A note on the paper "Eternal security in graphs" by Goddard, Hedetniemi, and Hedetniemi (2005)

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Abstract

In the paper "Eternal security in graphs" by Goddard, Hedetniemi and Hedetniemi (2005, [4]), the authors claimed that, for any Cayley graph, the *eternal m-security number* equals the minimum cardinality of a *dominating set*. However, the equality is false. In this note, we present a counterexample and comment on the eternal m-security number for Cayley graphs.

1 Introduction

Goddard, Hedetniemi, and Hedetniemi [4] studied the problem of determining the *eternal* 1*-security number* of a graph. They defined this problem as finding the minimum cardinality of an *eternal 1-secure set* of the graph. This cardinality was first considered by Burger et al. [2].

The same authors also investigated a related problem: determining the *eternal m-security number* of a graph, denoted by σ_m (some authors denote the eternal m-security number by γ_m^∞). This problem consists in finding the minimum cardinality of an *eternal m-secure set* of the graph. For a better understanding, we state the problem in a formal way. To this end, given a simple graph $G = (V, E)$, we first define the concepts of a *shift* and of a *dominating set.*

Take two sets of vertices $A, B \subseteq V$. A shift from A to B is a bijective function $f : A \to B$ such that, if $f(u) = v$, then $u = v$ or $uv \in E$. Notice that there is a shift from A to B only if $|A| = |B|$.

A set $D \subseteq V$ is a dominating set of G if, for each $v \in (V \setminus D)$, there is a vertex $u \in D$ such that $uv \in E$. We denote the minimum cardinality of a dominating set of G by $\gamma(G)$.

A dominating set $D_0 \subseteq V$ is an eternal m-secure set of G if, for any sequence of vertices $v_1, v_2, \ldots \in V$, one can construct a sequence of dominating sets $D_1, D_2, \ldots$ of G such that, for $i = 1, 2, \ldots$:

1. There is a shift from D_{i-1} to D_i;
2. $v_i \in D_i$.

The problem of determining σ_m admits the following interpretation. Consider guards placed on the vertices of a graph with at most one guard per vertex. Suppose that an attack occurs at a vertex. To defend the attack, one guard must move from an adjacent vertex to the attacked one, unless it already had a guard. The other guards may move to prepare to defend a next attack. The problem is to find the minimum number of guards so that attacks can be defended indefinitely. The computation of the eternal 1-security number of a graph corresponds to a version of this problem in which only one guard can move per defense. We shall use this interpretation in later arguments since it is more intuitive, although not strictly formal.

In Goddard et al. [4], the authors established the value of σ_m for graphs from several classes. They also presented bounds on σ_m for general graphs. One class studied by the authors is that of Cayley graphs, for which they stated Theorem 1 reproduced below. However, we found that this result is not valid. Prior to write down the theorem and exhibit the counterexample, we recall the definition of a Cayley graph.

A Cayley graph is a simple graph $G = (V, E)$ defined as follows. Consider a *group* Γ and a set C of elements of Γ satisfying:

(i) C does not contain the identity of Γ;

(ii) If $x \in C$, then $x^{-1} \in C$ (x^{-1} is the inverse of element x in Γ).

The Cayley graph $G = CG(\Gamma, C)$ of Γ with respect to C is such that $V = \Gamma$ and $xy \in E$ if and only if $x = hy, h \in C$. For connecting the vertices of G, the elements of C and C itself are called *connectors* and *connecting set.* One can prove that G is connected if and only if C *generates* Γ.

The theorem presented in Goddard et al. [4] follows. We disprove it by showing a Cayley graph for which $\gamma < \sigma_m$.

Theorem 1 (Goddard et al. [4, Theorem 10]). *For any Cayley graph G, $\gamma(G) = \sigma_m(G)$.*

This paper is organized as follows. In the next section, we briefly describe the groups used for constructing the graph that invalidates Theorem

1. In Section 3, we discuss the proof given by Goddard et al. [4]. In Section 4, we present the counterexample that we encountered. In Section 5, we comment on computational tests we carried out with Cayley graphs in an attempt to establish the exact relation between γ and σ_m for this class. Finally, in Section 6, we make some final remarks.

2 Groups D_6, $\mathbb{Z}_3$ and $D_6 \times \mathbb{Z}_3$

In this section, we define the groups D_6, $\mathbb{Z}_3$ and $D_6 \times \mathbb{Z}_3$. For more details on these groups, see, for example, the textbook by Dummit and Foote [3].

A regular polygon of n sides have $2n$ symmetries: n rotations and n reflections. Because of this number, the set of its symmetries is denoted by D_{2n}. The set D_{2n} under the operation of composition is a group. This group is called the *dihedral group* and, with some abuse of notation, is also denoted only by D_{2n}.

The elements of the dihedral group D_{2n}, in multiplicative notation, are given by

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\},$$

with $1, r, r^2, \ldots, r^{n-1}$ corresponding to the n rotations and $s, sr, sr^2, \ldots,$ sr^{n-1} corresponding to the n reflections of the polygon. This group admits the following *presentation*:

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

From this presentation, one can obtain the result of operations on elements.

The group D_6 is the defined by the symmetries of a triangle. To facilitate the arguments made further in this paper, we display the multiplication table of this group in Figure 1.

Now, the set of possible remainders after dividing an integer by 3 is $\mathbb{Z}_3 = \{0, 1, 2\}$. Under the operation of addition modulo 3, this set is a group. This group is an instance of the *cyclic group* and is denoted $\mathbb{Z}_3$, like the set itself.

The groups D_6 and $\mathbb{Z}_3$ can be used to form a new group through their *direct product*. The direct product of D_6 and $\mathbb{Z}_3$ is denoted $D_6 \times \mathbb{Z}_3$ and is defined as follows.

The set of elements of the group $D_6 \times \mathbb{Z}_3$ is given by the Cartesian product of the sets D_6 and $\mathbb{Z}_3$ – also denoted $D_6 \times \mathbb{Z}_3$. We have that

$$\begin{aligned} D_6 \times \mathbb{Z}_3 = \{&(1,0),(1,1),(1,2),(r,0),(r,1),(r,2),\\ &(r^2,0),(r^2,1),(r^2,2),(s,0),(s,1),(s,2),\\ &(sr,0),(sr,1),(sr,2),(sr^2,0),(sr^2,1),(sr^2,2)\}. \end{aligned}$$

	1	r	r^2	s	sr	sr^2
1	1	r	r^2	s	sr	sr^2
r	r	r^2	1	sr^2	s	sr
r^2	r^2	1	r	sr	sr^2	s
s	s	sr	sr^2	1	r	r^2
sr	sr	sr^2	s	r^2	1	r
sr^2	sr^2	s	sr	r	r^2	1

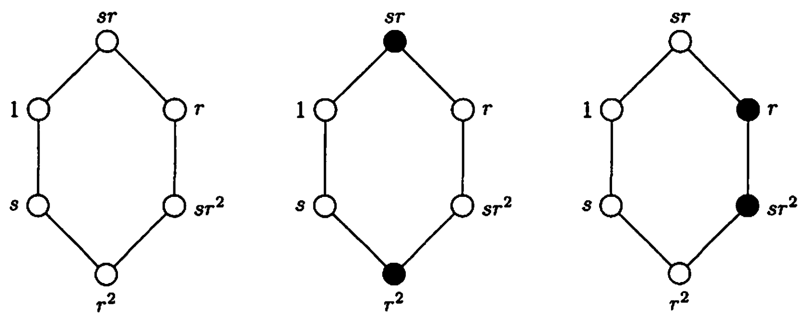
Figure 1: Compositions of symmetries for D_6

The operation of the group occurs componentwise: the result for the first half of the elements is shown in Figure 1 and the result for the second half is given by the addition modulo 3.

3 The proof by Goddard et al. [4]

We use the group D_6 to exhibit a flaw in the proof by Goddard et al. [4] to Theorem 1. The group D_6 is non-abelian (non-commutative) and it is its non-commutative property that allows us to reach our goal.

First of all, let us define a Cayley graph $G_1 = CG(D_6, C_1)$ of this group. For that, we choose the connecting set $C_1 = \{s, sr\}$. From the table in Figure 1, we have that G_1 is the graph depicted in Figure 2(a).



(a) G_1 (b) D_1 (black vertices) (c) hD_1 (black vertices)

Figure 2: The Cayley graph G_1 and the sets of vertices D_1 and hD_1

Now, let us follow the proof given by Goddard et al. [4]. Consider the dominating set $D_1 = \{r^2, sr\}$ pictured in Figure 2(b). Suppose an attack occurs at r. The only vertex in D_1 adjacent to r is sr. As they are adjacent, for some $h \in C_1$, $r = h(sr)$. By Figure 1, $h = s$.

Let xA and Ax, for an element $x \in D_6$ and a set $A = \{a_1, a_2, \dots, a_k\} \subseteq D_6$, stand for

$$xA = \{xa_1, xa_2, \dots, xa_k\} \text{ and}$$
$$Ax = \{a_1x, a_2x, \dots, a_kx\} \text{ respectively.}$$

Goddard et al. [4] claimed that

$$hD_1 = sD_1 = \{sr^2, s(sr)\} = \{sr^2, r\}$$

is a dominating set. However, as it is clear from Figure 2(c), this is not true.

We raise the possibility that the authors mistakenly considered the mapping

$$f(v) = hv, \text{ for each vertex } v \text{ of } G_1 \tag{1}$$

an automorphism of G_1. Instead, by a known result for Cayley graphs [1], it is the mapping

$$f(v) = vh, \text{ for each vertex } v \text{ of } G_1 \tag{2}$$

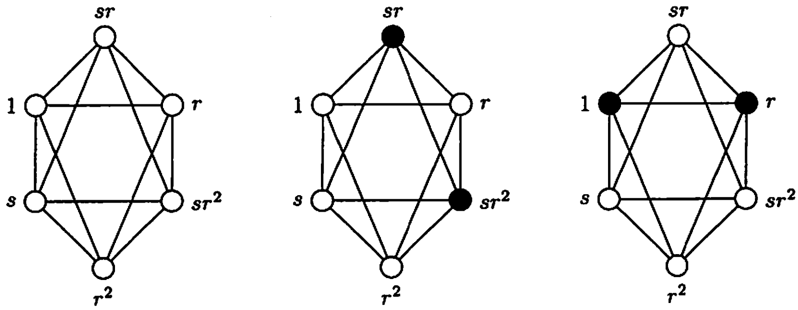
which is an automorphism of G_1.

We also observe that, in general, a mapping of the form (2) does not correspond to a shift from one dominating set to another. As an example, let us consider applying a mapping of the form (2) to a dominating set of the Cayley graph $G_2 = CG(D_6, C_2)$ defined by the connecting set $C_2 = \{s, sr, r, r^2\}$ - see a drawing of G_2 in Figure 3(a). The dominating set is $D_2 = \{sr, sr^2\}$ - shown in Figure 3(b). Let us choose $h = sr^2$. The outcome is: sr is mapped to the adjacent vertex r, but sr^2 is mapped to the non-adjacent vertex 1 – the resulting dominating set is shown in Figure 3(c).

At last, we point out that the proof by Goddard et al. [4] and their result are valid for a subclass of Cayley graphs. This subclass consists of graphs defined as follows.

A Cayley graph G is *obtainable from an abelian group* if there is an abelian group Γ and a connecting set C such that $G = CG(\Gamma, C)$. Note there may also be a non-abelian group Γ' and a connecting set C' satisfying $G = CG(\Gamma', C')$. For graphs defined in this way, mappings of the forms (1) and (2) coincide (because the elements commute). For this reason, the following theorem holds.

Theorem 2. *For any Cayley graph G obtainable from an abelian group, $\gamma(G) = \sigma_m(G)$.*



(a) G_2 (b) D_2 (black vertices) (c) D_2h (black vertices)

Figure 3: The Cayley graph G_2 and the dominating sets D_2 and D_2h

4 A counterexample

We present a Cayley graph G such that $\gamma(G) < \sigma_m(G)$. The construction of G follows immediately. In the sequel, we prove our claim.

We construct $G = CG(D_6 \times \mathbb{Z}_3, C)$ from the group $D_6 \times \mathbb{Z}_3$. For simplicity, we label the vertices of G as $v_1, v_2, \ldots, v_{18}$ according to the correspondence shown in Table 1. We choose the connecting set

$$C = \{(s,1),(s,2),(sr,1),(sr,2),(r,1),(r^2,2)\}$$

to provide the edges of G. The graph is pictured in Figure 4.

v_1	v_2	v_3	v_4	v_5	v_6
$(1,0)$	$(1,1)$	$(1,2)$	$(r,0)$	$(r,1)$	$(r,2)$
v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
$(r^2,0)$	$(r^2,1)$	$(r^2,2)$	$(s,0)$	$(s,1)$	$(s,2)$
v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
$(sr,0)$	$(sr,1)$	$(sr,2)$	$(sr^2,0)$	$(sr^2,1)$	$(sr^2,2)$

Table 1: Labels of vertices of G

We prove our claim in Theorem 6. Before doing so, we provide three lemmas. In the first one, we observe that $\gamma(G) = 3$. In the second lemma, we show that vertex v_1 is contained in only one minimum dominating set. In the last lemma, we argue that no vertex is contained in more than one minimum dominating set. Finally, after the latter, we exhibit all minimum dominating sets of G.

Lemma 3. *It holds that $\gamma(G) = 3$.*

Proof. As G is 6-regular, two vertices dominate at most 14 vertices. Since G has 18 vertices, we have that $\gamma(G) > 2$.

Observing Figure 4, we can see the set $\{v_1, v_6, v_8\}$ is a dominating set. Therefore, $\gamma(G) = 3$. □

Lemma 4. *The vertex v_1 is contained in only one minimum dominating set.*

Proof. Let us construct a minimum dominating set D that contains v_1. By Lemma 3, we have that D has three vertices. So, our task is to choose two more vertices.

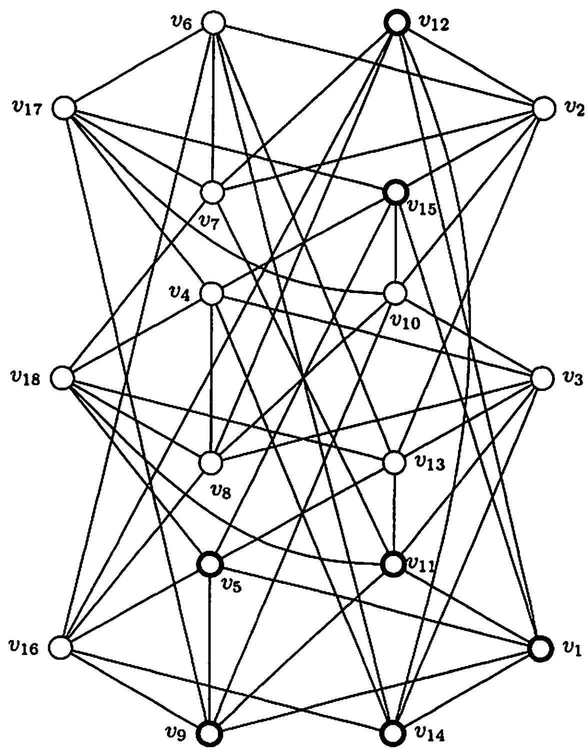


Figure 4: A Cayley graph G such that $\gamma(G) = 3$ and $\sigma_m(G) = 4$.

Since D contains v_1, we can see in Figure 4 that 7 vertices are already dominated by D: the vertices with thick border. Also in Figure 4, we can see that every vertex except v_6 and v_8 is adjacent to at least two already dominated vertices. Thus, two such vertices dominate at most 10 more vertices (summing up 17 vertices). Therefore, to construct D, we must choose at least one of v_6 and v_8.

Suppose we choose v_6. By Figure 4, we can see the only way of choosing one more vertex and dominating all vertices not yet dominated, is by selecting v_8. Then, suppose we pick v_8. By Figure 4, we can see v_6 must also be chosen for us to end up with a minimum dominating set. Hence $D = \{v_1, v_6, v_8\}$. □

Lemma 5. *No vertex of G is contained in more than one minimum dominating set.*

Proof. Suppose some vertex v_i is contained in two different minimum dominating sets $D_1 = \{v_i, v_j, v_k\}$ and $D_2 = \{v_i, v_l, v_m\}$. Since G is vertex-transitive (by a known result for Cayley graphs [1]), there is an automorphism α mapping v_i to v_1. But, then, $D_1' = \{v_1, \alpha(v_j), \alpha(v_k)\}$ and $D_2' = \{v_1, \alpha(v_l), \alpha(v_m)\}$ are two different dominating sets containing v_1, which contradicts Lemma 4. □

We can state, from Figure 4, that the following 6 sets are dominating sets of G:

$$\begin{aligned}&\{v_1, v_6, v_8\}, \{v_3, v_5, v_7\}, \{v_2, v_4, v_9\},\\ &\{v_{11}, v_{15}, v_{16}\}, \{v_{10}, v_{14}, v_{18}\}, \{v_{12}, v_{13}, v_{17}\}.\end{aligned} \tag{3}$$

By Lemma 5, these are the only minimum dominating sets.

Theorem 6. *It holds that $\gamma(G) < \sigma_m(G)$.*

Proof. To prove the theorem, we consider three guards defending the vertices of G. First of all, it is obvious that, to defend the vertices of a graph, guards must always be placed on vertices that form a dominating set, because an attack at an undominated vertex cannot be defended. In the case of three guards defending the vertices of G, they must always be placed on a minimum dominating set of G.

Consider then an attack at vertex v_1. By Lemma 4, $D_1 = \{v_1, v_6, v_8\}$ is the only minimum dominating set containing v_1. This fact implies that, after defending an attack at v_1, guards must be placed on the vertices of D_1.

Consider now an attack at vertex v_{13}. As listed in (3), $D_2 = \{v_{12}, v_{13}, v_{17}\}$ is the only minimum dominating set containing v_{13}. As a consequence, after defending an attack at v_{13}, guards must be placed on the vertices of D_2. However, as we can see from Figure 4, guards cannot move from vertices v_1, v_6 and v_8 to v_{12}, v_{13} and v_{17}, because v_{13} and v_{17} are adjacent to v_6 but to neither v_1 nor v_8.

Hence, there is a sequence of three attacks that cannot be defended by the three guards. So, at least four guards are needed to defend the vertices of G. Therefore, $\sigma_m(G) \geq 4 > 3 = \gamma(G)$ and the theorem is proved. □

5 Computational testing

The graph presented in the previous section was found through extensive computational testing. In this section, we describe some elements of this experiment and report relevant information.

5.1 Data

We searched 7871 Cayley graphs of non-abelian groups of order up to 31 and of order 33 catalogued by Royle [5]. These graphs have degree (recall that Cayley graphs are regular) less than half of the number of vertices. We denote them by *set 1*.

We also searched 7871 Cayley graphs which are the complements of the graphs of set 1. These are graphs having degree greater or equal than half of the number of vertices. We refer to them as *set 2*.

5.2 Results

One interesting outcome of our experiments is that, for almost all graphs, we found that $\gamma = \sigma_m$. Just for 61 out of 7871, i.e., 0.77% of them, we obtained a different result. Another relevant fact is that, in these cases, the result was always that $\gamma + 1 = \sigma_m$. We also noted that for all graphs of set 2, $\gamma = \sigma_m$.

Motivated by the above findings, we searched for a graph for which $\gamma + 1 < \sigma_m$. However, as determining σ_m becomes much more time-consuming as the graphs get bigger, an exhaustive search over a huge number of Cayley graphs of non-abelian groups rapidly becomes impractical. We then moved to the strategy of generating specific graphs to attain our goal.

One successful example is the graph G_2 consisting of two disconnected copies of the graph G presented in Section 4. This graph is a Cayley graph of the group $(D_6 \times \mathbb{Z}_3) \times \mathbb{Z}_2$. One can see that $\gamma(G_2) = 6 < 8 = \sigma_m(G_2)$. Moreover, it is possible to generate the graphs

- G_4 consisting of four disconnected copies of G;
- G_8 consisting of eight disconnected copies of G;
- and so forth.

For $G_i, i = 4, 8, \ldots, \sigma_m(G_i) - \gamma(G_i) = i$.

It is worth emphasizing, however, that G_i is not a connected graph. We could not discover a connected Cayley graph such that $\gamma + 1 < \sigma_m$.

6 Conclusion

We disproved a result by Goddard et al. [4] on the eternal m-security number of Cayley graphs. We did this by presenting a Cayley graph for which $\gamma < \sigma_m$. We remarked, however, that the result of Goddard *et al.* is valid for a large subclass of Cayley graphs: the Cayley graphs obtainable from abelian groups.

We also determined computationally the value of σ_m for 7871 Cayley graphs of non-abelian groups. For almost all of them, we got that γ is indeed equal to σ_m. For the remaining graphs, we found that $\gamma + 1 = \sigma_m$. We leave open the question of whether there exists a connected Cayley graph having $\gamma + 1 < \sigma_m$.

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References

[1] A. Bondy and U. S. R. Murty. *Graph Theory*. Graduate Texts in Mathematics. Springer, 2008.

[2] A. P. Burger, E. J. Cockayne, W. R. Gründlingh, C. M. Mynhardt, J. H. van Vuuren, and W. Winterbach. "Infinite order domination in graphs". In: *J. Combin. Math. Combin. Comput.* 50 (2004), pp. 179–194.

[3] D. S. Dummit and R. M. Foote. *Abstract Algebra*. Wiley, 2003.

[4] W. Goddard, S. Hedetniemi, and S. Hedetniemi. "Eternal security in graphs". In: *J. Combin. Math. Combin. Comput.* 52 (2005), pp. 169–180.

[5] G. Royle. *Cayley graphs*. [Online; accessed 30-November-2013]. Available at: `http://school.maths.uwa.edu.au/~gordon/remote/cayley/index.html`. 1998.