Magical Mathematical Connections with the Borda Voting Method

Melissa Erdmann

November 6, 2013

Abstract

We examine the Borda voting method, which has numerous interesting mathematical properties. We determine when a candidate can win a Borda election with all ith place votes and present a method of constructing ballots that yield such a victory. Then we present a connection between Borda elections and semi-magic squares. We show how a Borda election result gives rise to a semi-magic square, and we show that given any semi-magic square there exists at least one Borda election result corresponding to it.

1 Introduction

Election outcomes often have important and far-reaching consequences. Whether it be a country deciding on its next leader, a business selecting a health insurance provider for its employees, or an academic department determining who receives a scholarship, many lives are significantly affected by election results. Indeed, an election outcome depends not only on the votes of the people, but it also is a direct result of the election method employed to arrive at the outcome.

One such method is the Borda method, and in this paper we focus on mathematics connected to the Borda method. We determine when a candidate can win a Borda election with all ith place votes and present a method of constructing ballots that yield such a victory. Then we present a connection between Borda elections and semi-magic squares. We will show how a Borda election result gives rise to a semi-magic square, and we show that given any semi-magic square there exists at least one Borda election result corresponding to it.

2 The Borda Voting Method

The eighteenth century French mathematician Jean Charles de Borda proposed a weighted voting method to select members of the French Academy of Sciences; the French Academy adopted this method and used it for several years [4]. When using this method, which has come to be called the Borda voting method, each voter determines a strict ranking of the n candidates. Such a ranking is called a preference ballot. On each ballot the first-ranked candidate receives $n-1$ points, the second-ranked candidate receives $n-2$ points, and so on, down to the last place candidate receiving 0 points. The ith ranked candidate receives

$n-i$ points. The points from all ballots are totaled, and the candidate with the highest total wins. If there is a tie, something outside the Borda voting method must be used to break it. For example, the number of first place ranks of the tied candidates could be compared.

Examination of the Borda Election Method leads to some interesting mathematical discoveries. First, we look at the various place rankings that Borda winners receive.

2.1 Borda Elections and Place Rankings

Our first proposition is a simple observation.

Proposition 1. *In a Borda election with V voters and n candidates, the winning candidate receives more than $V\frac{(n-1)}{2}$ points.*

Proof. The proof is a simple application of the pigeonhole principle. Any voter's ballot is made up of $0+1+2+\cdots+n-1=\frac{n(n-1)}{2}$ points to distribute. Thus, the total number of points to be distributed in the Borda election is $V\frac{n(n-1)}{2}$.

If all n candidates receive $\leq V\frac{(n-1)}{2}$ points, then the total number of points allotted would be less than $V\frac{n(n-1)}{2}$, unless all candidates tie by receiving exactly $V\frac{(n-1)}{2}$ points. Thus, the Borda winner must receive more than $V\frac{(n-1)}{2}$ points.

□

We will investigate what sort of votes (first place, second place, and so on) a Borda winner can receive.

The next corollaries follow easily from the above proposition.

Corollary 1. *In a three-candidate election, the Borda winner must receive at least one first place vote.*

One first place vote is insignificant in a three-candidate Borda election with a large number of voters. However, in an election with a relatively small number of voters, it is noteworthy that the winner must be the first choice of at least one voter.

Corollary 2. *In a three-candidate election, the Borda winner has more first place rankings than last place rankings.*

Corollary 3. *If the winner of a three-candidate Borda election has received only one first place vote, then this candidate must have received no last place votes.*

Corollary 4. *In order for a Borda winner to achieve victory amongst n candidates with only ith place votes, we must have $i<\frac{n+1}{2}$.*

Proof. By Proposition 1,

$$(n-i)V>\frac{n-1}{2}V \implies i<\frac{n+1}{2}.$$

□

This means that in order for a candidate to win with all ith place votes, the voters must rank the candidate better than half of the total number of candidates. For example, if we have an election with 4 candidates, then a candidate could possibly win with all first or all second place votes but not with all third or last place votes. Similarly, if we have an election with 5 candidates, then a candidate could not win with all third, fourth, or last place votes.

Lemma 1. *If candidate A receives all the ith place rankings with $i < \frac{n+1}{2}$, then $\frac{n^2-3n+2i}{2} < (n-1)(n-i)$.*

Proof. Rearranging the second inequality gives the first inequality, and the steps are reversible.

□

Next we show that a candidate can win with all ith place votes, if $i < \frac{n+1}{2}$.

Theorem 1. *If $i < \frac{n+1}{2}$ (or equivalently, $\frac{n-1}{2} < n - i$), then there exists a Borda election with V voters and n candidates (with $V \geq n$) where candidate A wins with all ith place votes.*

Proof. If $i = 1$, the candidate has received all first place votes. In this case, the result is clear, so we consider the cases where $i > 1$.

Let n be the number of candidates, V the number of voters, $V \geq n$, and suppose $1 < i < \frac{n+1}{2}$ and candidate A receives all of the ith place rankings.

Write $V = c(n-1) + r$ with c and r integers, $0 \leq r < n-1, c > 0$.

First, let us consider the case where $r = 0$. We can divide up the voters into groups of size c. All voters give $n - i$ points to candidate A. All of the voters in the first group of size c can assign their ranks so that each of the remaining $n-1$ candidates receives points $0, 1, \ldots, n-i-1, n-i+1, \ldots, n-1$. For example, candidate B could receive a 0 from Voter 1, a 1 from Voter 2, a 2 from Voter 3, ..., a $n-1$ from Voter c. We can do the same thing with the second group of voters of size c and so on. In the end each candidate receives each of the possible ranks c times.

Thus, each non-A candidate receives a total of $\frac{c(n^2-3n+2i)}{2}$ points, and A receives $c(n-1)(n-i)$ points.

By Lemma 1, we know $\frac{c(n^2-3n+2i)}{2} < c(n-1)(n-i)$.

Thus, for $r = 0$ we have created a Borda election where A wins with all ith place votes.

Now let us consider the case where $r = 1$, that is $V = c(n-1) + 1$. We distribute points according to the same method as for $r = 0$ except that each non-A candidate also receives one additional value from $\{0, 1, \ldots, n-1\} \setminus \{n-i\}$.

We show that $c(\frac{n(n-1)}{2} - (n-i)) + n - 1 < (c(n-1)+1)(n-i)$, and so every candidate will have fewer points than candidate A.

This is equivalent to showing $0 < \frac{cn^2}{2} - cni - i + \frac{cn}{2} + 1$.

Depending on the parity of i, the largest that i can be is $i = \frac{n}{2}$ or $i = \frac{n-1}{2}$. If we can find an election where A wins with i equal to these large values, we can find an election where A wins with i equal to smaller values.

Substituting $i = \frac{n}{2}$ into $0 < \frac{cn^2}{2} - cni - i + \frac{cn}{2} + 1$, we have $0 < \frac{n}{2}(c-1) + 1$. This is true for all positive n and all $c \geq 1$.

Similarly, substituting $i = \frac{n-1}{2}$ into $0 < \frac{cn^2}{2} - cni - i + \frac{cn}{2} + 1$, we have $0 < (c - \frac{1}{2})n + \frac{3}{2}$. This is true for all positive n and all $c \geq 1$.

Thus, we can find a Borda election where A wins with all ith place votes and $V = c(n-1) + 1$.

Now we consider the cases where $1 < r < n - 1$.

If $r = 2$, we again distribute points as in the case where $r = 0$ and then add two remaining rankings to each candidate's total number of points. Choose these two rankings for each candidate so that they add to $n-1$ or to $n-2$. For example, $(n-1)+0$, $(n-2)+1, \ldots,$ would be the sums of the two remaining rankings. Upon reaching the point where a ranking's pairing would have been $n-i$, add the next smaller ranking. If, for example $n-i = n-3$, then we pair $n-4$ with 2, and so on.

Thus, $c(\frac{n(n-1)}{2} - (n-i)) + n - 1 < c(n-1)(n-i) + 2(n-i)$, since $\frac{n-1}{2} < n-i$.

If $r = 3$, we may add $n-1$ to the left of this inequality and $n-i$ to the right by using an analogous argument to that for $r = 1$. If $r = 4$, we have a similar situation to that when $r = 2$, and so on.

Thus, for all r with $0 \leq r < n-1$, we have found an election where candidate A wins with all ith place votes.

□

We will now express this proof as an algorithm.

Algorithm for Creating Borda Ballots where Candidate A Wins with All ith Place Votes, with $i < \frac{n+1}{2}$ and $V = c(n-1) + r, 0 \leq r < n-1$.

1. Place candidate A in ith place on all V ballots.

2. Set aside r ballots. On the remaining $c(n-1)$ ballots, place the remaining $n-1$ candidates in each of the remaining places c times, so that each of these candidates receives a total of $c\left(\left(\sum_{k=0}^{n-1} k\right) - (n-i)\right)$ points.

3. Now consider the r ballots that have been set aside. These r ballots have candidate A in ith place and $n-1$ unfilled spots remain. Now assign each remaining candidate to pairs of ballots so that this candidate receives points that sum to $n-1$ or $n-2$ per pair of ballots. If r is odd, one unfilled ballot will be leftover. Put each of the remaining $n-1$ candidates in any spot on this ballot.

This algorithm allows us to prove a more general version.

Corollary 5. *If we have V voters and n candidates, $V \geq n$, together with a list of integers k_j such that $1 \leq k_j < \frac{n+1}{2}$ for $j = 1, 2, \ldots, V$, then we can find Borda ballots with candidate A in places k_j such that candidate A wins.*

Proof. Let us first consider a list of integers k_j such that $1 \leq k_j < \frac{n+1}{2}$ for $j = 1, 2, \ldots, V$. If all of the k_j are equal, the result follows immediately from Theorem 1. Otherwise, let $i = \max(k_j)$. By the previous algorithm, we can construct V Borda ballots where A wins with all ith place votes. Now on each ballot we switch candidate A with the candidate in k_jth place for $j = 1, 2, \ldots, V$.

Candidate A's points will increase and the other candidates' points will decrease or remain the same. Thus, we have created an election where A is the winner. □

Some questions arise as to how far we can push this idea.

Remark 1. *If A wins a Borda election, must A receive more votes in ith place with $1 \leq i < \frac{n+1}{2}$ than in ith place with $\frac{n+1}{2} \leq i \leq n$?*

The answer to this question is no, as seen in the example in Table 1. Consider the situation where there are 8 voters and 8 candidates, and candidate A receives 2 first place ranks and 6 fifth place ranks. For the rankings in Table 1, candidate A is the winner.

Table 1: Candidate A Wins with Two First Place Votes.

Rankings	Candidate	Score
$A > C > E > G > H > F > D > B$	A	32
$A > B > D > F > H > G > E > C$	B	31
$G > F > E > D > A > C > B > H$	C	31
$F > E > D > C > A > B > G > H$	D	31
$E > D > C > B > A > G > F > H$	E	31
$D > C > B > G > A > F > E > H$	F	31
$C > B > G > F > A > E > D > H$	G	31
$B > G > F > E > A > D > C > H$	H	6

The next point, Remark 2, illustrates the importance of re-voting if a candidate is added to a pool, even though the added candidate may have no chance of winning. Similarly, if a candidate withdraws from an election, it is important that the new outcome be found.

Remark 2. *If we have a Borda election where A wins and we add a candidate X to the Borda rankings such that A receives more total points than X, must A still win the Borda election?*

The answer to this question is no, and we provide an example in Table 2 to illustrate this.

Table 2: Candidate A Wins Only Without Candidate X.

Number of Voters	Ranking	New Ranking
5	$A > B > C$	$A > B > X > C$
2	$A > C > B$	$A > C > B > X$
4	$B > A > C$	$B > X > A > C$
3	$B > C > A$	$B > C > X > A$
4	$C > A > B$	$C > X > A > B$
2	$C > B > A$	$C > B > X > A$

We notice that in the original election A is the winner with 22 points, followed by B with 21 points, and C receives 17 points. After adding X to the rankings,

A receives 29 points, more than X who receives 26 points. However, now the winner of the election is B with 37 points, and C receives 28 points.

3 Borda Elections and Semi-Magic Squares

In this section we examine the connections between Borda Elections and semi-magic squares.

We introduce the following definitions:

Definition 1. *A **candidate distribution** for a Borda election with n candidates and V voters is an n-tuple* $(a_{i1}, a_{i2}, \ldots, a_{in})$ *where* a_{ij} = *the number of jth place rankings for candidate i. Note that* $\sum_{j=1}^{n} a_{ij} = V$ *for all i.*

Definition 2. *A **place distribution** for a Borda election with n candidates and V voters is an n-tuple* $(a_{1j}, a_{2j}, \ldots, a_{nj})$ *where* a_{ij} = *the number of jth place rankings for candidate i. Note that* $\sum_{i=1}^{n} a_{ij} = V$ *for all j.*

Definition 3. *A **Borda matrix** is an* $n \times n$ *matrix whose rows are candidate distributions for a Borda election and whose columns are place distributions for a Borda election.*

Remark: We note that any permutation matrix can be thought of as corresponding to a single Borda ballot. For example, if candidate A is represented by row 1, candidate B is represented by row 2, and candidate C is represented by row 3, the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

corresponds to the ballot with election ranking $C > A > B$.

Definition 4. *[3] A **semi-magic square** is a square matrix whose entries are non-negative integers and whose rows and columns sum to the same number.*

The following theorem, attributed to Garrett Birkhoff, is useful in the context of semi-magic squares.

Theorem 2. *[6] Let A be an* $n \times n$ *semi-magic square with row sum V. Then A is the sum of V permutation matrices.*

Remark: The expression of A as a sum of permutation matrices is not unique in general.

Since any permutation matrix corresponds to a Borda ballot, we have the following corollary:

Corollary 6. *Given an* $n \times n$ *semi-magic square A with row sum V, we can create a Borda election that matches its candidate and place distributions.*

Remark: In general, these election rankings are not unique. For example, we have three sets of election rankings in Table 3 corresponding to the same semi-magic square S.

Table 3: Three Borda Elections Corresponding to the Same Semi-Magic Square.

Number of Voters	Ranking	Number of Voters	Ranking	Number of Voters	Ranking
6	$A > B > C$	9	$A > B > C$	4	$A > B > C$
4	$A > C > B$	1	$A > C > B$	6	$A > C > B$
3	$B > A > C$	0	$B > A > C$	5	$B > A > C$
2	$B > C > A$	5	$B > C > A$	0	$B > C > A$
5	$C > A > B$	8	$C > A > B$	3	$C > A > B$
4	$C > B > A$	1	$C > B > A$	6	$C > B > A$

$$S = \begin{bmatrix} 10 & 8 & 6 \\ 5 & 10 & 9 \\ 9 & 6 & 9 \end{bmatrix}$$

The following theorem follows readily from the previous definitions and Corollary 6.

Theorem 3 (Borda Matrices and Semi-Magic Squares).

1. *Every Borda matrix is a semi-magic square.*

2. *Every semi-magic square is a Borda matrix.*

We now present an application involving Borda matrices and semi-magic squares.

3.1 Application

Example 1. *In an election with 4 candidates and 2 voters, exactly 300 different combinations of ballots can result.*

There are 24 different elections where the ballots are identical. If the ballots are not identical, then there are 24 choices for the first one, 23 choices for the second one, and the order in which the ballots appear does not matter, so this gives $\frac{24*23}{2} = 276$ different combinations of ballots. Altogether, this gives 24+276=300 different combinations of ballots.

Example 2. *There are 282* 4×4 *semi-magic squares with row sum 2.*

In 1973 Ehrhart and Stanley both proved that the number of $n \times n$ semi-magic squares with row sum t is a polynomial of degree $(n-1)^2$ with rational coefficients [1, 5]. By using such a polynomial of Beck and Pixton, we find there are 282 4×4 semi-magic squares with row sum 2 [2].

Open Question 1. *Determine how many different Borda ballot combinations correspond to a given semi-magic square. In other words, determine how many different Borda elections correspond to a given Borda matrix.*

For example, as we have seen, in a 4-candidate, 2-voter Borda election there are 300 possible elections but only 282 corresponding semi-magic squares. In this case, one can easily show that each of the 18 semi-magic squares with two pairs of equal columns has two different corresponding elections.

4 Conclusion

The voting method employed in an election is a matter of great importance. Preference ballots contain valuable information; the Borda method uses such ballots. We investigate numerous mathematical connections related to the Borda voting method.

We show that in order for one of n candidates to win an election with all ith place votes it is necessary that $\frac{n-1}{2} < n - i$. Given such an i, we find an algorithm to create winning Borda ballots for a candidate with all ith place votes.

A Borda election result yields a Borda matrix, that is, a semi-magic square. Conversely, given any semi-magic square, there exist preference ballots that correspond to it. In some cases, different Borda election ballots correspond to the same semi-magic square. It is an open problem to determine the number of different elections that correspond to the same semi-magic square.

The vote is in, mathematics connected to the Borda voting method is magical.

5 Acknowledgement

Many thanks to Gary McGuire of University College Dublin for his guidance and support throughout the preparation of this article.

References

[1] Eugène Ehrhart. Sur les Carrès Magiques. *Reports of the Academy of Sciences, Paris*, A-B 227:A651–A654, 1973.

[2] Matthias Beck and Dennis Pixton. The Ehrhart Polynomial of the Birkhoff Polytope. *Discrete and Computational Geometry*, October, Volume 30, Issue 4:623–637, 2003.

[3] Mattias Beck, Moshe Cohen, Jessica Cuomo, and Paul Gribelyuk. The Number of "Magic" Squares, Cubes, and Hypercubes. *The American Mathematical Monthly*, 110:707–717, 2003.

[4] Donald G. Saari. *Chaotic Elections! A Mathematician Looks at Voting.* American Mathematical Society, United States of America, 2001.

[5] Richard P. Stanley. Linear Homogeneous Diophantine Equations and Magic Labelings of Graphs. *Duke Mathematical Journal*, 40:607–632, 1973.

[6] J.H. van Lint and R.M. Wilson. *A Course in Combinatorics.* Cambridge University Press, United Kingdom, 1992.