21]_0[22]_0 &= L_1(m-1) \\
L_3(m)[21]_0[22]_-[31]_0 &= L_2(m-2) = L_3(m)[21]_-[22]_0[41]_0 \\
L_3(m)[21]_0[22]_-[31]_- &= L_4(m-2) \\
L_3(m)[21]_-[22]_-[32]_0 &= L_1(m-1)[21]_0[22]_- \\
L_3(m)[21]_-[22]_-[32]_- &= T(m-4,4) = L_3(m)[21]_-[22]_0[41]_-[32]_0 \\
L_3(m)[21]_-[22]_0[41]_-[32]_- &= L_2(m-3) \\
L_4(m)[21]_-[22]_0 &= L_1(m)[21]_0[22]_- \\
L_4(m)[21]_-[22]_- &= L_3(m)[21]_0[22]_-
\end{aligned}$$

This glorious miasma of relationships reduces to the following set of recurrence relations on the Conway polynomials of $T(m,4), L_1(m), L_2(m), L_3(m)$, and $L_4(m)$:

$$\begin{aligned}
\nabla(T(m,4)) &= z^3\nabla(T(m-1,4)) + 3z^2\nabla(L_2(m)) + 2z\nabla(L_1(m)) + \\
&\qquad z\nabla(L_4(m)) + \nabla(L_3(m)) \\
\nabla(L_1(m)) &= z^2\nabla(T(m-3,4)) + z^2\nabla(L_2(m-1)) + z\nabla(T(m-4,4)) + \\
&\qquad 2z\nabla(L_2(m-2)) + z\nabla(L_4(m-1)) + \nabla(L_2(m-3)) \\
\nabla(L_2(m)) &= z^2\nabla(T(m-2,4)) + 2z\nabla(L_2(m-1)) + \nabla(L_1(m-1)) \\
\nabla(L_3(m)) &= 2z^2[\nabla(T(m-4,4)) + \nabla(L_2(m-2))] + z^2\nabla(L_1(m-1)) + \\
&\qquad 2z\nabla(L_2(m-3)) + z\nabla(L_4(m-2)) + \nabla(T(m-4,4)) \\
\nabla(L_4(m)) &= z^2\nabla(T(m-3,4)) + z\nabla(T(m-2,4)) + 2z\nabla(L_2(m-2)) + \\
&\qquad \nabla(L_4(m-2))
\end{aligned}$$

We can first eliminate $\nabla(L_3(m))$ and then $\nabla(L_1(m))$ by substitution, leaving only three recurrence relations:

$$
\begin{aligned}
\nabla(T(m,4)) &= z^3\nabla(T(m-1,4)) + 3z^2\nabla(L_2(m)) + z\nabla(L_4(m)) \\
&\quad +(z^4+4z^2+1)\nabla(T(m-4,4)) + (z^4+6z^2)\nabla(L_2(m-2)) \\
&\quad +(2z^3+4z)\nabla(L_2(m-3)) + (z^3+z)\nabla(L_4(m-2)) \\
&\quad +2z^3\nabla(T(m-3,4)) + 2z^3\nabla(L_2(m-1)) + 2z^2\nabla(L_4(m-1)) \\
&\quad +z^3\nabla(T(m-5,4)) + z^2\nabla(L_2(m-4)) \\
\nabla(L_2(m)) &= z^2\nabla(T(m-2,4)) + 2z\nabla(L_2(m-1)) + z^2\nabla(T(m-4,4)) \\
&\quad +z^2\nabla(L_2(m-2)) + z\nabla(T(m-5,4)) + 2z\nabla(L_2(m-3)) \\
&\quad +z\nabla(L_4(m-2)) + \nabla(L_2(m-4)) \\
\nabla(L_4(m)) &= z^2\nabla(T(m-3,4)) + z\nabla(T(m-2,4)) + 2z\nabla(L_2(m-2)) \\
&\quad +\nabla(L_4(m-2))
\end{aligned}
$$

Substituting $A_m = L_2(m)$ and $B_m = L_4(m)$ and rearranging terms yields the desired result. □

Unfortunately, even with copious numbers of hand-calculated $\nabla(T(m,4))$, $\nabla(L_2(m))$, and $\nabla(L_4(m))$ for small $m$, *Mathematica* does not return a closed form for these recurrence relations.

# 5 Conclusion

Explicit formulas were previously known for the Conway and Alexander polynomials for $(m,3)$ torus knots, but their discoverers used technical or deep methods to obtain those formulas. Moreover, these formulas only applied directly to torus *knots*, not torus *links*, and so our formula for the Conway polynomial of the $(m,3)$ torus links streamlines previous results.

While explicit formulae for the Conway and Alexander polynomials for $(m,4)$ torus links seem out of reach, for any particular value of $m$ the Conway (and thus Alexander) polynomial can be built from our recurrence relations.

As far as we can tell (please prove us wrong!) repeated use of crossing changes with the skein relation yields nothing intelligible for the Conway polynomial of $(m,n)$ torus knots with $n \geq 5$. We still hope that this elementary approach will prove applicable to other nicely expressible classes of knots.

# References


[1] C. Adams; M. Hildebrand; and J. Weeks. Hyperbolic Invariants of Knots and Links. *Trans. Amer. Math. Soc.* **326** (1991), 1–56.

[2] M. Bayram; H. Şimşek; N. Yıldırım. Automatic calculation of Alexander polynomials of $(3,k)$-torus knots. *Appl. Math. Comput.* **136** (2003), no. 2–3, 505–510.

[3] J. Hoste; M. Thistlethwaite; J. Weeks. The First 1701936 Knots. *Math. Intell.* **20** (Fall 1998), 33–48.

[4] V. F. R. Jones. Hecke Algebra Representations of braid groups and link polynomials, *Ann. of Math.* **126** (1987), 335–388.

[5] L. H. Kauffman. The Conway polynomial. *Topology*, **20** (1981), 101–108.

[6] A. Kawauchi. *A survey of knot theory.* Birkhäuser Verlag, 1996.

[7] F. B. Kronheimer; T. S. Mrowka. Gauge Theory for Embedded Surfaces I. *Topology* **32** (1993), 773–826.

[8] F. B. Kronheimer; T. S. Mrowka. Gauge Theory for Embedded Surfaces II. *Topology* **34** (1995), 37–97.

[9] W. B. R. Lickorish. *An Introduction to Knot Theory.* Springer, 1997.

[10] J. M. F. Labastida; M. Mariño. The HOMFLY polynomial for torus links from Chern-Simons gauge theory. *Internat. J. Modern Phys. A* **10** (1995), 1045–1089.

[11] H. R. Morton. The coloured Jones function and Alexander polynomial for torus knots. *Math. Proc. Cambridge Philos. Soc.* **117** (1995), no. 1, 129–135.

[12] K. Murasugi. On the Braid Index of Alternating Links. *Trans. Amer. Math. Soc.* **326** (1991), 237–260.

[13] L. Murasugi; J. Przytycki. The Skein Polynomial of a Planar Star Product of Two Links. *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 273–276.

[14] P. Ozsvath; Z. Szabo. Knot Floer homology and the four-ball genus. *Geometry & Topology* **7** (2003), 615–639.

[15] D. P. Rowland. Twisting with Fibonacci. *The Harvard College Mathematics Review.* **2** (Spring 2008), 66–74.

[16] R. F. Williams. The Braid Index of an Algebraic Link. *Braids (Santa Cruz, CA, 1986).* Providence, RI: Amer. Math. Soc., 1988.