Constructing the Spectrum of Packings and Coverings for the Complete Graph with 4-stars

Danny Dyer*, **Sadegheh Haghshenas**†, **and Nabil Shalaby**‡
Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada, A1C 5S7

Abstract: The packing and covering numbers for the 4-stars were determined by Roditty in 1986. In this paper we improve and extend these results by finding a corresponding maximum packing and minimum covering of the complete graph with 4-stars for every possible leave graph and excess graph.

1 Introduction

For basic graph theory definitions refer to [4]. All graphs we use are finite and contain no loops or multiple edges unless otherwise stated. Let G be a graph. A *G-decomposition* of a graph H is a partition of the edge set of H into graphs isomorphic to G. A *G-design* of order n and index λ is a G-decomposition of the complete multigraph K_n^λ. In this paper we only deal with $\lambda = 1$ and hence, consider a G-design of order n as a G-decomposition of the complete graph K_n. In fact, a G-design is a generalization of a $BIBD(n, k, 1)$ where G is the clique K_k. G-designs were first introduced by Rosa and Hell in 1972, in their attempt to solve the spectrum problem for P_3, a path on three vertices [5].

The *spectrum problem* for a graph G is to find the set D of positive integers n such that there exists a G-design of order n if and only if $n \in D$. The obvious necessary conditions for the existence of a G-decomposition of K_n are $|V(G)| \leq n$ for $n > 1$, $n(n-1) \equiv 0 \pmod{2|E(G)|}$, and $n - 1 \equiv 0 \pmod{d}$ where d is the greatest common divisor of the degrees of the vertices in G.

In 1975, Wilson proved that these necessary conditions are asymptotically sufficient [13]. However, in order to completely solve the spectrum problem for a particular graph G, it still remains to determine the specific conditions for n such that a G-design of order n exists. The spectrum problem has been considered for many classes of graphs [1].

In 1978 Huang and Rosa solved the spectrum problem for trees of nine vertices or less [6]. Stars are one of the infinite classes of trees for which the spectrum problem is completely solved. The *k-star* S_k is a connected graph on $k + 1$ vertices with one

*dyer@mun.ca
†ssh631@mun.ca
‡nshalaby@mun.ca

vertex of degree k and k vertices of degree 1. We call the vertex of degree k the *center* of the star and the vertices of degree one the *leaves* of the star. A k-star with center x and leaves $y_1, y_2, \ldots, y_{k-1}, y_k$ is denoted by $(x; y_1, y_2, \ldots, y_{k-1}, y_k)$.

In 1975 Yamamoto et al. [14] and in 1978 Tarsi [11] solved the spectrum problem for k-stars where $k \geq 1$ by proving the theorem below.

Theorem 1 ([14]). *For $k \geq 1$, K_n has an S_k-decomposition if and only if $n = 1$ or $n \geq 2k$, and $n(n-1) \equiv 0$ (mod $2k$).*

For the complete graphs which cannot be decomposed into stars, we are still interested in getting as close to a decomposition as we can. This leads to the notions of packing and covering. A *G-packing* of a graph H is a set of subgraphs of H such that each subgraph is isomorphic to G and every edge of H is contained in at most one subgraph. Those edges which are not contained in any of the subgraphs of the packing form a graph called the *leave graph.* A *maximum G-packing* of H is a packing with the smallest possible number of edges in the leave graph. A *G-covering* of a graph H is a set of subgraphs of H such that each subgraph is isomorphic to G and every edge of H is contained in at least one subgraph. Those edges which are contained in more than one subgraph of the packing form a graph called the *excess graph.* A *minimum G-covering* of H is a covering with the smallest possible number of edges in the excess graph. We assume that the leave graph and excess graph have no isolated vertices.

For graphs G and H, the *G-packing number* (*G-covering number*) of H is the number of copies of G in a maximum G-packing (minimum G-covering) of H. The packing (covering) problem for a graph G is to determine the G-packing number (G-covering number) of K_n. Roditty solved the problem for all trees of order seven or less [7], [8], [9], and [10]. In particular, he proved that for $n \geq 2k$ and $k \leq 6$, the S_k-packing number of the complete graph K_n is $\lfloor \frac{n(n-1)}{2k} \rfloor$ and the S_k-covering number of K_n is $\lceil \frac{n(n-1)}{2k} \rceil$. However, he did not determine all the possible leaves and excesses in his constructions. We refer to this problem as the *spectrum problem for packing and covering.*

In 1997 and 1998, Caro and Yuster established a Wilson-like result for the packing and covering problems. In fact, in 1997 [2], they proved that for any graph H with h edges, there exists a positive integer $n_0(H)$ such that for all integers $n > n_0(H)$ the H-packing number of K_n is $\lfloor \frac{dn}{2h} \lfloor \frac{n-1}{d} \rfloor \rfloor$, where d is the greatest common divisor of all degrees of H, unless $n \equiv 1$ (mod d) and $\frac{n(n-1)}{d} \equiv b$ (mod $\frac{2h}{d}$) where $1 \leq b \leq d$ in which case the packing number is $\lfloor \frac{dn}{2h} \lfloor \frac{n-1}{d} \rfloor \rfloor - 1$. They also proved in 1998 [3] that for any graph H with h edges, there exists a positive integer $n_0(H)$ such that for all integers $n > n_0(H)$ the H-covering number of K_n is $\lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil$, where d is the greatest common divisor of all degrees of H, unless d is even, $n \equiv 1$ (mod d) and $\frac{n(n-1)}{d} + 1 \equiv 0$ (mod $\frac{2h}{d}$), in which case the covering number is $\lceil \frac{\binom{n}{2}}{h} \rceil + 1$.

Let m and n be positive integers. The *disjoint union* of graphs G and H, denoted $G + H$, is the union of graphs G and H with disjoint vertex sets. The *join* of simple graphs G and H, denoted $G \vee H$ is the graph obtained from the disjoint union $G + H$ by adding the edges $\{\{x, y\} | x \in V(G), y \in V(H)\}$. Also for any graph G, mG is the graph consisting of m pairwise disjoint copies of G. Furthermore, we denote the complete multigraph on n vertices with multiplicity m by K_n^m [12].

We will use the following lemmas in the proof of our main theorems.

Lemma 2. *Let s be a positive odd integer. The graph $K_s \vee \frac{3(s-1)}{2} K_1$ has an S_4-decomposition.*

Proof. Label the vertices of K_s with the elements of $\mathbb{Z}_s$ having the subscript 2 and the remaining vertices with the elements of $\mathbb{Z}_{\frac{3(s-1)}{2}}$ having the subscript 1. Then, the following stars form a decomposition for $K_s \vee \frac{3(s-1)}{2} K_1$ (see Figure 1). For numbers with subscript 1 the computations are done modulo $\frac{3(s-1)}{2}$ and for those with subscript 2 the computations are done modulo s.

$$(i_2; (i + j + 1)_2, 3j_1, (3j + 1)_1, (3j + 2)_1),\ i \in \mathbb{Z}_s,\ j = 0, 1, \ldots, \frac{s-3}{2}.$$

□

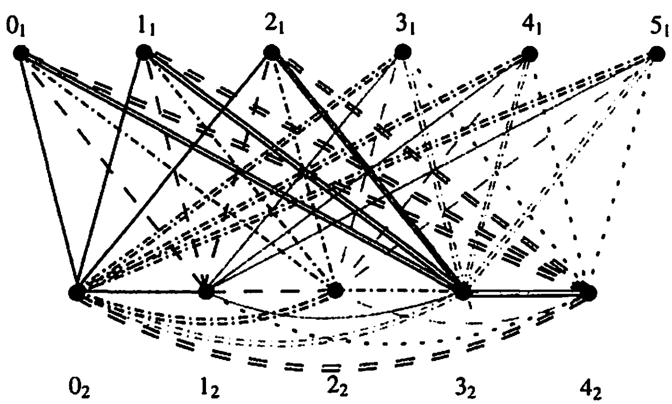

Figure 1: S_4-decomposition of $K_5 \vee 6K_1$

Lemma 3. *Let s be a positive odd integer and sK_2 be the union of s disjoint edges. For positive integers s and t with $s \le t$ the complete bipartite graph $K_{s,t}$ can be packed with $(t-1)$-stars with an sK_2 as the leave graph.*

Proof. Label the vertices of the part of size t with elements of $\mathbb{Z}_t$ having subscript 1 and the vertices of the other part with elements of $\mathbb{Z}_s$ having subscript 2. The following stars form a maximum packing of $K_{s,t}$ with $(t-1)$-stars with the s edges $\{0_1, 1_2\}, \{1_1, 2_2\}, \ldots, \{(s-2)_1, (s-1)_2\}$, and $\{(t-1)_1, 0_2\}$ as the leave graph (see Figure 2). For numbers with subscript 1 the computations are done modulo t and for those with subscript 2 the computations are done modulo s.

$$(i_2; i_1, (i+1)_1, \ldots, (i+t-2)_1), \; i = 0, 1, \ldots, s-1.$$

□

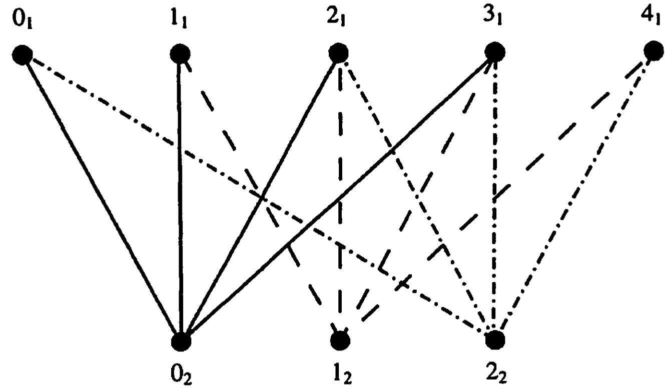

Figure 2: S_4-packing of $K_{3,5}$ with the leave $3K_2$

Lemma 4. *For any positive integers s and k the complete bipartite graph $K_{s,4k}$ has an S_4-decomposition.*

The proof is trivial.

2 Main Results

In 1986 Roditty solved the problem of packing and covering the complete graph K_n with 4-stars.

Theorem 5 ([9]). *The S_4-packing number of the complete graph K_n is $\lfloor \frac{n(n-1)}{8} \rfloor$ for $n \geq 7$.*

Theorem 6 ([9]). *The S_4-covering number of the complete graph K_n is $\lceil \frac{n(n-1)}{8} \rceil$ for $n \geq 7$.*

Here, we find a corresponding maximum packing and minimum covering of the complete graph with 4-stars for every possible leave graph and excess graph.

2.1 All the Possible Leave Graphs in the S_4-packing of K_n

Theorem 7. *Let $n \geq 7$ be an integer and the leave graph in a maximum packing of the complete graph K_n with 4-stars have i edges. For any graph H with i edges there exists a maximum packing of K_n with 4-stars such that the leave graph is isomorphic to H.*

Proof. By Theorem 1, K_n has an S_4-decomposition for $n \equiv 0$ or 1 (mod 8). We show that for the remaining cases we have maximum packings with all the possible leave graphs.

Case 1. $n \equiv 2 \pmod 8$

By Theorem 5, the leave is a single edge and the proof is complete in this case.

Case 2. $n \equiv 3 \pmod 8$

In this case, the leave graph has three edges. The non-isomorphic possible leave graphs are S_3, K_3, $S_2 + K_2$, P_4, and $3K_2$.

In order to get an S_3 as the leave, write $K_n = K_3 \vee K_{n-3}$. Since $n \equiv 3 \pmod 8$, we have $n - 3 \equiv 0 \pmod 8$ and hence K_{n-3} has an S_4-decomposition, R, by Theorem 1. Label the vertices of K_{n-3} with the elements of $\mathbb{Z}_{n-3}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Now, the vertices $0_1, 1_1, 2_1, 0_2, 1_2, 2_2$, the nine edges between the vertices with different subscripts, and the three edges between the vertices with subscript 2 form a $K_3 \vee 3K_1$. By Lemma 2, $K_3 \vee 3K_1$ has an S_4-decomposition, S. Now, the vertices $3_1, 4_1, \ldots, (n-5)_1$, the vertices $0_2, 1_2, 2_2$, and the edges between these two sets of vertices form a complete bipartite graph which has one part of size a multiple of 4. Therefore, by Lemma 4 this complete bipartite graph has an S_4-decomposition, T. Hence, $R \cup S \cup T$ form a maximum packing of K_n with 4-stars with the 3-star $((n-4)_1; 0_2, 1_2, 2_2)$ as the leave graph.

In order to obtain $3K_2$ as the leave, again write $K_n = K_3 \vee K_{n-3}$. Label the vertices as above and let R and S be the same decompositions. Now, the vertices $3_1, 4_1, \ldots, (n-9)_1$, the vertices $0_2, 1_2, 2_2$, and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 4, this complete bipartite graph has an S_4-decomposition, T. Now, the vertices $(n-8)_1, (n-7)_1, (n-6)_1, (n-5)_1, (n-4)_1$, the vertices $0_2, 1_2, 2_2$, and the edges between these two sets of vertices form a $K_{3,5}$. By Lemma 3, $K_{3,5}$ has a maximum packing, Q, with the leave $3K_2$. Hence, $R \cup S \cup T \cup Q$ forms a maximum packing of K_n with 4-stars with the leave graph $3K_2$.

Now, to get K_3 as the leave, write $K_n = K_1 \vee K_{n-1}$. Label the vertices of K_{n-1} with the elements of $\mathbb{Z}_{n-1}$ and the single vertex of K_1 with ∞. Since $n \equiv 3 \pmod 8$, by Theorem 5, K_{n-1} has a maximum packing with 4-stars, R, with the edge $\{(n-3)_1, (n-2)_1\}$ as the leave graph. Moreover, the vertices $0_1, 1_1, \ldots, (n-4)_1$, the vertex ∞, and the edges between these two sets, form a graph $K_{n-1,1}$, which has an S_4-decomposition, S, by Lemma 4. Therefore, $R \cup S$ forms a maximum S_4-packing of K_n with the leave K_3.

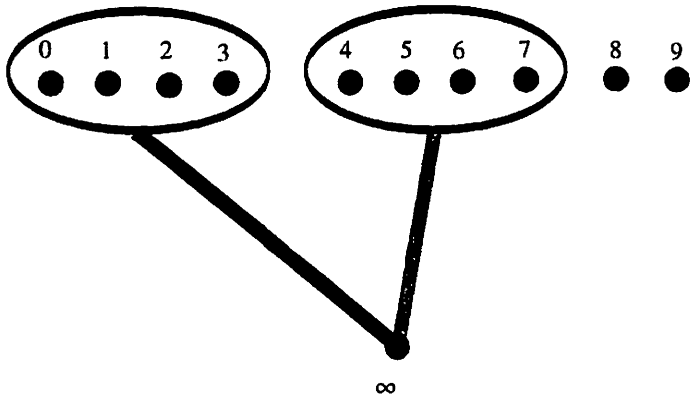

Figure 3: S_4-packing of K_{11} with the leave K_3

In order to get P_4 as the leave, again write $K_n = K_1 \vee K_{n-1}$, label the vertices as above, and let R be the same maximum packing with the same leave as above. The vertices $1, 2, \ldots, n-3$, the vertex ∞, and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 4, this complete bipartite graph has an S_4-decomposition, S. Therefore, $R \cup S$ forms a maximum packing of K_n with 4-stars where the three edges $\{n-3, n-2\}$, $\{n-2, \infty\}$, and $\{\infty, 0\}$ are left, which form a P_4 (see Figure 4, in which each thick line demonstrates a 4-star).

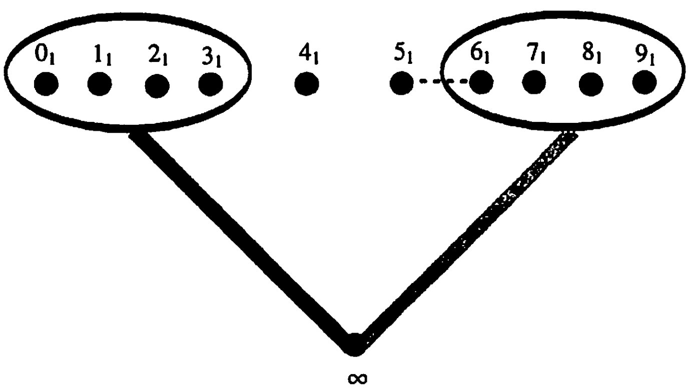

Figure 4: S_4-packing of K_{11} with the leave P_4

Finally, to get $S_2 + K_2$ as the leave, again write $K_n = K_1 \vee K_{n-1}$, label the vertices as above, and let R be the same maximum packing with the same leave as above. The vertices $2, 3, \ldots, n-2$, the vertex ∞, and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma

4, this complete bipartite graph has an S_4-decomposition, S. Therefore, $R \cup S$ forms a maximum packing of K_n with 4-stars where the three edges $\{n-2, n-3\}$, $\{\infty, 0\}$, and $\{\infty, 1\}$ are left which form an $S_2 + K_2$ (see Figure 5, in which each thick line demonstrates a 4-star). This completes the proof in this case.

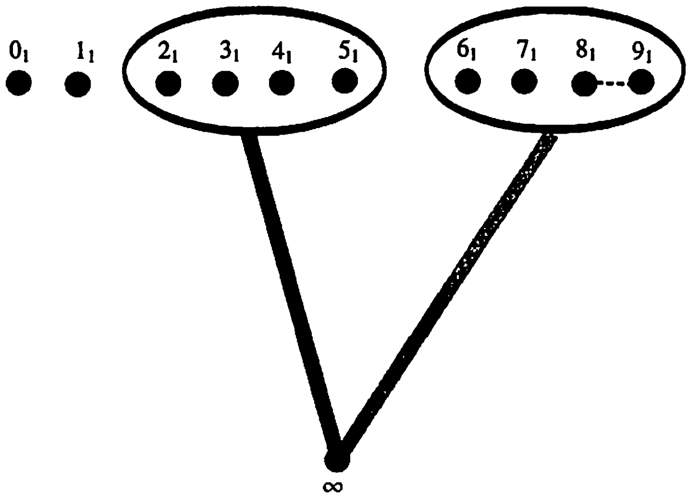

Figure 5: S_4-packing of K_{11} with the leave $K_2 + S_2$

Case 3. $n \equiv 4 \pmod 8$

By Theorem 5 the leave graph has two edges in this case. Hence, the possible leaves are S_2 and $2K_2$. In order to get S_2 as the leave, write $K_n = K_1 \vee K_{n-1}$. Label the vertices of K_{n-1} with the elements of $\mathbb{Z}_{n-1}$ and the single vertex of K_1 with ∞. Since $n \equiv 4 \pmod 8$, K_{n-1} has a maximum packing, R, with 4-stars with an S_3 as the leave as stated in case 2. Let the edges in this leave be $\{n-2, n-3\}$, $\{n-2, n-4\}$, and $\{n-2, n-5\}$. Now, the vertices $0, 1, \ldots, n-5$, the vertex ∞, and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 4, this complete bipartite graph has an S_4-decomposition, S. Therefore, we are left with the edges $\{n-2, n-3\}$, $\{n-2, n-4\}$, $\{n-2, n-5\}$, $\{\infty, n-4\}$, $\{\infty, n-3\}$, and $\{\infty, n-2\}$. Therefore, $R \cup S \cup \{(n-2; n-3, n-4, n-5, \infty)\}$ forms a maximum packing of K_n where the two edges $\{\infty, n-3\}$ and $\{\infty, n-4\}$ are left which form an S_2.

In order to get $2K_2$ as the leave, write $K_n = K_4 \vee K_{n-4}$. Label the vertices of K_{n-4} with the elements in $\mathbb{Z}_{n-4}$ having subscript 1 and the vertices of K_4 with the elements of $\mathbb{Z}_4$ having subscript 2. Since $n \equiv 4 \pmod 8$, K_{n-4} has an S_4-decomposition, R. The vertices $0_1, 1_1, 2_1$, the vertices $0_2, 1_2, 2_2, 3_2$, the edges within the set with subscript 2, and the edges between these two sets of vertices form a graph which we call H. The following 4-stars form a maximum packing of H, S, with 4-stars and leave the edges $\{0_2, 2_2\}$ and $\{1_2, 3_2\}$ which form a $2K_2$. For numbers with subscript 2 the computations are done modulo 4.

$$(i_2;(i+1)_2,0_1,1_1,2_1);\, i \in \mathbb{Z}_4.$$

Now, the vertices $3_1, 4_1, \ldots, (n-5)_1$, the vertices $0_2, 1_2, 2_2, 3_2$, and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 4, this complete bipartite graph has an S_4-decomposition, T. Therefore, $R \cup S \cup T$ forms a maximum packing of K_n where the edges $\{0_2, 2_2\}$ and $\{1_2, 3_2\}$ are left which form a $2K_2$. This completes the proof in this case.

Case 4. $n \equiv 5 \pmod 8$

In this case, again by Theorem 5, the leave graph has two edges. Write $K_n = K_1 \vee K_{n-1}$. Since $n \equiv 5 \pmod 8$, we have both of the possible leave graphs for K_{n-1} by the previous case. Let H be one of the leaves. Since $n-1$ is a multiple of 4, by Lemma 4, $K_{1,n-1}$ has an S_4-decomposition. So, the leave graph is H and the proof is completed in this case.

Case 5. $n \equiv 6 \pmod 8$

By Theorem 5, the leave has 3 edges in this case. Write $K_n = K_3 \vee K_{n-3}$. Since $n \equiv 6 \pmod 8$, we have all the possible leaves of S_4-packings of in K_{n-3} from case 2. Let H be one of those leaves and R be the corresponding packing. Label the vertices of K_{n-3} with the elements of $\mathbb{Z}_{n-3}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. The vertices $0_1, 1_1, 2_1$, the vertices $0_2, 1_2, 2_2$, the edges between these two sets of vertices, and the edges between the vertices in the second set form a $K_3 \vee 3K_1$. By Lemma 2, this graph has an S_4-decomposition, S. Now, the vertices $3_1, 4_1, \ldots, (n-4)_1$, the vertices $0_2, 1_2, 2_2$, and the edges between these two sets of vertices form a complete bipartite graph with one part of size $n-6$. Since $n \equiv 6 \pmod 8$, $n-6$ is a multiple of 4 and hence, this complete bipartite graph has an S_4-decomposition, T, by Lemma 4. Therefore, $R \cup S \cup T$ forms a maximum packing of K_n with 4-stars with the leave H and this completes the proof in this case.

Case 6. $n \equiv 7 \pmod 8$

In this case, again by Theorem 5, the leave graph is a single edge and the proof is complete. □

Note that for $n = 6$ the only possible leave will be K_3, which shows that the condition $n \geq 7$ in the theorem is necessary. We prove our statement as follows. Label the vertices of K_6 with the elements of $\mathbb{Z}_6$. Any maximum packing contains 3 stars. Without loss of generality we assume the first star to be $(0; 1, 2, 3, 4)$. We have two options for the next star center.

Assume we choose vertex 5 as the center of our next star. We can choose the leaves of the star to be the vertices 1, 2, 3, and 4 or choose one of the leaves to be the vertex 0 and the others to be three of the vertices 1, 2, 3, and 4. The first choice is impossible since every vertex will have degree at least two and we cannot add the third star. Hence, without loss of generality assume the second star to be $(5;0,1,2,3)$ and we have to choose $(4;1,2,3,5)$ as the third star and the leave graph will be the triangle with the edges $\{1,2\}$, $\{2,3\}$, and $\{3,1\}$.

Now, assume we choose one of the vertices of degree one to be the center of our second star. Without loss of generality we can take $(1;2,3,4,5)$ as the second star. Hence, the only possibility for the third star will be $(5;0,2,3,4)$ which gives a triangle with the edges $\{2,3\}$, $\{3,4\}$, and $\{4,2\}$ as the leave graph which completes the proof of our statement.

2.2 All the Possible Excess Graphs in the S_4-covering of K_n

In the previous subsection we illustrated how we can achieve all the possible leave graphs in an S_4-packing of K_n. Now, we show that we can obtain every possible excess graph in a minimum S_4-covering of K_n as well.

Theorem 8. *Let $n \geq 8$ be an integer and the excess graph in a minimum covering of the complete graph K_n with 4-stars have i edges. For any graph H with i edges there exists a minimum covering of K_n with 4-stars such that the excess graph is isomorphic to H.*

Proof. Again we know that for $n \equiv 0$ or $1 \pmod 8$, K_n has an S_4-decomposition. We show that for the remaining cases we have minimum coverings with all the possible excess graphs.

Case 1. $n \equiv 2 \pmod 8$

By Theorem 6 the excess graph has three edges in this case. The possible excesses with three edges are S_3, K_3, P_4, $3K_2$, $S_2 + K_2$, K_2^3, D, and F, where D is the graph K_2^2 with an edge attached to one of its vertices and F is the disjoint union of the graphs K_2^2 and K_2.

We can obtain the excess S_3 from a maximum packing of K_n with the leave K_2, adding a 4-star which has the leave of the packing as an edge.

For the excesses K_3, $S_2 + K_2$, $3K_2$, and P_4 we use the following construction. Write $K_n = K_3 \vee K_{n-3}$. Label the vertices of K_{n-3} by the elements of $\mathbb{Z}_{n-3}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Since $n \equiv 2 \pmod 8$, by case 6 in the proof of Theorem 7, K_{n-3} has an S_4-packing with a single edge as the leave graph; call the packing R. Let $\{(n-5)_1, (n-4)_1\}$ be that single edge. Consider the vertices $0_1, 1_1$, the vertices $0_2, 1_2, 2_2$, the edges between the vertices in the second set, and the edges between these two sets. The following 4-stars form a minimum covering called S with the triangle formed by the edges $\{0_2, 1_2\}$, $\{0_2, 2_2\}$, and $\{1_2, 2_2\}$ as the excess.

$$(0_2; 1_2, 2_2, 0_1, 1_1),$$
$$(1_2; 0_2, 2_2, 0_1, 1_1),$$
$$(2_2; 0_2, 1_2, 0_1, 1_1).$$

Now, Consider the vertices $2_1, 3_1, \ldots, (n-5)_1$, the vertices $0_2, 1_2, 2_2$, and the edges between these two sets. These form a complete bipartite graph with one part of size a multiple of 4 since $n \equiv 2 \pmod 8$. Hence, by Lemma 4, this complete bipartite graph has an S_4-decomposition, T. Therefore, $R \cup S \cup T \cup ((n-4)_1; (n-5)_1, 0_2, 1_2, 2_2)$ forms a minimum covering of K_n with a K_3 as the excess (see Figure 6, in which the thick line demonstrates a 4-star).

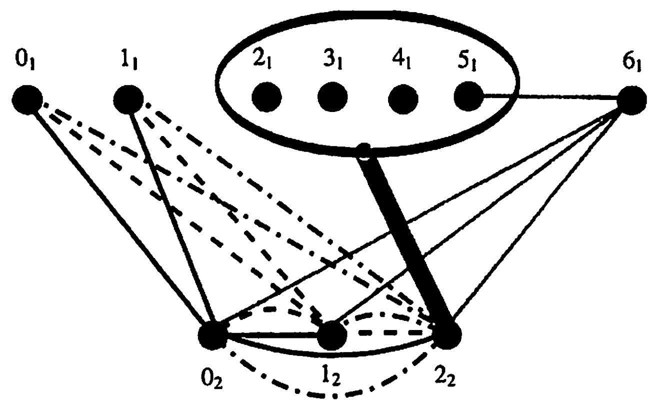

Figure 6: S_4-packing of K_n with the excess K_3

Consider the stars in the minimum covering above. Replacing the star $(2_2; 0_2, 1_2, 0_1, 1_1)$ with $(2_2; 0_2, (n-4)_1, 0_1, 1_1)$ gives the path P_4 as the excess graph.

Replacement of the same star with $(2_2; (n-4)_1, (n-5)_1, 0_1, 1_1)$ leads to the excess $S_2 + K_2$.

If we replace the stars $(0_2; 1_2, 2_2, 0_1, 1_1)$, $(1_2; 0_2, 2_2, 0_1, 1_1)$, and $(2_2; 0_2, 1_2, 0_1, 1_1)$ with $(0_2; 1_2, 2_1, 0_1, 1_1)$, $(1_2; 3_1, 2_2, 0_1, 1_1)$, and $(2_2; 0_2, 4_1, 0_1, 1_1)$, then the excess graph will be a $3K_2$.

For the remaining possible excesses, we use the following construction. Again write $K_n = K_3 \vee K_{n-3}$. Label the vertices of K_{n-3} with the elements of $\mathbb{Z}_{n-3}$ having subscript 1 and the vertices of K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. Since $n \equiv 2 \pmod 8$, we have $n - 3 \equiv 7 \pmod 8$. Hence, the leave graph in the packing of K_{n-3} has one edge by Theorem 5. Let R be a maximum packing of K_{n-3} with 4-stars and the single edge $\{(n-5)_1, (n-4)_1\}$ be the corresponding leave graph. The following stars along with the ones in R form a minimum covering of K_n with 4-stars with the three multiple edges $\{0_2, 1_2\}$ which is a K_2^3 (see Figure 7, in which the thick line demonstrates a 4-star).

$(0_2; 1_2, i_1, (i+1)_1, (i+2)_1), i = 0$ and 3

$(1_2; 0_2, i_1, (i+1)_1, (i+2)_1), i = 0$ and 3

$(i_2; (4j+6)_1, (4j+7)_1, (4j+8)_1, (4j+9)_1), 0 \leq i \leq 1, 0 \leq j \leq \frac{n-14}{4}, i, j \in \mathbb{Z}$

$(2_2; (4j)_1, (4j+1)_1, (4j+2)_1, (4j+3)_1), 0 \leq j \leq \frac{n-10}{4}, j \in \mathbb{Z}$

$((n-4)_1; (n-5)_1, 0_2, 1_2, 2_2),$

$(2_2; 0_2, 1_2, (n-6)_1, (n-5)_1).$

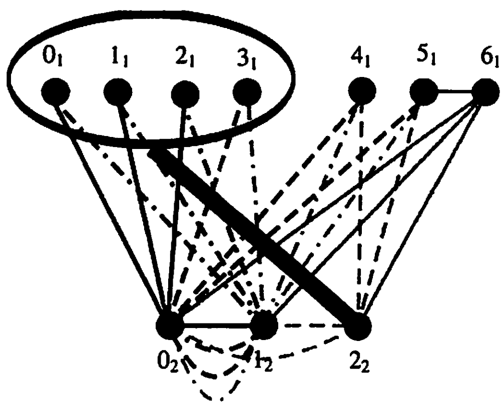

Figure 7: S_4-packing of K_n with the excess K_2^3

In the same covering as above, replace the star $(0_2; 1_2, 0_1, 1_1, 2_1)$ with $(0_2; 2_2, 0_1, 1_1, 2_1)$ to achieve the excess D.

Consider the covering with excess D and replace the stars $(1_2; 0_2, 0_1, 1_1, 2_1)$ and $(1_2; 0_2, 3_1, 4_1, 5_1)$ with $(1_2; (n-4)_1, 0_1, 1_1, 2_1)$ and $(1_2; (n-4)_1, 3_1, 4_1, 5_1)$ to give the excess F. This proves the theorem in the first case.

Case 2. $n \equiv 3 \pmod 8$

By Theorem 6, the excess graph is a single edge and the proof is complete in this case.

Case 3. $n \equiv 4 \pmod 8$

Again by Theorem 6, the excess graph has two edges. The possible graphs with two edges are S_2, $2K_2$, and K_2^2. The excess S_2 is easily obtained from a maximum packing with the leave S_2.

In order to get the excess $2K_2$ write $K_n = K_1 \vee K_{n-1}$. Label the vertex of K_1 with ∞ and the vertices of K_{n-1} with the elements of $\mathbb{Z}_{n-1}$. Since $n \equiv 4 \pmod 8$, we have $n-1 \equiv 3 \pmod 8$ and hence, the excess graph of a covering of K_{n-1} with 4-stars has a single edge. Let that single edge be $\{n-3, n-2\}$. The following stars along with those in the minimum covering of K_{n-1} form a minimum covering for K_n with 4-stars with the excess $2K_2$.

$$(\infty; 4i, 4i+1, 4i+2, 4i+3), i \in \left\{0, 1, \ldots, \frac{n-8}{4}\right\},$$
$$(\infty; 0, n-4, n-3, n-2).$$

The construction below gives the excess K_2^2. Write $K_n = K_3 \vee K_{n-3}$. Since $n \equiv 4 \pmod 8$, K_{n-3} has an S_4-decomposition. Partition the vertices of K_{n-3} into a set of three vertices, a set of two vertices, and a set of $n-8$ vertices. First, consider the set of three vertices. By Lemma 2, $K_3 \vee 3K_1$ has an S_4-decomposition. Now, consider the set of $n-8$ vertices. Since $n \equiv 4 \pmod 8$, $n-8$ is a multiple of 4. Hence, by Lemma 4, $K_{3,n-8}$ has an S_4-decomposition. Label the vertices of the set of two vertices with the elements of $\mathbb{Z}_2$ having subscript 1 and the vertices of the K_3 with the elements of $\mathbb{Z}_3$ having subscript 2. The following stars along with those in the decompositions of K_{n-3}, $K_3 \vee 3K_1$, and $K_{3,n-8}$ form a minimum covering of K_n with the excess K_2^2.

$$(0_1; 1_1, 0_2, 1_2, 2_2),$$
$$(1_1; 0_1, 0_2, 1_2, 2_2).$$

Case 4. $n \equiv 5 \pmod 8$

By Theorem 6, the excess has two edges. Let H be one of the possible graphs with two edges. Write $K_n = K_1 \vee K_{n-1}$. Since $n \equiv 5 \pmod 8$, by case 3, K_{n-1} has a minimum covering with the excess H. Since $n-1$ is a multiple of 4, $K_{1,n-1}$ has an S_4-decomposition by Lemma 4. Hence, the stars in the decomposition of $K_{1,n-1}$ along with those in the minimum covering of K_{n-1} form a minimum covering of K_n with the excess H.

Case 5. $n \equiv 6 \pmod 8$

By Theorem 6, the excess graph is a single edge and the proof is complete in this case.

Case 6. $n \equiv 7 \pmod 8$

In this case, the excess has three edges. For $n > 7$, write $K_n = K_5 \vee K_{n-5}$. Let H be any possible graph with three edges where multiple edges are allowed as well. Since $n \equiv 7 \pmod 8$, K_{n-5} has a minimum covering with excess H by case 1. Partition the vertices of K_{n-5} into a set of six vertices and a set of $n-11$ vertices. Consider the set of $n-11$ vertices. Since $n \equiv 7 \pmod 8$, $n-11$ is a multiple of 4. Hence, by Lemma

4, $K_{5,n-11}$ has an S_4-decomposition. Now, consider the set of six vertices. By Lemma 2, $K_5 \vee 6K_1$ has an S_4-decomposition. The stars in the decompositions of $K_{5,n-11}$ and $K_5 \vee 6K_1$ along with those in the minimum covering of K_{n-5} form a minimum covering of K_n with the excess H. □

The table below summarizes our results. In this table, D denotes the graph K_2^2 with an edge attached to one of its vertices and F denotes the disjoint union of the graphs K_2^2 and K_2.

Table 1: All Possible Leaves (Excesses) in the Packings (Coverings) of K_n with 4-stars

n (mod 8)	Leave (for $n \geq 7$)	Excess (for $n \geq 8$)
0	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$
2	K_2	$S_3, K_3, P_4, 3K_2, S_2 + K_2, K_2^3, D$, and F
3	$S_3, K_3, P_4, 3K_2$, and $S_2 + K_2$	K_2
4	S_2 and $2K_2$	$S_2, 2K_2$, and K_2^2
5	S_2 and $2K_2$	$S_2, 2K_2$, and K_2^2
6	$S_3, K_3, P_4, 3K_2$, and $S_2 + K_2$	K_2
7	K_2	$S_3, K_3, P_4, 3K_2, S_2 + K_2, K_2^3, D$, and F

3 Conclusions and Future Directions

In this paper, we achieved all possible leaves and excesses for the maximum packing and minimum covering of the complete graph K_n with 4-stars. As the next step, we will try to get all the possible leaves and excesses for the maximum packing and minimum covering of K_n with 5-stars. This case will be more complicated than 4-stars since we deal with leaves and excesses of size four as well.

It is also tempting to work on generalizations of Theorems 5 and 6 for any k-star. The problem seems to be solvable by similar constructions as illustrated in this paper for the cases when $n \equiv 2, 3, k$, or $2k - 1 \pmod{2k}$.

As another direction, we are going to find the spectrum for packings and coverings of the complete graph K_n with all trees of five vertices.

Acknowledgement

The authors would like to thank the referee for their comprehensive reading and useful comments.

References

[1] Adams, P.; Bryant, D., and Buchanan, M., *A survey on the existence of G-designs*, J. Combin. Des. 16, 2008, 373-410.

[2] Caro, Y. and Yuster, R., *Packing Graphs: the Packing Problem Solved.*, Electron. J. Combin. 4, no. 1, 1997.

[3] Caro, Y. and Yuster, R., *Covering Graphs: the Covering Problem Solved.*, J. Combin. Theory Ser. A 83, 1998, 273-282.

[4] Colbourn, C. and Dinitz, J., The CRC Handbook of Combinatorial Designs (second edition), CRC Press, Boca Raton, 2007.

[5] Hell, P. and Rosa, A., *Graph Decompositions, Handcuffed Prisoners and Balanced P-designs*, Discrete Math. 2, no. 3, 1972, 229-252.

[6] Huang, C. and Rosa, A., *Decomposition of complete graphs into trees*, Ars. Combin. 5, 1978, 23-63.

[7] Roditty, Y., *Packing and covering of the complete graph with a graph G of four vertices or less*, J. Combin. Theory Ser. A 34, no. 2, 1983, 231-243.

[8] Roditty, Y., *Packing and covering of the complete graph. II. The trees of order six*, Ars. Combin. 19, 1985, 81-93.

[9] Roditty, Y., *The packing and covering of the complete graph. I. The forests of order five*, Internat. J. Math. Math. Sci. 9, no. 2, 1986, 277-282.

[10] Roditty, Y., *Packing and covering of the complete graph. IV. The trees of order seven*, Ars. Combin. 35, 1993, 33-64.

[11] Tarsi, M., *Decomposition of complete multigraphs into stars*, Discrete Math. 26, 1979, 273-278.

[12] West D.: Introduction to Graph Theory (second edition). Prentice Hall (2001).

[13] Wilson, R. M., *Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph*, Congr. Numer. 15, 1976, 647-659.

[14] Yamamoto, S.; Ikedo, H.; Shige-eda, S.; Ushio, K.; and Hamada, N., *On claw-decompositions of complete graphs and complete bigraphs*, Hiroshima Math. J. 5, 1975, 33-42.