The reconstruction number of a lexicographic sum of cliques

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Abstract

The clique sum $\Sigma = G[G_1, G_2, \ldots, G_n]$ is the lexicographic sum over G where each fibre G_i is a clique. We show the reconstruction number of Σ is three unless Σ is vertex transitive and G has order at least two. In the latter case it follows that $\Sigma = G[K_m]$ is a lexicographic product and the reconstruction number is $m + 2$. This complements the bounds of Brewster, Hahn, Lamont, and Lipka. It also extends the work of Myrvold and Molina.

1 Introduction

All graphs in this paper are assumed to be simple, finite, and undirected. We follow the notation of [4].

Given a graph G and one of its vertices, v, the *vertex-deleted subgraph* $G - v$ is the subgraph obtained by deleting v and all the edges incident with v. The collection of all (unlabelled) vertex-deleted subgraphs is called the *deck* of G, denoted $\mathbb{D}(G)$. The individual members are *cards*. In general $\mathbb{D}(G)$ may contain several isomorphic cards, prompting some authors to refer to it as a multiset; however, we simply use set notation. A *reconstruction of* G is a graph H such that G and H have the same deck. The graph G is *reconstructible* if every reconstruction is isomorphic to G. The *Graph Reconstruction Conjecture (GRC)* states that every simple, finite, undirected graph G with at least three vertices is reconstructible. It was posed by Kelly and Ulam [7, 15]. In the premier issue of the Journal

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of Graph Theory (1977) Harary described the conjecture as [one of] the foremost unsolved problems in the field.

We say G is *reconstructible from* $\mathbb{C} \subseteq \mathbb{D}(G)$ if $G \cong H$ for any graph H such that $\mathbb{C} \subseteq \mathbb{D}(H)$. The *reconstruction number of* G, denoted $rn(G)$, is the minimum m such that G is reconstructible from some m cards in its deck. Reconstruction numbers were introduced in an attempt to understand how much information is required to reconstruct a graph. They are also referred to as the *existential* or *ally reconstruction numbers* [6, 12, 11]. A survey on reconstruction can be found at [3].

In 1990, Bollobás [2] proved almost all graphs have reconstruction number three. From this result one obtains a natural question: which graphs have reconstruction number greater than three. Such graphs are said to have a *high reconstruction number.*

McKay [8] verified the GRC for all graphs with at most eleven vertices using *Nauty.* McMullen [9] and Baldwin [1] calculated the reconstruction numbers of all graphs with fewer than eleven vertices. From this McMullen and Radziszowski [10] identified several classes of graphs with high reconstruction numbers. Many of their classes existed already in the literature, particularly in the work of Myrvold [12] and Harary and Plantholt [6].

In [5], Brewster, Hahn, Lamont, and Lipka provided a framework which captures and generalizes all of these classes, many of which are lexicographic products over vertex transitive graphs. For example, Theorem 19 [5] shows that the reconstruction number of the lexicographic product of a vertex transitive graph G around a clique of order m satisfies $\mathrm{rn}(G[K_m]) \geq m+2$. In this article we complement the above work through a study of lexicographic sums and products around cliques. In particular we calculate their exact reconstruction numbers.

The objects of our study are *clique sums.* Clique sums are special lexicographic sums (defined below).

Definition 1.1. Given graphs $G, G_1, G_2, \ldots, G_n$ where G has order n, the *lexicographic sum* $\Sigma = G[G_1, \ldots, G_n]$ is the graph with

- $V(\Sigma) = \{(i,j) : i \in V(G) \text{ and } j \in V(G_i)\}$; and,
- $E(\Sigma) = \{\{(i,j),(k,l)\} : ik \in E(G) \text{ or } i = k \text{ and } jl \in E(G_i)\}$.

The graphs G_i are called *fibres* of the sum. In the case that all the G_i are isomorphic, the sum is the *lexicographic product* G *around* G_i, denoted $G[G_i]$.

Informally, Σ is obtained from G by replacing each vertex i in G with the graph G_i. For adjacent i and k in G, we put all possible edges between G_i and G_k.

A lexicographic sum $\Sigma = G[G_1, \ldots, G_n]$ where each fibre G_i is a clique of order at least 2 is called a *clique sum.* For the remainder of the paper Σ is used to denote a clique sum. In the case $\Sigma = G[K_m]$ we will call it a *clique product.*

In Section 2 we introduce the closed neighbourhood partition. In Section 3 we prove and state our main result. Section 4 contains open problems.

2 The Closed Neighborhood Partition

Given a graph G and a vertex v, the *neighborhood of* v, denoted $N_G(v)$, is the set $\{w \in V(G) | wv \in E(G)\}$. The *closed neighborhood of* v, denoted $N_G[v]$, is $\{v\} \cup N_G(v)$. We omit the subscript when the graph G is clear from context. For a set $S \subseteq V(G)$, $N[S] = \cup_{v \in S} N[v]$. The *closed neighborhood partition*, or *cn-partition* for short, is the partition of $V(G)$ induced by the equivalence relation $w \sim_G v$ if $N_G[w] = N_G[v]$. Suppose $\sim_G$ partitions $V(G)$ into t cells $C_1, C_2, \ldots, C_t$. Given cells C_i and C_j, define their *cn-difference* to be $N[C_i] \Delta N[C_j]$. (We thank the referee who pointed out the terms *interval*, *module*, and *homogeneous set* are also used for such a partition.)

Key to our work is understanding how the cn-partition of a graph G and of the cards in its deck are related. In particular, given $u \not\sim_G v$, under what conditions do we have $u \sim_{G-w} v$ in the card $G - w$?

Lemma 2.1. *Let G be a graph with cn-partition $C_1, C_2, \ldots, C_t$. Let u, v, and w be distinct vertices of G. Then*

(a) there is a set $\mathcal{I} \subseteq \{1, 2, \ldots, t\}$ such that the symmetric difference

$$N[u] \Delta N[v] = \bigcup_{i \in \mathcal{I}} C_i.$$

(b) if $u \sim_G v$, then $u \sim_{G-w} v$.

(c) if $|N[u] \Delta N[v]| \geq 2$ and $u \not\sim_G v$, then $u \not\sim_{G-w} v$.

Proof. (a) Suppose $x \in N[u] \backslash N[v]$. Further suppose $x \in C_i$ for some i. Let $y \in C_i$, i.e. $N[y] = N[x]$. Then $y \in N[u] \backslash N[v]$. Hence, we conclude $N[u] \Delta N[v]$ is a union of cells from the cn-partition.

(b) If $N_G[u] = N_G[v]$, then clearly $N_{G-w}[u] = N_{G-w}[v]$.

(c) Suppose $\{x, y\} \subseteq N[u] \Delta N[v]$. Without loss of generality, $y \neq w$, $y \in N[u]$ and $y \notin N[v]$. Then $y \in N_{G-w}[u]$ and $y \notin N_{G-w}[v]$. Thus $u \not\sim_{G-w} v$. □

Corollary 2.2. *Suppose the graph G has cn-partition $C_1, C_2, \ldots, C_t$ where $|C_j| \geq 2$ for all $1 \leq j \leq t$. Let $w \in C_i$. Then $G - w$ has cn-partition $C_1, C_2, \ldots, C_i - \{w\}, \ldots, C_t$.*

Proof. Let $u \sim_G v$. Then by Lemma 2.1(b), $u \sim_{G-w} v$. On the other hand, suppose $u \not\sim_G v$. Then by assumption each C_j has order at least 2. Thus by Lemma 2.1(a), $|N[u] \triangle N[v]| \geq 2$. We conclude $u \not\sim_{G-w} v$. □

Corollary 2.3. *Suppose u, v, and w are vertices of G such that $u \not\sim_G v$, but $u \sim_{G-w} v$. Then $\{w\} = C_i$ for some cell in the cn-partition of G, and $N[u] \triangle N[v] = C_i$.*

3 Reconstructing clique sums

We now focus on the clique sum $\Sigma = G[G_1, G_2, \ldots, G_n]$ (where, by definition, each G_i is a clique of order at least 2). We begin by examining the relationship between the cn-partition of G and Σ.

Lemma 3.1. *Let G be a graph and $\Sigma = G[G_1, \ldots, G_n]$ be a clique sum. Suppose $v, w \in V(G)$ and $(v, i), (w, j) \in V(\Sigma)$. Then $v \sim_G w$ if and only if $(v, i) \sim_\Sigma (w, j)$.*

Proof. Suppose $v \sim_G w$. Let $(z, k) \in V(\Sigma)$. Assume $(z, k) \in N_\Sigma[(v, i)]$. By definition, either $z = v$ or $zv \in E(G)$. In either case, $z \in N_G[v]$. Since $v \sim_G w$, $z \in N_G[w]$. Thus, either $zw \in E(G)$, or $z = w$, from which we obtain $(z, k) \in N_\Sigma[(w, j)]$. We note the latter case follows from the fact that k and j belong to the same fibre of Σ and all fibres are cliques. Similarly, $(z, k) \notin N_\Sigma[(v, i)]$ implies $(z, k) \notin N_\Sigma[(w, j)]$. Consequently, $(v, i) \sim_\Sigma (w, j)$.

On the other hand, assume $(v, i) \sim_\Sigma (w, j)$ and $z \in N[v]$. Then $(z, k) \in N_\Sigma[(v, i)]$ and thus $(z, k) \in N_\Sigma[(w, j)]$. Hence $z \in N[w]$. Similarly, if $z \notin N[v]$, then $z \notin N[w]$. The result follows. □

A consequence of this lemma is that if the cn-partition of G is not composed of singletons then we can represent Σ as a clique sum over a smaller graph. Suppose $n-1$ and n are vertices of G such that $(n-1) \sim_G n$. Then $(n-1, i) \sim_\Sigma (n, j)$ for all i and j. The subgraph of Σ induced by $G_{n-1} \cup G_n$ is a clique and all vertices in this clique have the same closed neighbourhood. Consequently, if we let $G' = G - n$ and G'_{n-1} be the clique $G_{n-1} \vee G_n$ (where $\vee$ is the join of the two graphs), then $\Sigma = G'[G_1, G_2, \ldots, G_{n-2}, G'_{n-1}]$. A clique sum is *minimal* if $\Sigma = G[G_1, \ldots, G_n]$ and n is minimum over all such representations. From the lemma and our discussion we see that a minimal sum has G with a cn-partition consisting of singletons. The cells of the cn-partition of Σ are precisely the fibres of

Σ and the quotient of the cn-partition of Σ must be G. In other words, the cn-partition of Σ uniquely determines the graph G in a minimal sum.

Vertex transitivity and the automorphism group of clique sums play an important role below. We require a classic result of Sabidussi; see [13, 14]. To a pair of graphs (G, H) we associate two conditions.

Condition $\mathbf{S_1}$: H is connected whenever G has two vertices $u \neq v$ such that $N(u) = N(v)$; and

Condition $\mathbf{S_2}$: $\overline{H}$ is connected whenever G has two vertices $u \neq v$ such that $N[u] = N[v]$.

Theorem 3.2 (Sabidussi [13]). *Let G and H be graphs. A necessary and sufficient condition for* $\mathrm{Aut}(G[H]) = \mathrm{Aut}(G) \wr \mathrm{Aut}(H)$ *is that the pair (G, H) satisfies both Condition S_1 and Condition S_2.*

The theorem above naturally extends to clique sums. Given a minimal clique sum $\Sigma = G[G_1, \ldots, G_n]$, the above theorem states that an automorphism of Σ must map fibres to fibres. Condition S_1 is satisfied since each fibre is a clique. Condition S_2 is satisfied in a minimal clique sum as there cannot be vertices $u \neq v$ such that $N[u] = N[v]$. Thus any automorphism of Σ induces an automorphism of the quotient graph G. Hence, if Σ is vertex transitive, we can conclude G is vertex transisitve and $\Sigma = G[K_m]$ for some m. This fact is used below.

We now study the cn-partitions of Σ, its cards, and potential reconstructions of cards of Σ. To ease notation we refer to *a cell of the cn-partition of a graph G* as simply *a cell of G*. In particular, if $\{w\}$ is a singleton cell in the cn-partition of G, we say that *$\{w\}$ is a singleton of G*.

Let $G_1, G_2, \ldots, G_n$ be the cn-partition of (a minimal clique sum) Σ. By Corollay 2.2, the card $\Sigma - v$ must have n cells in its partition. The cell $G_i - \{v\}$ is called the *deficient cell* of $\Sigma - v$ and is labelled $\{w\}$ when it is a singleton of $\Sigma - v$.

We now consider the set-up where $V_1 = \Sigma - v_1$ is a card of Σ and X is a graph obtained by adding v_1 to V_1 together with some edges incident with v_1. (Note $N_\Sigma(v_1) \neq N_X(v_1)$ in general.) By Corollary 2.2, if X has cn-partition $X_1, X_2, \ldots, X_t$, and the card $X - x$ has fewer than t cells, then $X_i = \{x\}$ for some i. Furthermore, if $X - x$ has $t - 2$ or fewer cells, then there are vertices u and v such that $u \not\sim_X v$ but $u \sim_{X-x} v$. By Corollary 2.3 there must be at least 2 cells of X, say X_j and X_k, neither equal to $X_i = \{x\}$, such that $N[X_i] \triangle N[X_j] = \{x\}$. The set of all vertices $x \in V(X)$ such that $X - x$ has at most $t - 2$ cells is called the *merging set of X*. If X_j and X_k are distinct cells of X and $X_j \cup X_k$ is a cell in $X - x$, we say that *x merges X_j and X_k*. Observe that x cannot merge 3 cells, say X_j, X_k, X_ℓ, as this would force two cells to have the same closed neighbourhoods, i.e. they would not be distinct cells in X. It is possible that x merges more than one pair. For example, X_j may merge with X_k and X_ℓ may merge with X_m.

We now examine how v_1 can join to vertices of V_1 to form X. As a convention, let $C_1, C_2, \ldots, C_n$ be the cn-partition of V_1. Note $C_\ell = G_\ell - v_1$ for some ℓ, and $C_j = G_j$ for all $j \neq \ell$. For each cell C_i, if $N_X[v_1] \cap C_i = C_i$ or $\emptyset$, we say that v_1 *joins regularly to* C_i. Otherwise, we say that v_1 *mutilates* C_i. In the case that v_1 mutilates C_i, define $A_i = N_X[v_1] \cap C_i$ and $\overline{A_i} = C_i \backslash A_i$. Note that A_i and $\overline{A_i}$ are cells of X. Also, $\{v_1\}$ is a singleton of X as it is the unique vertex in X that joins A_i and does not join $\overline{A_i}$. If $C_i \cup \{v_1\}$ is a cell in X, i.e. $N_X[v_1] = N_X[u]$ for all $u \in C_i$, we say that v_1 *mimics* C_i. In this case, v_1 must join regularly to each cell in V_1. Consequently, v_1 can mimic at most one cell. We call $C_i \cup \{v_1\}$ the *mimicked cell.*

We state our main result.

Theorem 3.3. *Let $\Sigma = G[G_1, \ldots, G_n]$ be a minimal clique sum. If Σ is not vertex transitive or $|V(G)| = 1$, then $rn(\Sigma) = 3$. Otherwise $\Sigma = G[K_m]$ and $rn(\Sigma) = m + 2$.*

We identify a subset $\mathbb{C}$ of $\mathbb{D}(\Sigma)$ such that for any graph H not isomorphic to Σ, $\mathbb{C} \not\subseteq \mathbb{D}(H)$. The set $\mathbb{C}$ is defined based on the cases: $|V(G)| = 1$; $|V(G)| \geq 2$ and Σ is vertex transitive; and Σ is not vertex transitive.

When $|V(G)| = 1$, $\Sigma = K_m$ for some m. It is well known that $rn(K_n) = 3$ for $n \geq 3$. Thus assume for the remainder that $|V(G)| \geq 2$.

As noted above, if Σ is vertex transitive, then Σ is a clique product $G[K_m]$, where G is vertex transitive. For this case, let $\mathbb{C} = \{V_1, V_2, \ldots, V_{m+2}\}$ be $m + 2$ cards of Σ. Note that all cards of $\mathbb{C}$ are isomorphic since Σ is vertex transitive.

For the third case, let $\mathbb{C} = \{V_1, V_2, V_3\}$, where $V_i = \Sigma - v_i$. Pick v_1 from a cell of minimum order (of Σ) and v_2, v_3 each from a cell of maximum order (of Σ). Furthermore, choose the elements of $\mathbb{C}$ to contain a pair of nonisomorphic graphs, which is possible since Σ is not vertex transitive. (Clearly, if G is vertex transitive, then all cards are isomorphic. On the other hand if all elements of $\mathbb{D}(G)$ are isomorphic, then it is easy to see that G is regular. Using this fact, one can extend an isomorphism of $G - v$ to $G - u$ to an automorphism of G mapping v to u.)

If $V_1 \notin \mathbb{D}(H)$, then clearly $\mathbb{C} \not\subseteq \mathbb{D}(H)$. Thus, we focus on graphs with V_1 in their deck. Specifically, as defined above let X be V_1 together with a vertex v_1 and some edges incident with v_1. We examine the different ways of joining v_1 to V_1 (with the restriction that X is not isomorphic to Σ). Our strategy is to show that cards of X have cn-partitions, $X_1, \ldots, X_t$, which preclude them from being in $\mathbb{C}$, i.e. in each case, $\mathbb{C} \not\subseteq \mathbb{D}(X)$.

Lemma 3.4. *(Case 1) Suppose v_1 mutilates two or more cells of V_1. Then $\mathbb{D}(X)$ contains at most two cards of $\mathbb{C}$.*

Proof. By Corollary 2.2, V_1 has cells of size at least 2 with the possible exception of the deficient cell $\{w\}$. As noted above, each mutilated cell C_i from V_1 gives rise to 2 cells A_i and $\overline{A_i}$ in X. Also, $\{v_1\}$ is a singleton in X. Hence $t \geq n+3$. It suffices to prove that the merging set of X has at most 2 vertices, for then any 3 cards of X will contain a card having more than n cells.

Suppose u merges X_j and X_k in $X - u$. Suppose to the contrary u is different from v_1 and w. As $\{u\}$ must be a cell of X, the only possibility is $\{u\}$ is one of the mutilated cells, say A_i or $\overline{A_i}$. (All cells of X other than $\{v_1\}$ and $\{w\}$ have order at least two.) Suppose $\{u\} = A_i$ and $\bar{u} \in \overline{A_i}$. Let $x_j \in X_j$ be adjacent to u and $x_k \in X_k$ be nonadjacent to u. The other cases are analogous. Since u merges X_j and X_k, $\bar{u}$ must be adjacent to both of x_j and x_k or nonadjacent to both. The only vertex that has neighbours in exactly one of A_i or $\overline{A_i}$ is v_1. Thus, one of X_j or X_k is $\{v_1\}$. However, v_1 mutilates at least two cells of V_1 and thus cannot cn-differ by a single vertex from any other cell, a contradiction. □

Lemma 3.5. *(Case 2) Suppose v_1 mutilates one cell of V_1. Then $\mathbb{D}(X)$ contains at most two cards of $\mathbb{C}$.*

Proof. Note that X has $t = n+2$ cells, without loss of generality the cn-partition is: $C_1, \ldots, C_{n-1}, A_n, \overline{A_n}, \{v_1\}$. Hence, for each card $X - x \in \mathbb{C}$, x must belong to the merging set of X. In particular, this means that each such x is a singleton in X. Observe that X has at most four singletons: $\{w\}$, $\{v_1\}$, A_n, $\overline{A_n}$. We first show that the set $\{X - v_1, X - A_n, X - \overline{A_n}\}$ is not a subset of $\mathbb{C}$. Then we show that $X - w$ is not an element of $\mathbb{C}$.

Suppose to the contrary $X - v_1$, $X - A_n$, and $X - \overline{A_n}$ belong to $\mathbb{C}$. Hence, A_n and $\overline{A_n}$ are both singletons in X. Consider the card $X - A_n$. In particular, the vertex in $\overline{A_n}$ and v_1 are nonadjacent. A cell $C_i, 1 \leq i \leq n-1$, and $\overline{A_n}$ cannot be merged by A_n for that would require vertices in C_i to be adjacent in X to $\overline{A_n}$ (to merge) and to be nonadjacent in X to A_n (to have $N_X[C_i] \triangle N_X[\overline{A_n}] = A_n$). However, in X each vertex (other than v_1) joins both A_n and $\overline{A_n}$ or neither. Thus, $\overline{A_n}$ is a singleton in $X - A_n$.

Suppose V_2 does not have a singleton. If Σ is vertex transitive, then all elements of $\mathbb{C}$ are isomorphic. None contain a singleton and thus $X - A_n$ does not belong to $\mathbb{C}$. On the other hand, if Σ is not vertex transitive, then $\mathbb{C} = \{V_1, V_2, V_3\}$. By construction V_3 cannot contain a singleton either. If $X - A_n$ is not isomorphic to V_1, then $X - A_n \notin \mathbb{C}$. If $X - A_n$ is isomorphic to V_1, then $\mathbb{C}$ contains only one copy of V_1 and at most two of $X - v_1, X - A_n, X - \overline{A_n}$ can belong to $\mathbb{C}$ as required.

Suppose V_2 has a singleton. Recall that for clique sums, terms have order at least two, and $V_2 = \Sigma - v_2$ where v_2 belongs to a cell of Σ with maximum order. We conclude this maximum order is two, and thus all cells in Σ have order two. In particular, V_1, V_2, and V_3 each have a unique singleton. (This is independent of whether Σ is vertex transitive or not, i.e. $\mathbb{C}$ has three or $m + 2$ elements.) Recall, the singleton of V_1 is $\{w\}$.

Since each card of Σ has only one singleton and $\overline{A_n}$ is a singleton of $X - A_n$, the vertex in A_n must merge $\{w\}$ with some cell X_i of X. Thus, $N_X[w] \triangle N_X[X_i] = A_n$. The only cell of X that could realize such a cn-difference is $X_i = \{v_1\}$. By Lemma 2.1(a) any other vertex that cn-differs from w by A_n also cn-differs by $\overline{A_n}$. This implies v_1 and w are adjacent, and w is not adjacent to $A_n \cup \overline{A_n}$. In particular w, v_1, A_n induce a path of length two in X.

Now consider $X - \overline{A_n}$. The induced path of length two ensures neither $\{w\}$ nor A_n merge with $\{v_1\}$. Similar to above, A_n is a singleton in $X - \overline{A_n}$. By the same reasoning $\{w\}$ is also a singleton. Therefore, $X - \overline{A_n}$ has at least two singletons and is not a card of Σ, a contradiction.

Finally, we rule out $X - w$ as a member of $\mathbb{C}$, after which the result follows. Suppose to the contrary $X - w$ is a member of $\mathbb{C}$. Since X has $n + 2$ cells, and all cards of Σ have n cells, w must belong to the merging set of X. Thus $\{w\}$ is a singleton of X. Now X has up to three other singletons: $\{v_1\}$, A_n, and $\overline{A_n}$. Consider the subgraph Y of X induced by these three cells. These 3 cells are the cn-partition of Y. Hence, removing w does not merge any of these three cells with each other. Also, $\{v_1\}$ cannot merge with any of $C_1, \dots, C_{n-1}$ since v_1 cn-differs with each of these cells by either A_n or $\overline{A_n}$. This means the unique singleton of $X - w$ is $\{v_1\}$. As above, without loss of generality, V_2 has a singleton which implies all cells of Σ have order 2. In particular, all elements of $\mathbb{C}$ have $n - 1$ cells of order 2. Then in X, there are $n - 2$ cells of order 2 and four singletons. Since singletons cannot merge in $X - w$, and $\{v_1\}$ is the unique singleton of $X - w$, the cell A_n must merge with one of $C_1, \dots, C_{n-1}$. The result is a cell of order at least 3 in $X - w$ and hence it is not a member of $\mathbb{C}$. □

Lemma 3.6. *(Case 3) Suppose v_1 joins regularly to each cell of V_1 and X has no mimicked cell. Then $\mathbb{D}(X)$ contains at most 2 cards of $\mathbb{C}$.*

Proof. Since v_1 does not mimic any cell of V_1, the graph X has $t = n+1$ cells in its cn-partition: $C_1, C_2, \dots C_n, \{v_1\}$. Also, X has at most 2 singletons since at most one of $C_1, C_2, \dots, C_n$ is a singleton. Since only singletons $\{x\}$ in X have the property that $X - x$ has fewer than t cells, no 3 cards of X can belong to $\mathbb{C}$. □

Lemma 3.7. *(Case 4) Suppose v_1 mimics a cell of V_1. Then $\mathbb{D}(X)$ does not contain $\mathbb{C}$.*

Proof. First suppose that Σ has cells of the same order m. Then X has a cell of order $m+1$, the mimicked cell. Since each element of $\mathbb{C}$ contains no cell of order $m+1$, any x such that $X-x$ is in $\mathbb{C}$ must be from the mimicked cell. In particular, all such cards must be isomorphic. If Σ is vertex transitive, $\mathbb{C}$ has $m+2$ cards and the result follows. If Σ is not vertex transitive, then $\mathbb{C}$ has cards that are not all isomorphic and the result follows.

Now suppose Σ has cells of different orders and minimum order k. Then the deficient cell of V_1 and, consequently, X has order $k-1$. Since v_1 mimics a cell, X has n cells and any card of X with n cells must have a cell of order $k-1$ or $k-2$. However, V_2 and V_3 have no cell of order $k-1$ (or smaller). The result follows. □

Proof of Theorem 3.3. Lemmas 3.4, 3.5, 3.6, and 3.7 above show $rn(\Sigma) \leq |\mathbb{C}|$. If $|\mathbb{C}| = 3$, then we are done. On the other hand if Σ is vertex transitive, then $\Sigma = G[K_m]$. The work above shows $rn(\Sigma) \leq m+2$ and Theorem 19 of [5] shows $rn(\Sigma) \geq m+2$. □

4 Conclusion

This article extends the results of [12, 11], as unions of components may be viewed as the lexicographic sum $\overline{K_n}[G_1, \dots, G_n]$, in the case where each fibre is a clique. A natural question is to study lexicographic sums of other graphs.

Question 4.1. *Can the methods above be applied to lexicographic products $G[H]$ where G and H are vertex transitive?*

Question 4.2. *Can upper bounds be found for other classes identified in [5].*

Key to our proofs is the fact that in clique sums each fibre has order at least two. In general, the cn-partition of a graph may contain a mix of singletons and larger cliques.

Question 4.3. *What is the existential reconstruction number for clique sums (or general graphs) with a cn-partition containing a mix of singletons and larger cliques?*

Question 4.4. *Can the techniques is this paper be applied to other partitions?*

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