On r-Regular Compositions

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ABSTRACT: If the integer $r \geq 2$, say that a composition of the natural number n is *r-regular* if no part is divisible by r. Let $c_r(n)$ denote the number of r-regular compositions of n (with $c_r(0) = 1$). We show that $c_r(n)$ satisfies a linear recurrence of order r. We also obtain asymptotic estimates for $c_r(n)$, and we evaluate $c_r(n)$ for $2 \leq r \leq 5$ and $1 \leq n \leq 10$.

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1. Introduction

A *composition* of the natural number n is a representation of n as a sum of one or more natural numbers. (Representations that differ only in the order of terms are considered distinct.) If the integer $r \geq 2$, say that a composition is *r-regular* if none of its parts is divisible by r. Let $c_r(n)$ denote the number of r-regular compositions of n. Formulas for $c_r(n)$ have previously been obtained in the cases $r = 2, 3$. In particular, $c_2(n)$ is the number of compositions of n into odd parts . It is known that $c_2(n) = F_n$, the n^{th} Fibonacci number. This statement appears to have been first made by Cayley [1]; it appears as an exercise in [6], and follows immediately from a result in [5]. The case $r = 3$ was settled in [4]. In this note, we generalize these earlier results by showing in Theorem 3 below that $c_r(n)$ satisfies a linear recurrence of order r. We thereby prove a conjecture made in [4]. We also obtain asymptotic estimates for $c_r(n)$ for large n, and we evaluate $c_r(n)$ for $2 \leq r \leq 5$ and $1 \leq n \leq 10$.

2. A recursive formula for $c_r(n)$

Our first theorem concerns a recurrence that enables the recursive computation of the number of compositions of the natural number n all of whose parts satisfy a given condition.

Theorem 1 Let $\{a_n\}$ be a strictly increasing sequence of natural numbers. If n is a natural number, let $g(n)$ denote the number of compositions of n whose parts all belong to $\{a_n\}$. Define $g(0) = 1$, and $g(\alpha) = 0$ if α is not a non)-negative integer. Then for all $n \geq 1$, we have

$$g(n) = \sum_{k \geq 1} g(n - a_k) \ .$$

Proof: Let the generating function for compositions with exactly k elements from $\{a_n\}$ be

$$g_k(x) = (\sum_{i \geq 1} x^{a_i})^k \ .$$

Let $G(x)$ be the generating function for all compositions of n with summands from $\{a_n\}$. Then we have

$$G(x) = 1 + \sum_{k=1}^{\infty} g_k(x) = (1 - \sum_{i=1}^{\infty} x^{a_i})^{-1} \ ,$$

and also

$$G(x) = \sum_{n=0}^{\infty} g(n)x^n \rightarrow (1 - \sum_{i=1}^{\infty} x^{a_i})(\sum_{n=0}^{\infty} g(n)x^n) = 1 \ .$$

The conclusion now follows if we equate coefficients of like powers of x. ■

In Theorem 2 below, we use Theorem 1 to obtain a formula for $c_r(n)$.

Theorem 2

$$c_r(n) = \sum\{c(n-k) : r \not| k\} .$$

Proof: This follows immediately from Theorem 1 and the definition of $c_r(n)$. ■

In Theorem 3 below, we obtain a more convenient recurrence for $c_r(n)$, that is, a recurrence with a fixed number of terms.

Theorem 3 $c_r(n) = 2^{n-1}$ for $1 \le n \le r-1; c_r(r) = 2^{r-1} - 1$

$$c_r(n) = \sum_{k=1}^{r} c_r(n-k) \ for \ n \ge r+1 .$$

Proof: The statement is true by inspection for $n \le r$. If $n \ge r+1$, then Theorem 2 implies $c_r(n) = \sum\{c(n-k) : r \not| k\}$, so that

$$c_r(n) = c_r(n-1) + c_r(n-2) + \cdots + c_r(n-r+1) + \sum\{c(n-r-k) : r \not| k\} .$$

But Theorem 2 also implies that $c_r(n-r) = \sum\{c(n-r-k) : r \not| k\}$, from which the conclusion now follows. ■

In the table below, we list $c_2(n), c_3(n), c_4(n), c_5(n)$ for $1 \le n \le 10$.

n	1	2	3	4	5	6	7	8	9	10
$c_2(n)$	1	1	2	3	5	8	13	21	34	55
$c_3(n)$	1	2	3	6	11	20	37	68	125	230
$c_4(n)$	1	2	4	7	14	27	52	100	193	372
$c_5(n)$	1	2	4	8	15	30	59	116	228	448

3. Asymptotics

Before we present an asymptotic estimate for $c_r(n)$, we do so for a similar r^{th} order recurrence , $\{u_r(n)\}$, which has a simpler generating function. Then we express $c_r(n)$ in terms of $u_r(n)$. This easily leads to an asymptotic estimate for $c_r(n)$.

Lemma 1 If the integer $r \geq 2$, let the sequence $\{u_r(n)\}$ satisfy the recurrence:

$$u_r(n) = \sum_{j=1}^{r} u_r(n-j)$$

for $n \geq r$, with initial conditions:

$$u_r(1) = 1 \ , \ u_r(n) = \sum_{j=1}^{n-1} u_r(n-j) \ for \ 2 \leq n \leq r-1 \, , .$$

Let $a_r(x) = \sum_{j=1}^{r} x^j$. Let ρ_r be the real root of $a_r(x) = 1$, with $.5 < \rho_r < 1$. Then

$$u_r(n) \sim \frac{\rho_r^{-n}}{a_r'(\rho_r)} \ .$$

Proof: The generating function for $\{u_r(n)\}$ is given by:

$$g_r(x) = \sum_{n=1}^{\infty} u_r(n) x^n = \frac{x}{1 - a_r(x)} \ .$$

Since $a_r'(x) = 1 + \sum_{j=2}^{r} j x^{j-1}$, we know $a_r'(\rho_r) \neq 0$, so $g_r(x)$ has a simple pole at $x = \rho_r$. From the local expansion of $g_r(x)$ at this dominant pole, we have

$$g_r(x) \sim \frac{x/\rho_r}{a_r'(\rho_r)(1 - x/\rho_r)} = \frac{1}{a_r'(\rho_r)} \sum_{n=1}^{\infty} (\frac{x}{\rho_r})^n \ .$$

The conclusion now follows. ■

Remarks: This method of estimation is mentioned on p. 225 of [2], and was used in another context in [3].

Lemma 2 If $r \geq 2$ and $n \geq 1$, then

$$c_r(n) = \sum_{j=0}^{n-1} u_r(n-j) \ .$$

Proof: This is easily established by induction on n, making use of Theorem 3 and Lemma 1. ■

Theorem 4

$$c_r(n) \sim \frac{\rho_r^{-n}(1-\rho_r^{r-1})}{a_r'(\rho_r)(1-\rho_r)}$$

Proof: This follows from Lemmas 1 and 2. ■

4. References

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