Counting the maximal independent sets in power set graphs

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Abstract

Counting the number of maximal independent sets is #P-complete even for chordal graphs. We prove that the number of maximal independent sets in a subclass G_n^R (Right power set graphs) of chordal graphs can be computed in polynomial time using Golomb's non-linear recurrence relation. We provide a recursive construction of G_n^R and prove that there are $2^{(\frac{|V(G_n^R)|+1}{4})}$ maximum independent sets in G_n^R. We also provide a polynomial time algorithm to solve the maximum independent set problem (MISP) in a superclass $\mathcal{F}_n$ of complement of G_n^R.

Keywords: Maximum independent set; Golomb's recurrence; Power set graphs

1 Introduction

Counting the number of independent sets and number of maximum independent sets in a graph is #P-complete [7] and counting the number of independent sets of size k in a graph is $\#W[1]$-complete [2]. Indeed, counting the number of maximal independent sets in chordal graphs is #P-complete [6]. In addition, counting the number of independent sets in a planar bipartite graph of maximum degree four is also #P-complete [8]. In this paper, we give a recursive construction of a subclass G_n^R of chordal graphs and count the number of maximal independent sets of G_n^R in polynomial (logarithmic) time using the following non-linear recurrence relation by Golomb [1, 3]

$$y_n = 1 + \prod_{i=1}^{n-1} y_i \quad ; \quad y_1 = 1.$$

This equation generates a sequence $\{y_n\} = \{1, 2, 3, 7, 43, 1807, 3263443, \ldots\}$ and it occurs in Lucas test for primality of Mersenne numbers [4]. We prove that there are $2^{(\frac{|V(G_n^R)|+1}{4})}$ maximum independent sets in G_n^R. Moreover, we provide a polynomial time algorithm to solve MISP in a superclass $\mathcal{F}_n$ of complement of G_n^R.
For graph terminologies, we refer [9]. The graphs considered in this paper are finite, simple and undirected. Here K_n denote the complete graph on n vertices. A *clique* (*independent set*) is a subset of vertices of a graph G which are pairwise adjacent(non-adjacent) in G. The cardinality of a maximum clique (independent set) in a graph G is called *clique* (*independence*) *number* and is denoted by $\omega(G)(\alpha(G))$. An independent set of a graph G is *maximal* if it is not properly contained in any other independent set of G. The *join* $G_1 \oplus G_2$ of vertex-disjoint graphs G_1 and G_2 is a graph with $V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \cup [V(G_1), V(G_2)]$ where $[V(G_1), V(G_2)] = \{(x, y) \ : \ x \in V(G_1), y \in V(G_2)\}$. Also, the *co-join*(or *disjoint union*) $G_1 \cup G_2$ of vertex-disjoint graphs G_1 and G_2 is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

2 Right power set graphs G_n^R

In this section, we discuss a subclass G_n^R of chordal graphs and its complement graph class G_n^L, and provide a recursive construction of G_n^R. Let us denote $\{1, 2, \ldots, n\}$ as $[n]$ and $\mathcal{P}([n])$ as the power set on $[n]$. For $A \in \mathcal{P}([n]) \setminus \{\emptyset\}$, let $m(A) := \min \{a \ : \ a \in A\}$. Let A_1, A_2 be two non-empty distinct subsets of $[n]$. We define Left$(A_1, A_2) := A_1$ if $m((A_1 \setminus A_2) \cup (A_2 \setminus A_1)) \in A_1$, else Left$(A_1, A_2) := A_2$. For a subset $A = \{b_1, b_2, b_3, \ldots, b_l\} \in \mathcal{P}([n]), l \geq 2$, where $b_1 < b_2 < \ldots < b_l$, we say the subsets of the form $\{b_2, b_3, \ldots, b_l\}, \{b_3, b_4, \ldots, b_l\}, \ldots, \{b_{l-1}, b_l\}$ and $\{b_l\}$ are right subsets of A. Note that for a right subset A_1 of A_2 (i) $A_1 \subset A_2$ (ii) $A_1 \setminus A_3$ is a right subset of $A_2 \setminus A_3$ for every proper subset A_3 of A_1 and (iii) if $a \in A_1$, then $b \in A_1$ for all $b \in A_2$ such that $b > a$. If a proper non-empty subset A_1 of A_2 is not a right subset of A_2, then there exists $b \in A_2 \setminus A_1$ such that $m(A_1) < b$.
The *right power set graph* G_n^R is a graph with vertex set $V(G_n^R) = \mathcal{P}([n]) \setminus \{\emptyset\}$ such that $(A_1, A_2) \in E(G_n^R)$ if and only if A_1 is the right subset of A_2 (or vice versa) where $A_1, A_2 \in V(G_n^R)$.

The *left power set graph* G_n^L is a graph with vertex set $V(G_n^L) = \mathcal{P}([n]) \setminus \{\emptyset\}$ and the edge set $E(G_n^L)$ defined as follows:

- For any $A_1, A_2 \in V(G_n^L)$, if $A_1 \setminus A_2 \neq \emptyset$ and $A_2 \setminus A_1 \neq \emptyset$, then $(A_1, A_2) \in E(G_n^L)$, and
- For any $A_1, A_2, A_3 \in V(G_n^L)$, if $A_1 \setminus A_2 \neq \emptyset, A_2 \setminus A_1 \neq \emptyset$, and $A_1, A_2 \subset A_3$, then $(A_3, \text{Left}(A_1, A_2)) \in E(G_n^L)$.

The Figure 1 depicts the graphs G_3^L and G_3^R. Note that the complement graph of G_3^L is G_3^R, which is true in general.

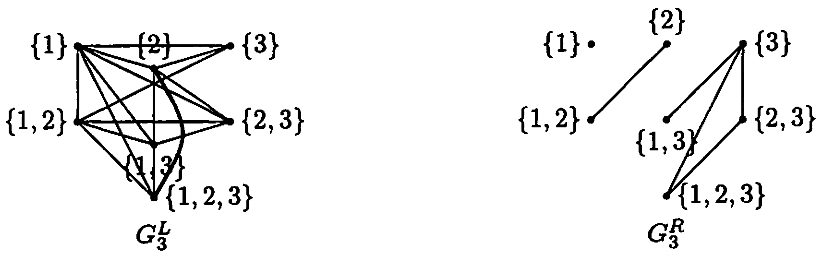

Figure 1: The graphs G_3^L and G_3^R

Lemma 2.1. *For a positive integer n, the complement graph of G_n^L is G_n^R.*

Proof. It is enough to prove that for any two distinct sets $A_1, A_2 \in \mathcal{P}([n]) \setminus \{\emptyset\}, (A_1, A_2) \in E(G_n^L)$ if and only if $(A_1, A_2) \notin E(G_n^R)$.
First, we prove that $(A_1, A_2) \in E(G_n^L)$ implies $(A_1, A_2) \notin E(G_n^R)$. There are two cases: (i) Neither $A_1 \subset A_2$ nor $A_2 \subset A_1$. So $A_1(A_2)$ is not a right subset of $A_2(A_1)$ and hence $(A_1, A_2) \notin E(G_n^R)$. (ii) W.l.o.g., assume $A_1 \subset A_2$. Since $(A_1, A_2) \in E(G_n^L)$ and $A_1 \setminus A_2 = \emptyset$, by definition, there exists a subset A_3 of A_2 such that $A_1 \setminus A_3 \neq \emptyset$, $A_3 \setminus A_1 \neq \emptyset$ and $\text{Left}(A_1, A_3) = A_1$. Note that $\text{Left}(A_1, A_3) = A_1$ implies $a = m(A_1 \setminus A_3) < b = m(A_3 \setminus A_1)$. Moreover, $a \in A_1, b \notin A_1$ and $b \in A_2$ implies A_1 is not a right subset of A_2. Since $A_1 \subset A_2$, A_2 is not a right subset of A_1. Hence $(A_1, A_2) \notin E(G_n^R)$.

Next, we prove that $(A_1, A_2) \notin E(G_n^L)$ implies $(A_1, A_2) \in E(G_n^R)$. Suppose $(A_1, A_2) \notin E(G_n^L)$. Then either $A_1 \setminus A_2 = \emptyset$ or $A_2 \setminus A_1 = \emptyset$. W.l.o.g., assume $A_1 \subset A_2$. It is enough to prove that A_1 is a right subset of A_2. On the contrary, suppose A_1 is not a right subset of A_2. Then there exists $b \in A_2 \setminus A_1$ such that $a = m(A_1) < b$. Define a set $A_3 = (A_1 \setminus \{a\}) \cup \{b\}$. It is clear that, A_1 and A_3 are proper subsets of A_2. Also $a \in A_1 \setminus A_3, b \in A_3 \setminus A_1$

and Left$(A_1, A_3) = A_1$. This contradicts $(A_1, A_2) \notin E(G_n^L)$. Therefore, A_1 is a right subset of A_2 and hence $(A_1, A_2) \in E(G_n^R)$. □

Next, we construct the components of G_n^R recursively by defining a sequence of graphs M_n, as follows (see Figure 2):

1. M_1 is a graph with $V(M_1) = \{1\}$ and $E(M_1) = \emptyset$.
2. For any positive integer $i > 1$, M_i is a graph with vertex set $V(M_i) = \{i\} \cup \{A \cup \{i\} : A \in \bigcup_{j=1}^{i-1} V(M_j)\}$ and edge set $E(M_i) = E_i \cup \{(A, \{i\}) : A \in V(M_i) \setminus \{i\}\}$ where $E_i = \{(A, B) : A, B \in V(M_i) \text{ and } (A \setminus \{i\}, B \setminus \{i\}) \in \bigcup_{j=1}^{i-1} E(M_j)\}$.

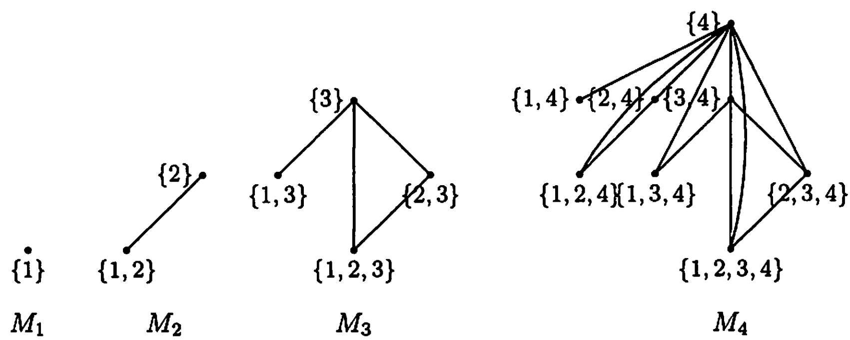

Figure 2: The graphs M_1, M_2, M_3, M_4

In the next section, we compute the clique number, independence number for G_n^R and G_n^L.

3 Enumeration of independent sets and cliques

Observation 3.1. *a. Let $A, B \in V(G_n^R)$ such that $A, B \neq \{n\}$. Then $(A, B) \in E(G_n^R)$ if and only if $(A \setminus \{n\}, B \setminus \{n\}) \in E(G_{n-1}^R)$.*

b. $V(M_1 \cup M_2 \cup \ldots \cup M_n) = \mathcal{P}([n]) \setminus \{\emptyset\}$, $V(M_n) = \mathcal{P}([n]) \setminus \mathcal{P}([n-1])$ for $n > 1$.

c. *For a clique $A_1, A_2, \ldots, A_p$ in M_n, there exists a chain of subsets $A_1 \subset A_2 \subset \ldots \subset A_p$ where A_i is a right subset of A_j, $1 \le i < j \le p$.*

Theorem 3.1. *For a positive integer n,*

1. *M_n is isomorphic to the join of K_1 and $\bigcup_{i=1}^{n-1} M_i$.*
2. *$G_n^R = \bigcup_{i=1}^{n} M_i = M_n \cup G_{n-1}^R$.*
3. *M_n and G_n^R are subclasses of chordal graphs.*

Proof. Define $f : V(M_n) \mapsto V(K_1 \oplus \bigcup_{i=1}^{n-1} M_i)$ as follows: Let $V(K_1) = \{x\}$ For every $A \in V(M_n)$,

$$f(A) = \begin{cases} A \setminus \{n\}, & \text{if } A \neq \{n\} \\ x, & \text{if } A = \{n\} \end{cases}$$

It is easy to verify that f is bijective function which preserves adjacency. By the construction, M_n is the n^{th} component of G_n^R. Hence $G_n^R = \bigcup_{i=1}^{n} M_i = M_n \cup G_{n-1}^R$.

Also note that, M_n is constructed by taking co-join of $M_1, M_2, \ldots, M_{n-1}$ and finally applying join with K_1. It is clear that M_1 , M_2 and M_3 are chordal. If a graph G is chordal, then $K_1 \oplus G$ is also chordal. Hence M_n and G_n^R are chordal. □

Theorem 3.2. *For a positive integer n,*

1. *$\omega(M_n) = 1 + \omega(M_{n-1})$ for $n > 1$, $\omega(M_1) = 1$ and $\omega(M_n) = n$.*
2. *$\omega(G_n^R) = \alpha(G_n^L) = n$.*
3. *$\alpha(M_n) = 2\alpha(M_{n-1})$ for $n \ge 3$, $\alpha(M_1) = \alpha(M_2) = 1$, $\alpha(M_3) = 2$, and hence $\alpha(M_n) = 2^{n-2}$.*
4. *$\alpha(G_n^R) = \omega(G_n^L) = 2^{n-1}$, $n \ge 3$.*
5. *The number of maximum independent sets in M_n is $2^{2^{n-3}}$, $n \ge 3$. Also, the number of maximum independent sets in G_n^R is $2^{2^{n-2}}$. Similarly, the number of maximum cliques in G_n^L is $2^{2^{n-2}}$, $n \ge 2$.*
6. *$\mathcal{N}(M_n) = 1 + \prod_{i=1}^{n-1} \mathcal{N}(M_i)$ and $\mathcal{N}(G_n^R) = \mathcal{N}(G_{n-1}^R)[1 + \mathcal{N}(G_{n-1}^R)]$ where $\mathcal{N}$ is the number of maximal independent sets.*

7. *$[n]$ and all its right subsets forms a unique clique of size n in M_n. Also, they form a unique independent set of size n in G_n^L.*

Proof. The proof of $1, 2, 3, 4$ are simple applications of Theorem 3.1.
We know that $w(M_1) = 1, \quad w(M_2) = 2, \quad w(M_3) = 3$. Assuming the result for $n-1$,

$$\begin{aligned}\omega(M_n) &= \omega(K_1 \oplus \bigcup_{i=1}^{n-1} M_i) \\ &= \omega(K_1) + max\{\omega(M_1), \omega(M_2), \ldots, \omega(M_{n-1})\} \\ &= 1 + \omega(M_{n-1}) = n \qquad \text{(By induction)}\end{aligned}$$

Now,

$$\begin{aligned}\omega(G_n^R) &= \omega(\bigcup_{i=1}^{n} M_i) \\ &= max\{\omega(M_1), \omega(M_2), \ldots, \omega(M_n)\} \\ &= \omega(M_n) = n\end{aligned}$$

$$\begin{aligned}\alpha(M_n) &= \alpha(K_1 \oplus \bigcup_{i=1}^{n-1} M_i) \\ &= max\{\alpha(K_1), \alpha(M_1) + \alpha(M_2) + \ldots + \alpha(M_{n-1})\} \\ &= \sum_{i=1}^{n-1} \alpha(M_i) \\ &= \sum_{i=1}^{n-2} \alpha(M_i) + \alpha(M_{n-1}) = 2\alpha(M_{n-1}) \qquad \text{(By induction)}\end{aligned}$$

But we know that $\alpha(M_1) = 1, \quad \alpha(M_2) = 1$, and $\alpha(M_3) = 2$; hence $\alpha(M_n) = 2^{n-2}, n \geq 3$.
By Theorem 3.1,

$$\alpha(G_n^R) = \sum_{i=1}^{n} \alpha(M_i) = 2^{n-1}$$

Let us denote the number of maximum independent sets by *ni*. By the construction of M_n,

$$\begin{aligned}ni(M_n) &= ni(M_{n-1}) \underbrace{ni(M_{n-2}) \cdots ni(M_2)\, ni(M_1)} \\ &= ni(M_{n-1})\, ni(M_{n-1}) \\ &= (ni(M_{n-1}))^2\end{aligned}$$

Since $ni(M_1) = 1$ and $ni(M_2) = 2$, we get $ni(M_n) = 2^{2^{n-3}}$ for $n \geq 3$ and $ni(G_n^R) = 2^{2^{n-2}}$.
Again by Theorem 3.1 (1) , we can obtain Golomb's non-linear recurrence relation

$$\mathcal{N}(M_n) = 1 + \prod_{i=1}^{n-1} \mathcal{N}(M_i) \quad ; \quad \mathcal{N}(M_1) = 1. \tag{1}$$

Also by Theorem 3.1 (2), we obtain

$$\mathcal{N}(G_n^R) = \mathcal{N}(G_{n-1}^R)[1 + \mathcal{N}(G_{n-1}^R)] \tag{2}$$

Hence by Equations (1) and (2), the number of maximal independent sets in G_n^R can be computed in $O(log|V(G_n^R)|)$ time.
By Observation 3.1*(c)*, $\{1,2,3,\ldots,n\}, \{2,3,\ldots,n\}, \{3,\ldots,n\},\ldots,\{n-1,n\}$, $\{n\}$ forms a unique clique of size n in M_n. And hence these sets forms a unique independent set of size n in G_n^L. □

As a consequence of Theorem 3.2, we have

Corollary 3.1. *Counting the number of maximal independent sets in G_n^R can be done in polynomial ($log|V(G_n^R)|$) time. (By Theorem 3.2(6))*

Corollary 3.2. *There are $2^{(\frac{|V(G_n^L)|+1}{4})}$ maximum cliques in G_n^L. (By Theorem 3.2(5))*

4 Power set graphs

We discuss a superclass $\mathcal{F}_n$ (Power set graphs) of G_n^L and prove that the class admits a polynomial time algorithm to solve the MISP. A graph $G \in \mathcal{F}_n$ if $V(G) = \mathcal{P}([n]) \setminus \{\emptyset\}$ such that

1. for every $A_1, A_2 \in V(G)$, if $A_1 \setminus A_2 \neq \emptyset$ and $A_2 \setminus A_1 \neq \emptyset$, then $(A_1, A_2) \in E(G)$, and
2. for every $A_1, A_2, A_3 \in V(G)$, if $A_1, A_2 \subset A_3$, $A_1 \setminus A_2 \neq \emptyset$ and $A_2 \setminus A_1 \neq \emptyset$, then at least one of A_1 and A_2 is adjacent to A_3 in G.

Observation 4.1. *Let $L_i = \{A \in \mathcal{P}([n]) \ : \ |A| = i\}$ for $i \geq 1$. For a graph $G \in \mathcal{F}_n$, (i) $L_i \cap V(G)$ induces a clique, (ii) $V(G)$ can be partitioned into at most n cliques, (iii) every vertex in $L_i \cap V(G)$ is not adjacent to atmost one vertex in $L_j \cap V(G)$ for $1 \leq j < i \leq n$ and (iv) $\alpha(G) \leq n$ and $|V(G)| = 2^n - 1$.*

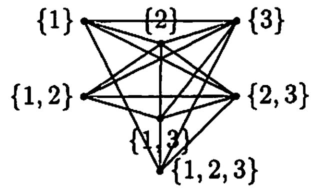

Figure 3: A graph $G \in \mathcal{F}_3$

Algorithm 1 Finding a maximum independent set of $G \in \mathcal{F}_n$

Input: A graph $G \in \mathcal{F}_n$
Output: A maximum independent set of the graph G
$I := \emptyset$
for all $i = n, n-1, \ldots, 1$ **do**
  for all $A \in L_i$ **do**
    $$S_A := \{B : (A, B) \notin E(G) \quad \& \quad B \in \bigcup_{j=1}^{i-1} L_j\}$$
    $S \leftarrow$ **MIS**(S_A)
    if $(|I| \leq |S|)$ **then**
      $I \leftarrow S \cup A$
    end if
  end for
end for
Return I

Observation 4.1 lead us to a polynomial time algorithm to compute MISP in $\mathcal{F}_n$. In Algorithm 1, **MIS**(S_A) finds the power set of $S_A (|S_A| \leq n = log(|V(G)| + 1))$ and computes a maximum independent subset of S_A in G by an exhaustive search which takes $O(|V(G)|^2 log|V(G)|)$. As this step is repeated for every vertex in G, the time complexity is $O(|V(G)|^3 log|V(G)|)$.

5 Conclusion

In this paper, we provided a subclass G_n^R of chordal graphs for which the number of maximal independent sets can be computed in $O(log|V(G_n^R)|)$ time. We gave a recursive construction of the class G_n^R and proved that there are $2^{(\frac{|V(G_n^L)|+1}{4})}$ maximum cliques in G_n^L. In addition, we proved that MISP for the class $\mathcal{F}_n$, a superclass of G_n^L can be solved efficiently.

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