FEW FAMILIES OF HARMONIC MEAN GRAPHS

C. David Raj[1], C. Jayasekaran[2] and S.S. Sandhya[3]

1. Department of Mathematics, Malankara Catholic College, Mariagiri, Kaliyakkavilai, Kanyakumari – 629 153, Tamil Nadu.

2. Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil, Kanyakumari, Pin:629 003, Tamil Nadu.

3. Department of Mathematics, Sree Ayyappa College for women, Chunkankadai, Kanyakumari, Pin:629 807, Tamil Nadu.

Abstract

A graph G = (V, E) with p vertices and q edges is called a Harmonic mean graph if it is possible to label the vertices v∈V with distinct labels f(v) from 1, 2, ..., q + 1 in such a way that when each edge e = uv is labeled with $f(e = uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$, then the edge labels are distinct. In this case f is called Harmonic mean labeling of G. In this paper, we investigate some new families of Harmonic mean graphs.

Key words: Graph, Harmonic mean labeling, Harmonic mean graphs.

1. Introduction: We begin with simple, finite, connected and undirected graph G(V, E) with p vertices and q edges. For a detailed survey of graph labeling we refer to Gallian[1]. For all other standared terminology and notations we follow Harary[2].

S. Somasundaram and S.S. Sandhya introduced Harmonic mean labeling of graphs in [3] and studied their behaviour in [4] and [5]. S.S. Sandhya, C. Jayasekaran and C. David Raj proved that $P_n \odot A(K_1)$, the step ladder graph $S(T_n)$, $P_n \odot K_2$ and $C_n \odot K_2$ are Harmonic mean graphs in [6]. In this paper, we investigate some new Harmonic mean graphs.

We now give the following definitions which are useful for the present investigation.

Definition 1.1. A graph G = (V, E) with p vertices and q edges is called a Harmonic mean graph if it is possible to label the vertices v∈V with distinct labels f(v) from 1, 2, ..., q + 1 in such a way that when each edge e = uv is labeled with $f(e = uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$, then the edge labels are distinct. In this case f is called Harmonic mean labeling of G.

Definition 1.2. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \odot G_2$ formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

Definition 1.3. H – graph is obtained from two paths $u_1u_2...u_n$ and $v_1v_2...v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}}\ v_{\frac{n+1}{2}}$ when n is odd or $u_{\frac{n}{2}+1}v_{\frac{n}{2}}$ when n is even.

Definition 1.4. A triangular snake T_n is obtained from a path $u_1u_2...u_n$ by joining u_i and u_{i+1} to a new vertex w_i for $1 \leq i \leq n-1$. That is, every edge of a path is replaced by a cycle C_3.

Definition 1.5. A Quadrilateral snake Q_n is obtained from a path $u_1u_2...u_n$ by joining u_i and u_{i+1} to two new vertices v_i and w_i respectively and joining v_i and w_i for $1 \leq i \leq n-1$. That is, every edge of a path is replaced by a cycle C_4.

Definition 1.6. The *Cartesian product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph G(V, E) with $V = V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \square G_2$ whenever ($u_1 = v_1$ and u_2 is adjacent to v_2) or ($u_2 = v_2$ and u_1 is adjacent to v_1). It is denoted by $G_1 \square G_2$.

Definition 1.7. The prism D_n, $n \geq 3$ is a trivalent graph which can be defined as the Cartesian product $P_2 \square C_n$ of a path on two vertices P_2 with a cycle on n vertices C_n. We denote a graph obtained by attaching P_3 at each vertex of outer cycle of D_n by (D_n; P_3).

Definition 1.8. The product $P_m \square P_n$ is called a *planar grid*. The product $P_2 \square P_n$ is called a *ladder*, and it is denoted by L_n.

Definition 1.9. A triangle ladder TL_n, $n \geq 2$, is a graph obtained from a ladder L_n by adding the edges u_iv_{i+1} for $1 \leq i \leq n-1$, where u_i and v_i, $1 \leq i \leq n$, are the vertices of L_n such that $u_1u_2...u_n$ and $v_1v_2...v_n$ are two paths of length n in L_n.

2. Main Results

Theorem 2.1. H – graph admits a Harmonic mean labeling

Proof. Let G be a H – graph. Then G is obtained from two paths $u_1u_2...u_n$ and $v_1v_2...v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ when n is odd or $u_{\frac{n}{2}+1}v_{\frac{n}{2}}$ when n is even. Define a function f: $V(G) \rightarrow \{1, 2, ..., q+1\}$ by

$f(u_i) = i$, $1 \leq i \leq n$;

$f(v_i) = n + i$; $1 \leq i \leq n$.

Then obviously the edge labels are distinct. Hence G admits a Harmonic mean labeling.

Example 2.2. A Harmonic mean labeling of H – graph when n = 6 and n = 7 are shown in figure 2.1 and 2.2, respectively.

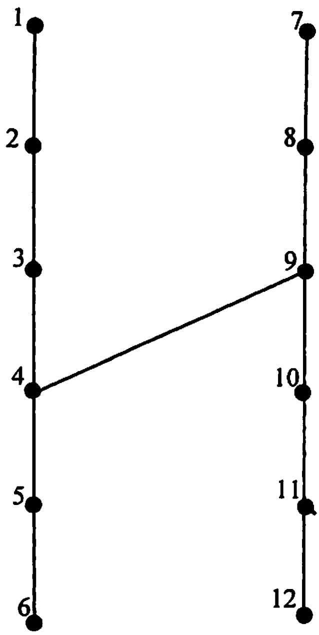

Figure. 2.1

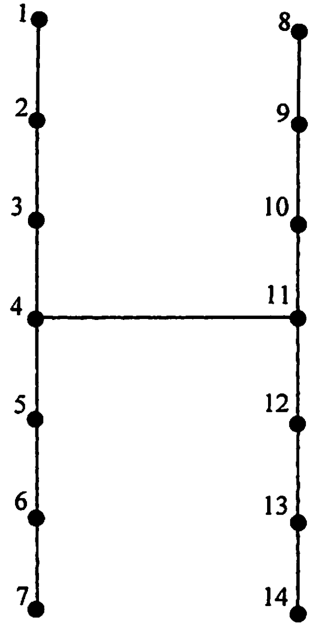

Figure. 2.2

Theorem 2.3. $D_n \odot K_1$ is a Harmonic mean graph.

Proof. Let u_i, v_i be the vertices of inner and outer cycle of D_n respectively in which u_i is joined with v_i. Let s_i and t_i be the vertices which are joined with u_i and v_i respectively, $1 \leq i \leq n$. The resultant graph is $D_n \odot K_1$ whose edge set is $E = \{u_nu_1, v_nv_1, u_iu_{i+1}, v_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{s_iu_i, u_iv_i, v_it_i / 1 \leq i \leq n\}$. Define a function $f: V(D_n \odot K_1) \rightarrow \{1, 2, \ldots, q+1\}$ by

$f(s_1) = 2$; $f(s_i) = 5i + 1$; $2 \leq i \leq n$;

$f(u_1) = 3$; $f(u_i) = 5i$, $2 \leq i \leq n$;

$f(v_i) = 5i - 1$, $1 \leq i \leq n$;

$f(t_1) = 1$; $f(t_i) = 5i - 3$, $2 \leq i \leq n$.

Then the edges get labels

$f(s_1u_1) = 2$; $f(s_iu_i) = 5i$, $2 \leq i \leq n$;

$f(u_1v_1) = 3$; $f(u_iv_i) = 5i - 1$, $2 \leq i \leq n$;

$f(v_1t_1) = 1$; $f(v_it_i) = 5i - 2$, $2 \leq i \leq n$;

$f(u_1u_2) = 4$; $f(u_iu_{i+1}) = 5i + 2$, $2 \leq i \leq n - 1$; $f(u_nu_1) = 5$;

$f(v_iv_{i+1}) = 5i + 1,\ 1 \le i \le n - 1;\ f(v_nv_1) = 7.$

Hence f is a Harmonic mean labeling for $D_n \odot K_1$. Therefore, $D_n \odot K_1$ is a Harmonic mean graph.

Example 2.4.. A Harmonic mean labeling of $D_n \odot K_1$ is given in figure 2.3.

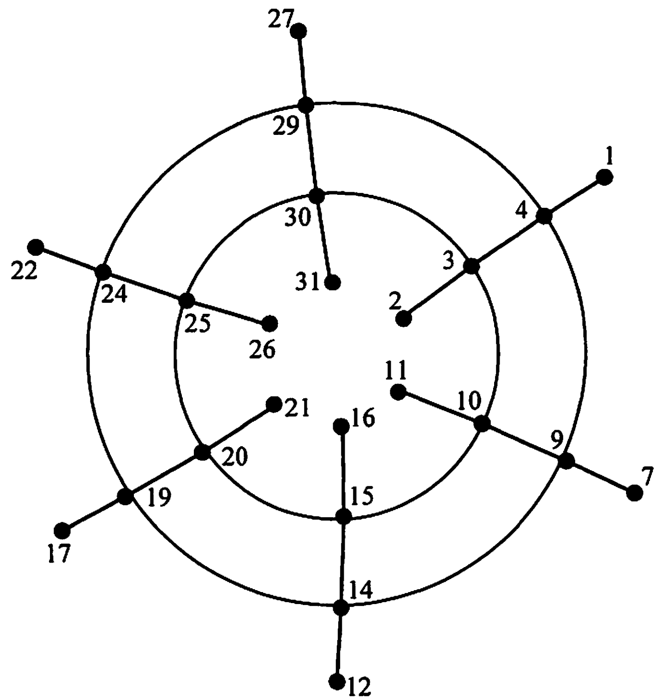

Figure. 2.3

Theorem 2.5.. $L_n \odot K_2^c$ is a Harmonic mean graph.

Proof. Let u_i, v_i, $1 \le i \le n$, be the vertices of L_n. $L_n \odot K_2^c$ is obtained from L_n by joining u_i to two new vertices s_i and t_i and v_i to two new vertices x_i and y_i and hence its edge set is $E = \{u_iu_{i+1}, v_iv_{i+1} / 1 \le i \le n - 1\} \cup \{u_is_i, u_it_i, u_iv_i, v_ix_i, v_iy_i / 1 \le i \le n\}$. Define a function $f: V(L_n \odot K_2^c) \to \{1, 2, \ldots, q + 1\}$ by

$f(s_1) = 1;\ f(s_i) = 7(i - 1),\ 2 \le i \le n;$

$f(t_i) = 7i - 5,\ 1 \le i \le n;$

$f(u_1) = 4;\ f(u_i) = 7i - 4,\ 2 \le i \le n;$

$f(v_1) = 5;\ f(v_i) = 7i - 3,\ 2 \le i \le n;$

$f(x_1) = 3;\ f(x_i) = 7i - 2,\ 2 \le i \le n;$

$f(y_i) = 7i - 1,\ 1 \le i \le n.$

Then the edges are labeled with

$f(s_iu_i) = 7i - 6,\ 1 \le i \le n;$

$f(t_iu_i) = 7i - 5,\ 1 \le i \le n;$

$f(u_iu_{i+1}) = 7i - 1,\ 1 \le i \le n - 1;$

$f(v_iv_{i+1}) = 7i,\ 1 \le i \le n - 1;$

$f(u_1v_1) = 4;\ f(u_iv_i) = 7i - 4,\ 2 \le i \le n;$

$f(v_1x_1) = 3;\ f(v_ix_i) = 7i - 3,\ 2 \le i \le n;$

$f(v_iy_i) = 7i - 2,\ 1 \le i \le n.$

Thus f provides a Harmonic mean labeling for $L_n \odot K_2^c$. Hence the theorem.

Example 2.6. A Harmonic mean labeling for $L_5 \odot K_2^c$ is shown in figure 2.4

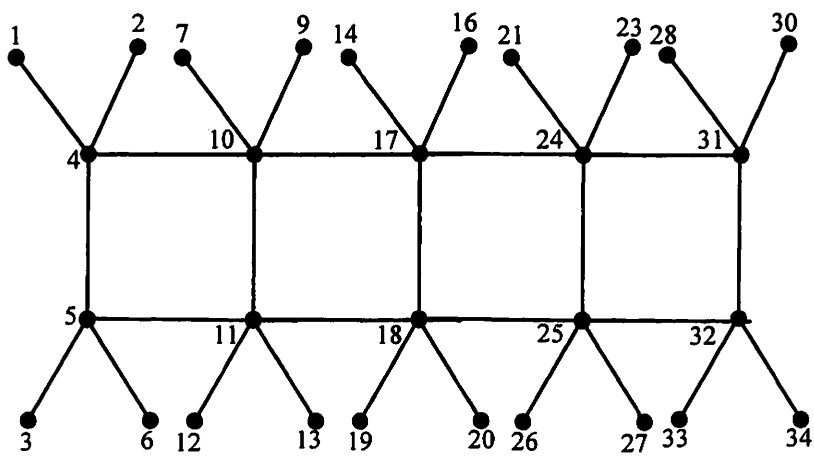

Figure 2.4

Theorem 2.7. Triangular ladder TL_n is a Harmonic mean graph.

Proof. Let u_i, v_i be the vertices of L_n, $1 \le i \le n$. Join u_i to v_{i+1}, $1 \le i \le n - 1$. The resultant graph is TL_n whose edge set is $E = \{u_iu_{i+1},\ v_iv_{i+1},\ u_iv_{i+1} / 1 \le i \le n - 1\} \cup \{u_iv_i / 1 \le i \le n\}$. Define a function $f: V(TL_n) \rightarrow \{1, 2, \ldots, q + 1\}$ by

$f(u_1) = 3;\ f(u_i) = 4i - 2,\ 2 \le i \le n;$

$f(v_1) = 1;\ f(v_i) = 4(i - 1),\ 2 \le i \le n.$

Then the edges are labeled with

$f(u_iu_{i+1}) = 4i,\ 1 \le i \le n - 1;$

$f(v_1v_2) = 1$; $f(v_iv_{i+1}) = 4i - 2, 2 \le i \le n - 1$;

$f(u_1v_1) = 2$; $f(u_iv_i) = 4i - 3, 2 \le i \le n$;

$f(u_1v_2) = 3$; $f(u_iv_{i+1}) = 4i - 1, 2 \le i \le n - 1$.

Thus f provides a Harmonic mean labeling for TL_n.

Example 2.8. A Harmonic mean labeling of TL_6 is given below.

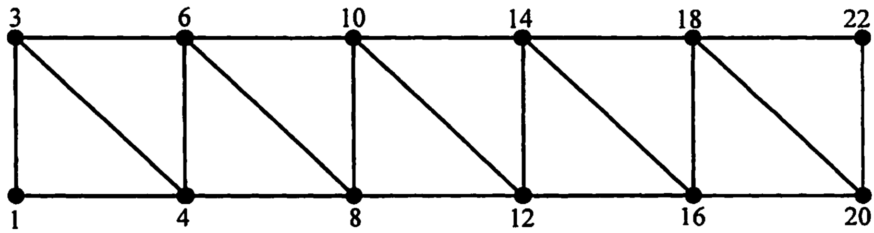

Figure 2.5

Theorem 2.9. $TL_n \odot K_1$ is a Harmonic mean graph.

Proof. Let u_i, v_i, $1 \le i \le n$ be the vertices of TL_n . Join u_i and v_i to the new vertices x_i and y_i respectively. The resultant graph is $TL_n \odot K1$ whose edge set is $E = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1} / 1 \le i \le n - 1\} \cup \{u_ix_i, u_iv_i, v_iy_i / 1 \le i \le n\}$..

Define a function f: $V(TL_n \odot K_1) \rightarrow \{1, 2, \ldots, q + 1\}$ by

$f(u_i) = 6i - 2, 1 \le i \le n$;

$f(v_1) = 3$; $f(v_i) = 6(i - 1) + 1, 2 \le i \le n$;

$f(x_1) = 1$; $f(x_i) = 6i - 3, 2 \le i \le n$;

$f(y_1) = 2$; $f(y_i) = 6(i - 1), 2 \le i \le n$.

Then the edges get the labels

$f(u_1u_2) = 6$; $f(u_iu_{i+1}) = 6i + 1, 2 \le i \le n - 1$;

$f(v_iv_{i+1}) = 6i - 2, 1 \le i \le n - 1$;

$f(u_iv_{i+1}) = 6i - 1, 1 \le i \le n - 1$;

$f(u_1v_1) = 3$; $f(u_iv_i) = 6i - 4, 2 \le i \le n$;

$f(u_1x_1) = 1$; $f(u_ix_i) = 6i - 3, 2 \le i \le n$;

$f(y_1v_1) = 2$; $f(y_2v_2) = 7$; $f(y_iv_i) = 6(i - 1), 3 \le i \le n$.

Thus f provides a Harmonic mean labeling for $TL_n \odot K_1$. Hence $TL_n \odot K_1$ is a Harmonic mean graph.

Example 2.10. A Harmonic mean labeling of $TL_6 \odot K_1$ is given in figure 2.6.

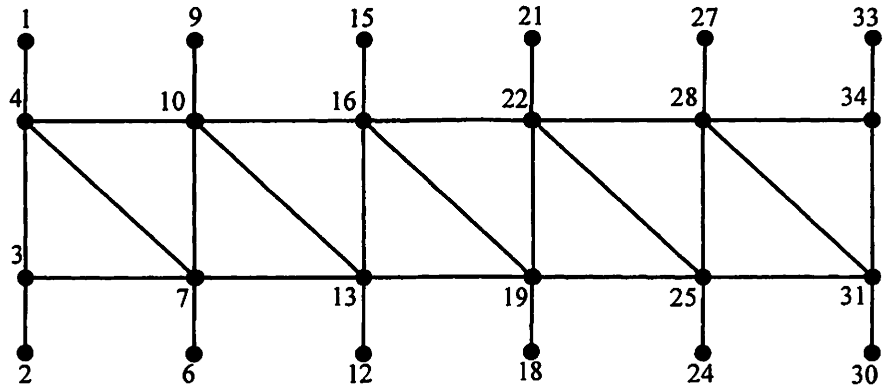

Figure 2.6

Theorem 2.11. $T_n \odot K_1$ is a Harmonic mean graph.

Proof. Let u_i, v_i, x_i and y_i be the vertices of $T_n \odot K_1$ such that u_i and u_{i+1} is joined to v_i, $1 \leq i \leq n-1$. Join x_i to u_i, $1 \leq i \leq n$ and y_i to v_i, $1 \leq i \leq n-1$. Then the edge set is $\{u_iu_{i+1}, u_iv_i, u_{i+1}v_i, v_iy_i / \ 1 \leq i \leq n-1\} \cup \{u_ix_i / \ 1 \leq i \leq n.\}$. Define a function $f: V(T_n \odot K_1) \rightarrow \{1, 2, \ldots, q+1\}$ by

$f(u_i) = 5i - 3, \ 1 \leq i \leq n;$

$f(v_i) = 5i - 2, \ 1 \leq i \leq n - 1;$

$f(x_1) = 1; \ f(x_2) = 5; \ f(x_i) = 5i - 4, \ 3 \leq i \leq n;$

$f(y_i) = 5i - 1, \ 1 \leq i \leq n - 1.$

Then the edges are labeled with

$f(u_1u_2) = 3; \ f(u_iu_{i+1}) = 5i - 1, \ 2 \leq i \leq n - 1;$

$f(u_ix_i) = 5i - 4, \ 1 \leq i \leq n;$

$f(u_iv_i) = 5i - 3, \ 1 \leq i \leq n - 1;$

$f(u_{i+1}v_i) = 5i, \ 1 \leq i \leq n - 1;$

$f(v_1y_1) = 4; \ f(v_iy_i) = 5i - 2, \ 2 \leq i \leq n - 1;$

Thus f provides a Harmonic mean labeling for $T_n \odot K_1$. Hence $T_n \odot K_1$ is a Harmonic mean Graph.

Example 2.12. Harmonic mean labeling of $T_6 \odot K_1$ is shoun in figure 2.7.

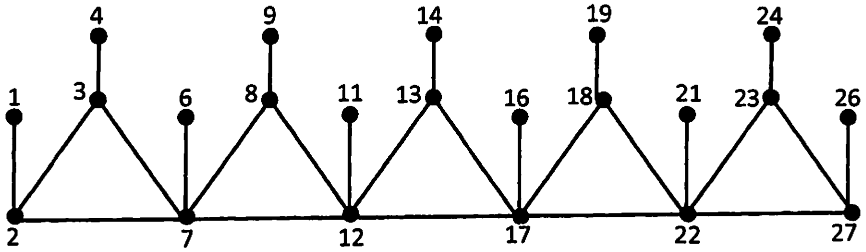

Figure. 2.7

Theorem 2.13. $Q_n \odot K_1$ is a Harmonic mean graph.

Proof. Consider a path $u_1u_2....u_n$. Join u_i and u_{i+1} to the vertices v_i and w_i respectively and then join v_i and w_i. Join u_i to x_i, $1 \leq i \leq n$, v_i to y_i and w_i to z_i, $1 \leq i \leq n-1$. The resultant graph is $Q_n \odot K1$ whose edge set is $\{u_iu_{i+1}, u_iv_i, v_iy_i, v_iw_i, w_iz_i, u_{i+1}w_i\} / 1 \leq i \leq n-1\} \cup \{u_ix_i / 1 \leq i \leq n\}$. Define a function $f: V(Q_n \odot K_1) \rightarrow \{1, 2, \ldots, q+1\}$ by

$f(x_1) = 1$; $f(x_i) = 7(i-1), 2 \leq i \leq n$;

$f(u_1) = 4$; $f(u_i) = 7i - 5, 2 \leq i \leq n$;

$f(v_i) = 7i - 4, 1 \leq i \leq n-1$;

$f(y_1) = 2$; $f(y_i) = 7i - 3, 2 \leq i \leq n-1$;

$f(w_i) = 7i - 2, 1 \leq i \leq n-1$;

$f(z_i) = 7i - 1, 1 \leq i \leq n-1$.

Then the edges get labels

$f(u_iu_{i+1}) = 7i - 2, 1 \leq i \leq n-1$;

$f(u_ix_i) = 7i - 6, 1 \leq i \leq n$;

$f(u_1v_1) = 3$; $f(u_iv_i) = 7i - 5, 2 \leq i \leq n-1$;

$f(v_1y_1) = 2$; $f(v_iy_i) = 7i - 4, 2 \leq i \leq n-1$;

$f(v_iw_i) = 7i - 3, 1 \leq i \leq n-1$;

$f(w_iz_i) = 7i - 1, 1 \leq i \leq n-1$;

$f(u_{i+1}w_i) = 7i, 1 \leq i \leq n-1$.

In the view of the above labeling pattern f provides a Harmonic mean labeling for $Q_n \odot K_1$. Hence the theorem.

Example 2.14. A Harmonic mean labeling of $Q_n \odot K_1$ is given in figure 2.8.

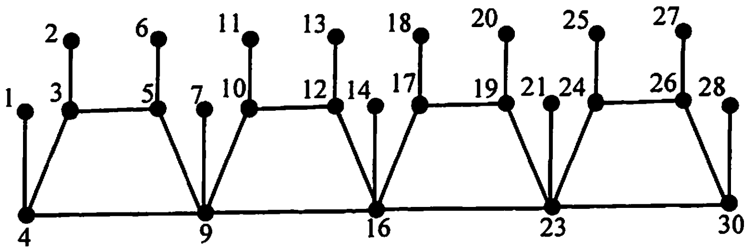

Figure. 2.8

Conclusion:

As all graphs are not Harmonic mean graphs, it is very interesting to investigate graphs which admits a Harmonic mean labeling. In this paper, we investigated some new Harmonic mean graphs. It is possible to investigate similar results for several other graphs in the context of different labeling techniques.

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