Spanning Eulerian subgraphs and Catlin's reduced graphs

Wei-Guo Chen*, Zhi-Hong Chen†

Abstract

A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph H_R of G whose set of odd degree vertices is R. A graph is *reduced* if it has no nontrivial collapsible subgraphs. Catlin [4] showed that the existence of spanning Eulerian subgraphs in a graph G can be determined by the reduced graph obtained from G by contracting all the collapsible subgraphs of G. In this paper, we present a result on 3-edge-connected reduced graphs of small orders. Then, we prove that a 3-edge-connected graph G of order n either has a spanning Eulerian subgraph or can be contracted to the Petersen graph if G satisfies one of the following:

(i) $d(u) + d(v) > 2(n/15 - 1)$ for any $uv \notin E(G)$ and n is large;

(ii) the size of a maximum matching in G is at most 6;

(iii) the independence number of G is at most 5.

These are improvements of prior results in [16], [18], [24] and [25].

1. Introduction

We shall use the notation of Bondy and Murty [3], except when otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. The graph of order 2 and size 2 is called a 2-cycle and denoted by C_2. For a graph G, $\kappa'(G)$ and $d_G(v)$ (or $d(v)$) denote the edge-connectivity of G and the degree of a vertex v in G, respectively. The set of vertices of degree i in G is denoted by $D_i(G)$. The maximum cardinality of an independent set of vertices in G is denoted by $\alpha(G)$. The size of a maximum matching in G is denoted by $\alpha'(G)$. Let $O(G)$ be the set of vertices of odd degree in G. A connected graph G is *Eulerian* if $O(G) = \emptyset$. An Eulerian subgraph H in G is a spanning Eulerian subgraph if $V(H) = V(G)$. A graph is *supereulerian* if it has a spanning Eulerian subgraph. A graph G is *collapsible* if for any even subset $R \subseteq V(G)$ or $R = \emptyset$,

*Email: 0815.chen@gmail.com, Guangdong Information Center, Guangzhou, China

†Email: chen@butler.edu, Butler University, Indianapolis, IN 46208, USA

JCMCC 96 (2016), pp. 41-63

G has a spanning connected subgraph H_R with $O(H_R) = R$. Examples of collapsible graphs include K_n and $K_{n,n} - e$ $(n \geq 3)$.

Throughout this paper, we use P for the Petersen graph and use P_{14} and P_{16} for the graphs defined in Figure 1.1. P, P_{14} and P_{16} are 3-edge-connected non-supereulerian graphs.

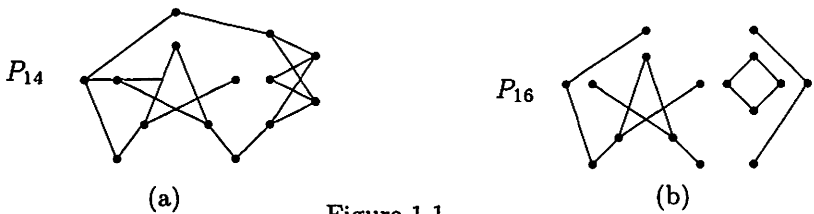


Figure 1.1

For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. If H is connected, then we use v_H denote the vertex in G/H to which H is contracted and v_H is called the contraction image of H. Thus, we regard $V(G/H) = (V(G) - V(H)) \cup \{v_H\}$ and $E(G/H) = E(G) - E(H)$.

Catlin's reduction method: In the study of graphs with spanning Eulerian subgraphs and other related graph theory problems such as Hamiltonian line graph and double cycle cover problems [7, 8, 9], Catlin [4] developed a reduction technique by contracting collapsible subgraphs. Catlin [4] showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs H_1, H_2, $\cdots$, H_c such that $V(G) = \cup_{i=1}^{c} V(H_i)$. The *reduction* of G is a graph obtained from G by contracting each H_i into a single vertex v_i $(1 \leq i \leq c)$ and is denoted by G'. For a vertex $v \in V(G')$, there is a unique maximal collapsible subgraph $H(v)$ in G such that v is the contraction image of $H(v)$ and $H(v)$ is the *preimage* of v. We regard K_1 as a collapsible and supereulerian graph, and having $\kappa'(K_1) = \infty$. A vertex $v \in V(G')$ is *trivial* if v is the contraction image of K_1. A graph is called Catlin's *reduced* or reduced if $G = G'$. We use $\mathcal{CL}$ and $\mathcal{SL}$ to denote the families of collapsible graphs and supereulerian graphs, respectively. Thus, $\mathcal{CL} \subset \mathcal{SL}$. By the definition of contraction, $\kappa'(G') \geq \kappa'(G)$.

Theorem A below shows the importance of Catlin's reduction method.

Theorem A (Catlin [4]). Let G be a connected graph. Let G' be the reduction of G. Let H be a collapsible subgraph of G. Then each of the following holds:

(a) $G \in \mathcal{CL}$ if and only if $G/H \in \mathcal{CL}$. Thus, $G \in \mathcal{CL}$ if and only if $G' = K_1$.

(b) $G \in \mathcal{SL}$ if and only if $G/H \in \mathcal{SL}$. Thus, $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.

Knowing the structures of reduced graphs of small orders is very important when using the reduction method. The following theorem has been used by many:
Theorem B (Chen and Lai [14, 18]). Let G be a connected graph with at most 11 vertices and $\delta(G) \geq 3$. Then either $G \in \mathcal{CL}$ or $G' \in \{K_2, P\}$.

Catlin had a conjecture on reduced graph of order at most 17.
Conjecture 1 (Catlin [9]). Any 3-edge-connected simple graph of order at most 17 is either supereulerian or is contractible to the Petersen graph.

Several conjectures (see [12, 9, 22]) are extended from Conjecture 1. In this paper, we prove the following theorem, a progress toward solving Conjecture 1.
Theorem 1.1. Let G be a 3-edge-connected graph with at most 15 vertices. Let G' be the reduction of G. Then each of the following holds:

(a) if $|V(G)| \leq 13$, then either $G \in \mathcal{SL}$ or $G' = P$;

(b) if $|V(G)| \leq 14$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$;

(c) if $|V(G)| = 15$, $G \notin \mathcal{SL}$ and $G' \notin \{P, P_{14}\}$, then $G = G'$ and G is a 2-connected and essentially 4-edge-connected reduced graph with girth at least 5 and $V(G) = D_3(G) \cup D_4(G)$ where $D_4(G)$ is an independent set and $|D_4(G)| = 3$.

Results like Theorem 1.1 play an important role in Catlin's reduction method, since using Catlin's reduction method, many problems related to the existence of spanning Eulerian subgraphs can be reduced to the same or similar problems of graphs with very few vertices. Using Theorem 1.1, we obtain the best possible conditions on $\alpha'(G)$ and $\alpha(G)$ for a graph G to be supereulerian. The $\alpha'(G)$ case is better than a conjecture in [25].

Theorem 1.2. Let G be a 3-edge-connected simple graph. Let G' be the reduction of G. Then each of the following holds:

(a) if $\alpha'(G) \leq 6$, then either $G \in \mathcal{SL}$ or $G' = P$;

(b) if $\alpha(G) \leq 5$, then either $G \in \mathcal{SL}$ or $G' = P$.

Next, we prove the following:
Theorem 1.3. Let G be a 3-edge-connected graph of order n and with girth g, where $g \in \{3, 4\}$. If n is large enough and

$$d(u) + d(v) > \frac{2}{g-2}\left(\frac{n}{15} - 4 + g\right) \text{ for any } uv \notin E(G) \qquad (1)$$

then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$. Thus, either $G \in \mathcal{SL}$ or G can be contracted to P.

2. Prior results and π-reduction method

For a graph G, we use $F(G)$ for the minimum number of extra edges that must be added to G to obtain a spanning supergraph having two edge-disjoint spanning trees.

A number of results on reduced graphs are summarized in the following.

Theorem C. Let G be a connected reduced graph. Then each of the following holds:

(a) (Catlin [4]). G is K_3-free with $\delta(G) \leq 3$ and any subgraph H of G is reduced. Furthermore, $|E(H)| \leq 2|V(H)| - 4$ unless $H \in \{K_1, K_2\}$;

(b) (Catlin [5]). $F(G) = 2|V(G)| - |E(G)| - 2$;

(c) (Catlin et al. [11]). If $F(G) \leq 2$, then $G \in \{K_1, K_2, K_{2,t}(t \geq 1)\}$.

(d) (Chen and Lai [19]). If $\delta(G) \geq 3$, then $\alpha'(G) \geq (|V(G)| + 4)/3$.

(e) (Chen [16]). If $\alpha(G) \geq 4$, then $\frac{\delta(G)\alpha(G)+4}{2} \leq |V(G)| \leq 4\alpha(G) - 5$.

Let G be a graph containing an induced 4-cycle $uvzwu$ and let $E = \{uv, vz, zw, wu\}$. Denote by G/π the graph obtained from $G - E$ by identifying u and z to form a vertex x, and by identifying v and w to form a vertex y, and by adding an edge $e_\pi = xy$. The way to obtain G/π from G is called *π-reduction method* [5].

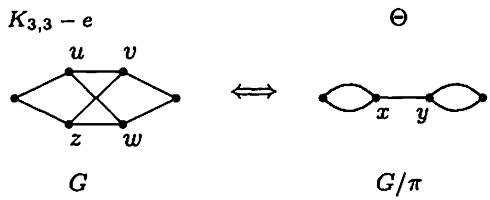


Figure 2.1

Define Θ as a graph shown in Figure 2.1.

Theorem D (Catlin [5]). Let G be a connected graph and let G/π be the graph defined above, then each of the following holds:

(a) If $G/\pi \in \mathcal{CL}$, then $G \in \mathcal{CL}$;

(b) If $G/\pi \in \mathcal{SL}$ then $G \in \mathcal{SL}$;

(c) If G is K_3-free and G/π contains Θ as subgraph, then G has $K_{3,3} - e$ as a collapsible subgraph.

Notation: Let $s_{1,2}, s_{2,3}, s_{3,1}, m, l$ and t be natural numbers with $t \geq 2$ and $m, l \geq 1$. Let $\Phi_a \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_{1,2}, s_{2,3}, s_{3,1})$ to be the graph obtained from Φ_a by adding $s_{i,j}$ vertices with neighbors $\{a_i, a_j\}$ ($1 \le i \ne j \le 3$). Note that $K_{1,3}(1,1,1)$ is the 3-cube minus a vertex. For graphs $K'_{2,t}$, $S(m,l)$, $J(m,l)$, and $J'(m,l)$, see the Figure 2.2 below. They are reduced graphs.

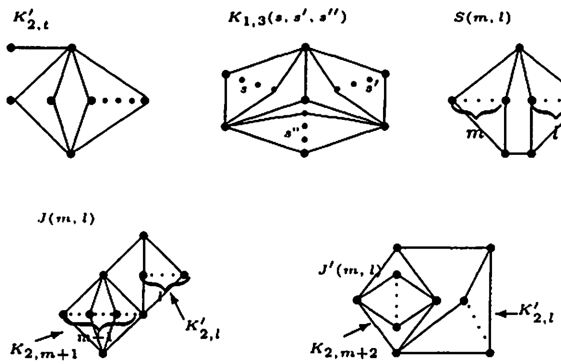


Figure 2.2

Let $\mathcal{F} = \{K_1,\ K_{2,t},\ K'_{2,t},\ K_{1,3}(s,s',s''),\ S(m,l),\ J(m,j),\ J'(m,l),\ P\}$.

Some prior results on reduced graphs of small orders are given in the following theorem:

Theorem E. Let G be a simple connected graph of order n.

(a) (Chen [14]). If $n \le 7$, $\kappa'(G) \ge 2$, and $|D_2(G)| \le 2$, then $G \in \mathcal{CL}$.

(b) (Catlin [10]). If $n \le 8$, $\kappa'(G) \ge 2$ and $|D_2(G)| \le 1$. Then $G \in \mathcal{CL}$.

(c) (Chen and Lai [18]). If G is reduced with $n \le 11$ and $F(G) \le 3$ then either $G \in \mathcal{F}$ or G is a tree with at most 3 edges.

The following corollary will be needed:

Corollary 2.1. Let G be a connected simple graph of order n with $\delta(G) \ge 2$. Let G' be the reduction of G.

(a) If $n \le 6$ and $\delta(G) \ge 2$ and $|D_2(G)| \le 2$, then $G \in \mathcal{CL}$.

(b) If $n \le 7$, $\delta(G) \ge 2$ and $|D_2(G)| \le 2$, then $G' \in \{K_1, K_2\}$.

(c) If $G \ne K_1$ is reduced, $n \le 7$, $\kappa'(G) \ge 2$ and $|D_2(G)| = 3$, then $G \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$.

(d) If $n \le 9$, $\delta(G) \ge 2$ and $|D_2(G)| \le 1$, then $G' \in \{K_1, K_2, K_{1,2}\}$.

(e) If $n \le 9$, $\kappa'(G) \ge 2$ and $|D_2(G)| \le 2$, then $G' \in \{K_1, K_{2,3}\}$. Furthermore, if G is K_3-free, $G \in \mathcal{CL}$.

Proof. For (a) and (b), if $\kappa'(G) \geq 2$, then by Theorem E(a), $G \in \mathcal{CL}$. We may assume that $\kappa'(G) = 1$. Let e be an edge-cut of G. Let G_1 and G_2 be the two components of $G - e$ and $|V(G_1| \leq |V(G_2)|$. Since $\delta(G) \geq 2$ with $|D_2(G)| \leq 2$, $|V(G_1)| \geq 3$.

If $n = 6$, then $|V(G_1)| = |V(G_2)| = 3$. But then $|D_2(G)| > 2$, a contradiction. Thus, (a) holds.

If $n = 7$, then $|V(G_1)| = 3$ and $|V(G_2)| = 4$. Since $|D_2(G)| \leq 2$, $G_1 = K_3$ and $G_2 = K_4$. Then $G' = (G/K_3)/K_4 = K_2$. Thus, (b) holds.

To prove (c), (d) and (e), we prove the following claim first:

Claim 1. If G is reduced and $n + |D_2(G)| \leq 11$, then $F(G) \leq 3$.

Counting the edges in G we have

$$|E(G)| = \frac{\Sigma_{v \in V(G)} d(v)}{2} \geq \frac{2|D_2(G)| + 3(n - |D_2(G)|)}{2} = \frac{3n - |D_2(G)|}{2}. \quad (2)$$

By Theorem C(b), (2), $n + |D_2(G)| \leq 11$,

$$F(G) = 2|V(G)| - |E(G)| - 2 \leq 2n - \frac{3n - |D_2(G)|}{2} - 2 \leq \frac{7}{2}.$$

Since $F(G)$ is an integer, $F(G) \leq 3$. Claim 1 is proved.

For (c), since G is reduced with $n \leq 7$ and $|D_2(G)| = 3$, $n + |D_2(G)| \leq 10$. By Claim 1, $F(G) \leq 3$. By Theorem E(c) and G is not a tree since $\delta(G) \geq 2$, $G \in \mathcal{F}$. Except for the graphs in $\{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$, other graphs G in $\mathcal{F}$ either have $|V(G)| > 7$, or $|D_2(G)| \neq 3$ or $\kappa'(G) < 2$. Corollary 2.1(c) is proved.

For (d) and (e), we only need to consider $8 \leq n \leq 9$.

If G is reduced, then since $|D_2(G)| \leq 2$, $n + |D_2(G)| \leq 11$. By Claim 1, $F(G) \leq 3$. By Theorem E(c) and G is not a tree, $G \in \mathcal{F}$. However, each graph in $\mathcal{F}$ has at least three vertices of degree 2, contrary to the fact that $|D_2(G)| \leq 2$. Thus, G cannot be reduced.

Let H be a nontrivial maximal collapsible subgraph of G. Let v_H be the contraction image of H in G/H. Then $|V(H)| \geq 3$ and so $|V(G/H)| \leq 7$. If $V(H) \cap D_2(G) \neq \emptyset$, then $|D_2(G/H)| \leq |D_2(G)| \leq 2$. By (b) proved above, $(G/H)' \in \{K_1, K_2\}$ and so $G' \in \{K_1, K_2\}$. Thus, (d) is proved for this case. For (e), since $\kappa'(G) \geq 2$, $G' \neq K_2$. Thus, $G' = K_1$. (e) is proved for this case too.

In the following, we assume that $V(H) \cap D_2(G) = \emptyset$.

Case 1. $|D_2(G)| \leq 1$ and $\delta(G) \geq 2$ as stated in (d).

If $d_{G/H}(v_H) \geq 2$, then since $|D_2(G)| \leq 1$, $|D_2(G/H)| \leq 2$. Then since $|V(G/H)| \leq 7$ with $|D_2(G/H)| \leq 2$, by (a) and (b) above, either $G/H \in \mathcal{CL}$ and so by Theorem A $G \in \mathcal{CL}$, or the reduction of $G/H = K_2$, and so $G' = K_2$. Hence, (d) is proved for this case.

If $d_{G/H}(v_H) = 1$, let $e = uv_H$ be the edge incident with v_H, which is an edge-cut of G. Let G_1 be the component of $G - e$ containing u. Then H is the other component of $G - e$. Since $V(H) \cap D_2(G) = \emptyset$, $|V(H)| \geq 4$.

If $d(u) > 2$, then $|V(G_1)| \leq 5$ with $\delta(G_1) \geq 2$ and $|D_2(G_1)| \leq 2$, and so by (a), $G_1 \in \mathcal{CL}$. Thus $G' = G/(H \cup G_1) = K_2$.

If $d(u) = 2$, then $d_{G_1}(u) = 1$. Since $|D_2(G)| = \{u\}$, $d_{G_1}(v) \geq 3$ for $v \in V(G_1) - \{u\}$. Let $H_1 = G_1 - u$. Then $|V(H_1)| \leq 4$ with $|D_2(H_1)| = 1$. By (a) again, $H_1 \in \mathcal{CL}$, thus $G/(H \cup H_1) = K_{1,2}$. Corollary 2.1(d) is proved.

Case 2. $|D_2(G)| \leq 2$ and $\kappa'(G) \geq 2$ as stated in (e).

We only need to consider the case that $|D_2(G)| = 2$. Let $G_0 = G/H$. Then $|V(G_0)| \leq 7$ and $|D_2(G_0)| \leq 3$. If $|D_2(G_0)| \leq 2$, then by (b) proved above, $G_0' \in \{K_1, K_2\}$. Since $\kappa'(G_0) \geq \kappa'(G) \geq 2$, $G_0' \neq K_2$. Thus, $G_0' = K_1$ and so $G' = K_1$. We are done in this case.

If $|D_2(G_0)| = 3$, then $d_{G_0}(v_0) = 2$. Therefore, there are only two edges from $G - E(H)$ adjacent to vertices in $V(H)$. Since $V(H) \cap D_2(G) = \emptyset$, $|V(H)| \geq 4$, and so $|V(G_0)| = |V(G/H)| \leq 6$.

If G_0 is reduced, then since $|D_2(G_0)| = 3$, $\kappa'(G_0) \geq 2$ and $|V(G_0)| \leq 6$, by (c) above, $G_0 = K_{2,3}$ and so $G' = G_0 = K_{2,3}$. (e) is proved for this case.

If G_0 is not reduced, let H_1 be a nontrivial maximal collapsible subgraph of G_0. Let $G_2 = G_0/H_1$ and let v_1 be the contraction image of H_1. Since $|V(G_0)| \leq 7$, $|V(G_2)| \leq 4$ if $d_{G_2}(v_1) \geq 3$; or $|V(G_2)| \leq 3$ if $d_{G_2}(v_1) = 2$. Then G_2 is collapsible. Thus, $G' = G_1' = K_1$. Thus, $G' \in \{K_1, K_{2,3}\}$.

If G is K_3-free, then any non-trivial collapsible subgraph H of G has order at least 6. Thus, $|V(G/H)| \leq 4$ which implies that $G' \neq K_{2,3}$ and so $G \in \mathcal{CL}$. Corollary 2.1(e) holds.

This completes the proof of Corollary 2.1. □.

Lemma 2.2. Let G be a simple and K_3-free connected graph of order n where $n \leq 15$ and $|D_2(G)| \leq 2$. Let $H_0 = uvzwu$ be a induced 4-cycle in G. Let G/π be the graph obtained from G as defined by the π-reduction method. Then each of the following holds.

(a) If $G/\pi = P$, then $G \in \mathcal{SL}$.

(b) If $\kappa'(G) \geq 2$, $\kappa'(G/\pi) = 1$ and $|D_2(G)| \leq 1$, then G is not reduced.

Proof. **(a)** Since $G/\pi = P$ is a 3-regular graph of order 10, by the definition of G/π, $|V(G)| = 12$ and $D_2(G) \subseteq V(H_0)$. Then G is one of the graphs in Figure 2.3 (up to isomorphic). As we can see G_a has a hamiltonian cycle $v_1v_2zvv_5v_3v_7v_6wuv_4v_8v_1$; G_b (and G_c) has a spanning closed trail: $v_1v_6v_7v_4uvzwuv_2v_3v_5v_8v_1$. Thus, $G \in \mathcal{SL}$.

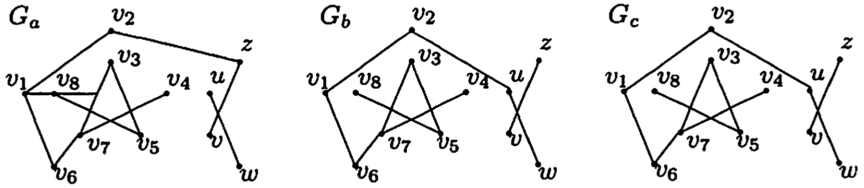


Figure 2.3

(b) Since $\kappa'(G) \geq 2$, e_π must be an edge-cut of $\kappa'(G/\pi)$. Since $|D_2(G)| \leq 1$, $G - E(H_0)$ has two non-trivial components, say G_1 and G_2, where $u, z \in V(G_1)$ and $v, w \in V(G_2)$ with $1 < |V(G_1)| \leq |V(G_2)|$. Since $|V(G)| \leq 15$, $|V(G_1)| \leq 7$. Since $|D_2(G)| \leq 1$ and $\kappa'(G) \geq 2$, G_1 has a 2-edge-connected subgraph H_1 with $|D_2(H_1)| \leq 2$. By Theorem E(a), $H_1 \in \mathcal{CL}$, and so G is not reduced. □

Corollary 2.3. Let G be a simple graph of order n with $\kappa'(G) \geq 2$.

(a) If $n \leq 10$ and $|D_2(G)| \leq 1$, then either $G \in \mathcal{CL}$ or $G = P$;

(b) If $n = 12$, $|D_2(G)| = 1$ and G is essentially 3-edge-connected, then either $G \in \mathcal{SL}$ or G has a maximal collapsible subgraph K_3 such that $G' = G/K_3 = P$.

Proof. **(a)** By Corollary 2.1(d), we only need to consider $|V(G)| = 10$. If $|D_2(G)| = 0$, then by $\kappa'(G) \geq 2$, Corollary 2.2 follows from Theorem B.

Next, we assume $|D_2(G)| = 1$. Let v be the only vertex in $D_2(G)$.
Case 1. G contains a K_3. Let H be a maximal collapsible graph containing a K_3. Then G/H is simple with $\kappa'(G/H) \geq 2$. If $H = K_3$, then $|V(G/H)| \leq 8$ and $|D_2(G/H)| \leq 1$, and so by Theorem E(b), $G/H \in \mathcal{CL}$. Thus $G \in \mathcal{CL}$. If $H \neq K_3$, then $|V(G/H)| \leq 7$ and $|D_2(G/H)| \leq 2$ and so by Theorem E(a) and by $\kappa'(G/H) \geq 2$, $G/H \in \mathcal{CL}$. Hence $G \in \mathcal{CL}$.

Case 2. G is K_3-free. Let $G_1 = G - v$. Since $|D_2(G)| \leq 1$, $|D_2(G_1)| \leq 2$. Hence $|V(G_1)| \leq 9$ and $|D_2(G_1)| \leq 2$. If $\kappa'(G_1) \geq 2$, then by Corollary 2.1(e), G_1 and then G are in $\mathcal{CL}$. Thus, we may assume that $\kappa'(G_1) = 1$.

Suppose that G_1 has a cut-edge e'. Let G_1' and G_1'' be the two components of $G_1 - e'$. By $|D_2(G)| \leq 1$, each of G_1' and G_1'' contains at least 3 vertices and so each satisfies the hypothesis of Corollary 2.1(a). Thus both G_1' and G_2'' are in $\mathcal{CL}$ and so $(G/G_1')/(G_1'') = K_3$. Hence $G \in \mathcal{CL}$.

(b) Suppose that $G \notin \mathcal{SL}$ and so $G' \neq K_1$. Let $D_2(G) = \{v\}$.
Case 1. G has a K_3 subgraph. Let H be a maximum collapsible subgraph containing a K_3. Let v_H be the contraction image of H in G/H.

If $H = K_3$ then $d_{G/H}(v_H) \geq 2$ if $v \in V(H)$; otherwise, $d_{G/H}(v_H) \geq 3$. Thus, $|V(G/H)| = 10$ and $|D_2(G/H)| \leq 1$. Since $G' = (G/H)' \neq K_1$, by (a) above, $G/H = G/K_3 = P$.

If $H \neq K_3$, then $|V(H)| \geq 4$, and so $|V(G/H)| \leq 9$ and $|D_2(G/H)| \leq 2$. By Corollary 2.1(e) and $G' \neq K_1$, $G' = (G/H)' = K_{2,3}$. Since $|D_2(G)| = 1$, at least two vertices of degree 2 in G' are contractions of nontrivial collapsible subgraphs of G. Thus, G has an essential edge-cut of size 2, contrary to the fact that G is essentially 3-edge-connected.

Case 2. G is K_3-free but has a 4-cycle H_0. Let G/π be the graph defined by the π-reduction method. Then $|V(G/\pi)| = 10$ and $|D_2(G/\pi)| \leq 1$.

If $\kappa'(G/\pi) \geq 2$, then by (a) above, either $G/\pi \in \mathcal{CL}$ or $G/\pi = P$. By Theorem D or by Lemma 2.2(a), $G \in \mathcal{SL}$, a contradiction.

If $\kappa'(G/\pi) = 1$, then since $|D_2(G/\pi)| \leq 1$, by Lemma 2.2(b), G is not reduced. Let H be a nontrivial maximal collapsible subgraph of G. Since G is K_3-free, $|V(H)| \geq 6$. Thus, $|V(G/H)| \leq 7$ with $|D_2(G/H)| \leq 2$ and $\kappa'(G/H) \geq \kappa'(G) \geq 2$. By Theorem E(a), $G/H \in \mathcal{CL}$, a contradiction.

Case 3. G has no 3- and 4-cycles. Since $|D_2(G)| = 1$ and G cannot have 11 vertices of degree 3, $\Delta(G) \geq 4$. Let z be a vertex of degree $\Delta(G)$. Let $N(z) = \{y_1, y_2, y_3, y_4, \cdots\}$. Since G has no 3- or 4-cycles, $(N(y_i) - z) \cap (N(y_j) - z) = \emptyset$ for any $i \neq j$ and $1 \leq i, j \leq 4$. Since $|D_2(G)| = 1$, at least 3 of $N(y_i) - z$ has at least 2 vertices. We may assume that $|N(y_1) - z| \geq 1$ and $|N(y_i) - z| \geq 2$ for $i = 2$, 3, or 4. Let $S = \cup_{i=1}^{4}(N(y_i) - z)$. Then $|S| \geq 7$. Since $\{z\} \cup N(z) \cup S \subseteq V(G)$, $12 = |V(G)| \geq 1 + 4 + 7 = 12$. Thus, $\Delta(G) = 4$, $|S| = 7$ and G has only one vertex of degree 4 which is adjacent to the vertex in $D_2(G)$, which is y_1. Furthermore, every vertex in S has degree 3. Let $G_S = G[S]$. Then $d_{G_S}(v) \geq 2$ for any $v \in S$. Since G has no 3- and 4-cycles, $G_S = C_7$. Thus, G must be isomorphic to the graph in Figure 2.4, which has a hamiltonian cycle: $zy_1s_1s_2y_2s_5s_4s_3y_3s_6s_7y_4z$, contrary to the fact that $G \notin \mathcal{SL}$. □

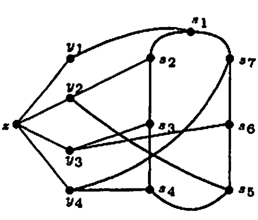


Figure 2.4

Define $\mathcal{ST}$ as the set of graphs H with the property that $\delta(H) = 2$ and for any two vertices $u, v \in D_2(H)$, H has a spanning (u, v)-trail. Note that $\{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\} \subset \mathcal{ST}$.

Lemma 2.4. Let $H \in \mathcal{ST}$. Let G_1 be a supereulerian graph with a vertex v of degree 2 or 3. Let G be a graph obtained from G_1 by replacing the vertex v by H with the edges incident with v joining different vertices in $D_2(H)$. Then $G \in \mathcal{SL}$.
Proof. Since $G_1 \in \mathcal{SL}$, G_1 has a spanning closed trail $T : ve_1v_1 \cdots v_ke_kv$ with e_1 and e_k incident with v in G_1. Since each edge incident with v in G_1 is incident with a vertex in $D_2(H)$ in G, let x_1 and x_2 be the two vertices in $D_2(H)$ incident with e_1 and e_k, respectively. Since H has a spanning (x_1, x_2)-trail, let T_H be a spanning (x_1, x_2)-trail in H. Therefore, $G[T \cup T_H]$ is a spanning closed trail in G. □

To avoid long and complicated case by case arguments, we will use the computer search results obtained by David Pike in [23]. David Pike [23] found all the Non-Hamiltonian Cubic 2-edge-connected graphs of order up to 16: 1 graph of order 10 (the Petersen graph), 1 graph of order 12 (which contains a K_3), 6 graphs of order 14 (only P_{14} is K_3-free), and 33 graphs of order 16. The completed list of those graphs can be found in [23]. We are only interested in reduced cubic 2-edge-connected graphs. After excluded non-reduced graphs from the list, we have the following:
Theorem F (Pike [23]). If G is a cubic 2-edge-connected Non-Hamiltonian reduced graph of order at most 16, then G can be contracted to the Petersen graph. Further more,

(a) If G is not the Petersen graph, G has girth 4;
(b) If $|V(G)| \le 12$, then G is the Petersen graph;
(c) If $|V(G)| = 14$, then G is the graph P_{14}.

3. Applications of Theorem 1.1

It was proved in [16] that if G is a 3-edge-connected simple graph with $\alpha(G) \le 4$, then either $G \in \mathcal{SL}$ or $G' = P$. We show that this is still true for $\alpha(G) \le 5$. For maximum edge independence number $\alpha'(G)$, it was proved in [18] that for a 3-edge-connected simple graph G with $\alpha'(G) \le 5$, either $G \in \mathcal{SL}$ or $G' = P$. Not knowing this result, Yan in [25] posted it as a conjecture. Theorem 1.2 is an improvement of these results.

Proof of Theorem 1.2. (a) By Theorem C(d), a connected reduced graph of order n with $\delta(G) \ge 3$ has $\alpha'(G) \ge (n+4)/3$. Thus, $(n+4)/3 \le 6$, and so $n \le 14$. By Theorem 1.1, either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$. But $\alpha'(P_{14}) = 7$. Theorem 1.2(a) is proved.

(b) By way of contradiction, suppose that $G \notin \mathcal{SL}$ and $G' \ne P$. By Theorem C(e) and $\alpha(G) \le 5$, $|V(G)| \le 4\alpha(G) - 5 \le 15$. Then by Theorem 1.1, $|V(G)| = 14$ or 15. If $|V(G)| = 14$, then $G = P_{14}$ and so $\alpha(G) = \alpha(P_{14}) = 6$, a contradiction.

If $|V(G)| = 15$, then by Theorem 1.1, G has girth at least 5 and $D_4(G)$ is an independent set with $|D_4(G)| = 3$. Let $v \in D_4(G)$. Let $N_G(v) = \{x_1, x_2, x_3, x_4\}$. Since $D_4(G)$ is independent, $D_4(G) \cap N_G(v) = \emptyset$, and so $d(x_i) = 3$ $(1 \le i \le 4)$. Let $N_G(x_i) - \{v\} = \{y_i^1, y_i^2\}$. Since G has girth at least 5, $(N_G(x_i) - \{x\}) \cap (N_G(x_j) - \{v\}) = \emptyset$ $(i \neq j)$. Let $S = \{y_1^1, y_1^2, y_2^1, y_2^2, y_3^1, y_3^2, y_4^1, y_4^2\}$. Since $|V(G)| = 15$, there are two vertices in $V(G) - (S \cup N_G(v) \cup \{v\})$. Let $V(G) - (S \cup N_G(v) \cup \{v\}) = \{z_1, z_2\}$. If $\{z_1, z_2\} \subset D_4(G)$, then $z_1 z_2 \notin E(G)$, and so $\{z_1, z_2, x_1, x_2, x_3, x_4\}$ is an independent set in G, a contradiction. Thus, at least one of $\{z_1, z_2\}$ (say z_1) has degree 3. Then there is a vertex in $\{x_1, x_2, x_3, x_4\}$ (say x_1) such that z_1 is not adjacent to any vertices in $N_G(x_1)$. Therefore, $\{z_1, y_1^1, y_1^2, x_2, x_3, x_4\}$ is an independent set in G, contrary to $\alpha(G) \le 5$. □.

The following theorem was proved by Catlin [6] and Chen [15]:

Theorem G (Chen [15]). Let $p \ge 2$ be an integer. Let G be a 2-edge-connected simple graph of order n with girth g, where $g \in \{3, 4\}$. Let G' be the reduction of G. If $n \ge 4(g-2)p^2$ and

$$d(u) + d(v) > \frac{2}{g-2}\left(\frac{n}{p} - 4 + g\right) \text{ for any } uv \notin E(G), \tag{3}$$

then either $G \in \mathcal{CL}$, or $G' \neq K_1$ is a graph of order less than p. In particular, either $G \in \mathcal{SL}$ or $G' \notin \mathcal{SL}$ with order less than p.

The Dirac degree condition below implies the degree-sum condition (3).

$$\delta(G) > \frac{1}{g-2}\left(\frac{n}{p} - 4 + g\right). \tag{4}$$

The case $p = 5$ in (4) was conjecture by Bauer [1], of which the case $g = 3$ was proved by Catlin [4], and the case $g = 4$ was proved by Lai [20].

The case $p = 2$ with $g = 3$ in (3) was proved by Lesniak-Foster and Williamson [21]. The case $p = 5$ with $g = 3$ in (3) was proved by Catlin [6], which was conjectured by Benhocine et al. [2].

For 3-edge-connected graphs, Chen [14] proved the case $p = 10$ with $g = 3$ in (4), Catlin [6] proved the case $p = 10$ with $g = 3$ in (3), and Chen [15] proved the case $p = 11$ in (3). Li et al [22] proved the following:

Theorem H (Li et al. [22]). Let G be a 3-edge-connected graph of order n. Then each of the following holds:

(a) If $\delta(G) \ge \frac{n-13}{12}$ and $n \ge 61$, then either $G \in \mathcal{SL}$ or $G' = P$.
(b) If G is K_3-free, $\delta(G) \ge \frac{n-25}{24}$ and $n \ge 97$, then either $G \in \mathcal{SL}$ or $G' = P$.

Theorem 1.3 is an improvement of Theorem H.

Proof of Theorem 1.3. Theorem 1.3 is the case $p = 15$ in Theorem G.

Let G' be the reduction of G. Suppose that G is not collapsible and so $G' \neq K_1$. Since $\kappa'(G') \geq \kappa'(G) \geq 3$, by Theorem G, $|V(G')| < 15$. By Theorem 1.1, either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$. Since P_{14} can be contracted to the Petersen graph by contracting the $K_{2,3}$ into a single vertex, either $G \in \mathcal{SL}$ or G can be contracted to P. Theorem 1.3 is proved. □.

Corollary 3.3. Let G be a 3-edge-connected graph of order n. Then when n is large, each of the following holds:

(a) If $\delta(G) > \frac{n}{15} - 1$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$.

(b) If G is K_3-free and $\delta(G) > \frac{n}{30}$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$.

Remark: Let G be a graph obtained from P_{16} by replacing each vertex by a $K_{n/16}$. Then G is a 3-edge-connected graph of order n with $\delta(G) \geq n/16-1$. However the reduction of G is P_{16}. Thus, the degree condition in Corollary 3.3 cannot be reduced to $\delta(G) \geq n/16 - 1$. But if we relax the conclusions of Theorem 1.3 and Corollary 3.3 from "the reduction of G is in $\{P, P_{14}\}$" to "G can be contracted to P", we have the following conjecture:

Conjecture 2 (Catlin et al [13]). Let G be a 3-edge-connected graph of order n with girth $g \in \{3, 4\}$. If $d(u) + d(v) > \frac{2}{g-2}\left(\frac{n}{18} - 4 + g\right)$ for any $uv \notin E(G)$ and n is large, then either $G \in \mathcal{SL}$ or G can be contracted to P.

Let G be a graph obtained from a Blanuša snark by replacing each vertex by a $K_{n/18}$ or $K_{n/36,n/36}$. Then $\delta(G) = \frac{2}{g-2}\left(\frac{n}{18} - 4 + g\right)$. But the reduction of G is the Blanuša snark and cannot be contracted to the Petersen graph. Thus, the degree condition in Conjecture 2 is the best possible.

4. An Associated Result

Theorem 4.1. Let G be a 3-edge-connected graph with $|V(G)| \leq 15$. Let G' be the reduction of G. Then one of the following holds:

(a) $G \in \mathcal{SL}$ or

(b) $G' \in \{P, P_{14}\}$, or

(c) G' is a graph satisfying each of the following:

  (i) G' is 2-connected, 3-edge-connected and essentially 4-edge-connected;

  (ii) G' has girth at least 5;

  (iii) $D_4^*(G') = \{v \in V(G') \mid d_{G'}(v) \geq 4\}$ is an independent set;

  (iv) $\Delta(G') \leq \lfloor \frac{|V(G')|-1}{3} \rfloor$.

Proof. By way of contradiction, suppose that

$$G \text{ is a counterexample to (a) and (b) with } |E(G)| \text{ minimized.} \quad (5)$$

Since the reduction method preserves the edge-connectivity, by Theorem B and (5), we may assume that $G = G'$ and $12 \leq |V(G)| \leq 15$.

Claim 1. G is 2-connected.

Suppose that G has a vertex cut v. Let H_1 and H_2 be the two components of $G - v$. Let $G_i = G[H_i \cup v]$ $(i = 1, 2)$. Since $\kappa'(G) \geq 3$, G_i is 3-edge-connected. We may assume that $|V(G_1)| \leq |V(G_2)|$. Then $|V(G_1)| \leq (|V(G)| + 1)/2 \leq 8$. Thus, by Theorem B, $G_1 \in \mathcal{CL}$, contrary to the fact that G is a reduced graph. Claim 1 is proved.

Claim 2. G is essentially 4-edge-connected.

Suppose that G has an essential edge cut $X \subseteq E(G)$ with $|X| = 3$. Let G_1 and G_2 be the two components of $G - X$ with $|V(G_1)| \leq |V(G_2)|$. Then $|V(G_1)| \leq 7$.

Since $\kappa'(G) \geq 3$ and $|X| = 3$, $\kappa'(G_1) \geq 2$. If $|D_2(G_1)| \leq 2$, then by Theorem E(a), G_1 is not reduced, a contradiction. Thus, the three edges in X must be incident with three different vertices in G_1 respectively, and $|D_2(G_1)| = 3$. Since $\kappa'(G_1) \geq 2$, $|V(G_1)| \leq 7$ and $|D_2(G_1)| = 3$, by Corollary 2.1(c), $G_1 \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\}$. Therefore, either $|V(G_1)| = 5$, or $|V(G_1)| = 7$, and so $|V(G_2)| = |V(G)| - |V(G_1)| \leq 10$.

Let $G_0 = G/G_1$ with v_0 as the contraction image of G_1. Then G_0 is the graph obtained from G_2 and v_0 by joining the edges in X from G_2 to v_0, and so $|V(G_0)| \leq 11$. Since $\kappa'(G) \geq 3$, $\kappa'(G_0) \geq 3$. By Theorem B, either $G_0 \in \mathcal{SL}$ or $G_0' = P$. If $G_0 \in \mathcal{SL}$, then since $G_1 \in \{K_{2,3}, K_{1,3}(1,1,1), J'(1,1)\} \subseteq \mathcal{ST}$, by Lemma 2.4, $G \in \mathcal{SL}$, a contradiction. If $G_0' = P$, then $G_1 = K_{2,3}$ and so $G = P_{14}$, a contradiction. Claim 2 is proved.

Claim 3. G has no 4-cycle.

By way of contradiction, suppose G has a 4-cycle, say $H_0 = uvzwu$. Let G/π be the graph defined by the π-reduction method with $e_\pi = xy$ as the new edge in G/π. Since $\kappa'(G) \geq 3$ and $12 \leq |V(G)| \leq 15$, by the definition of G/π, $\delta(G/\pi) \geq 3$, $\kappa'(G/\pi) \geq 1$ and

$$10 \leq |V(G/\pi)| = |V(G)| - 2 \leq 13.$$

Case 1. $\kappa'(G/\pi) \geq 3$. By (5), since G is a minimum counterexample and $|V(G/\pi)| < |V(G)|$, either $G/\pi \in \mathcal{SL}$ or $(G/\pi)' = P$. Since $G \notin \mathcal{SL}$, by Theorem D, $G/\pi \notin \mathcal{SL}$. Thus, the reduction of G/π is P.

Case 1A. $|V(G/\pi)| = 10$. Then $G/\pi = P$. By Lemma 2.2 $G \in \mathcal{SL}$, a contradiction.

Case 1B. $|V(G/\pi)| = 11$. Then since the reduction of G/π is P, G/π contains a cycle C_2 of length 2. Since G is K_3-free, the C_2 in G/π is formed by the π-reduction operation on G. By the definition of G/π and $\kappa'(G) \geq 3$, G must be one of the two graphs in Figure 4.2. Both are supereulerian (Ψ_1 has a spanning closed trail $x_1x_2u_1uvx_5x_3x_6x_8wzu_1x_4x_7x_1$, and Ψ_2 has a hamiltonian cycle $x_1x_2zvx_5x_3x_6x_8u_1wux_4x_7x_1$), a contradiction again.

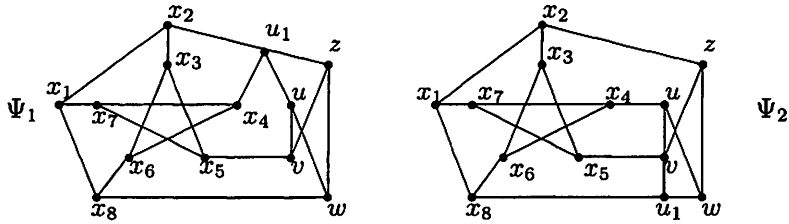
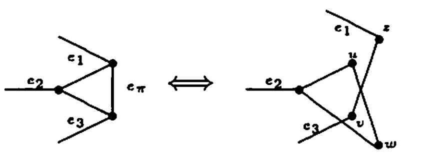


Figure 4.2

Case 1C. $|V(G/\pi)| = 12$. Since the reduction of G/π is P, G/π either contains a K_3 or two C_2 such that $(G/\pi)/K_3 = P$ or $(G/\pi)/(C_2 \cup C_2) = P$.

Case 1C(i). $(G/\pi)/K_3 = P$. If $e_\pi \in E(K_3)$, then since G is K_3-free with $\delta(G) \geq 3$, G is a graph with the structure as shown in Figure 4.3, which has an essential 3-edge-cut, contrary to the fact that G is essentially 4-edge-connected.



$\{e_1, e_2, e_3\}$ is
an essential 3 edge-cut

Figure 4.3

If $e_\pi \notin E(K_3)$, then G/π and G are the graphs as shown in Figure 4.4 below. Graph G in Figure 4.4 has a hamiltonian cycle: $C = x_1x_2x_3x_5x_{10}x_9x_6x_7uvx_4x_8wzx_1$, contrary to (5).

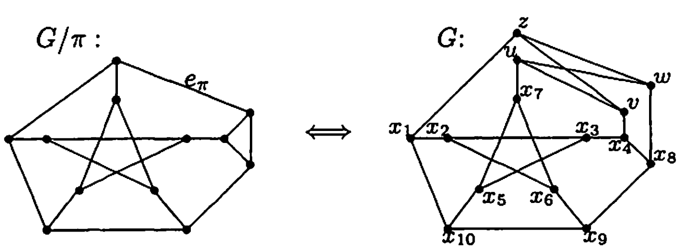


Figure 4.4

Case 1C(ii). $(G/\pi)/(C_2 \cup C_2) = P$. Since G is reduced, by Theorem D(c), G/π has no Θ as a subgraph. The two C_2 cycles must be incident with edge e_π in G/π as shown in Figure 4.5. By the definition of G/π, graph G is isomorphic to the graph shown in Figure 4.5.

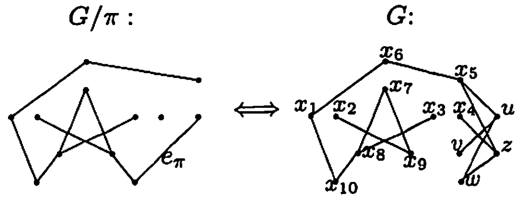


Figure 4.5

The subgraph $H = x_1x_2x_3x_8x_{10}wzx_4ux_5zvx_9x_7x_6x_1$ is a spanning Eulerian subgraph in G in Figure 4.5, a contradiction.

Case 1D. $|V(G/\pi)| = 13$. Since G is reduced, the reduction of G/π is P and G/π has no Θ as a subgraph, G/π either contains a K_3 and a C_2 such that $(G/\pi)/(K_3 \cup C_2) = P$ or contains a collapsible subgraph H of order 4 such that $(G/\pi)/H = P$.

Case 1D(i). G/π contains a K_3 and a C_2 such that $(G/\pi)/(K_3 \cup C_2) = P$.
Since G is reduced, K_3 and C_2 are generated by the π-reduction on G. Since $\delta(G) \geq 3$ and P is 3-regular, G/π has one of the two configurations shown in Figure 4.6. Thus, G is one of the graphs G_a and G_b in Figure 4.6.

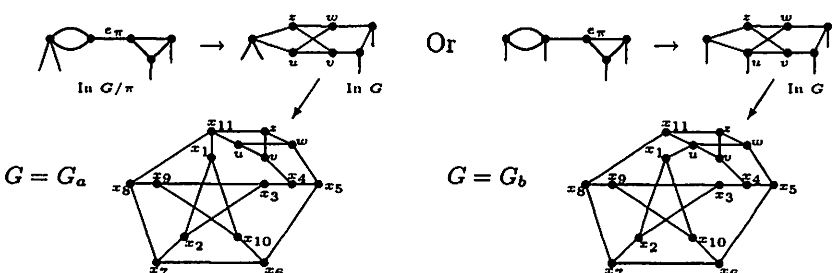


Figure 4.6

G_a has a spanning closed trail: $x_{11}x_1x_2x_3x_9x_{10}x_6x_7x_8x_{11}uvx_4x_5vzx_{11}$; and G_b has a hammiltonian cycle: $x_{11}x_8x_9x_3x_2x_7x_6x_{10}x_1uvx_4x_5wzx_{11}$. Thus, $G \in \{G_a, G_b\} \subset \mathcal{SL}$, a contradiction. Case 1D(i) is proved.

Case 1D(ii). G/π contains a collapsible subgraph H of order 4 such that $(G/\pi)/H = P$.

Let v_0 be the contraction image of H in P. There are exactly three edges incident with H. By Claim 2 above, G is essentially 4-edge-connected and so $e_\pi \notin E(H)$ and e_π is an edge in $E(P)$ incident with v_0. Let $E(v_0) = \{e_\pi, e_a, e_b\}$ be the set of three edges in P incident with v_0, and so the edges in $E(v_0)$ are the only three edges joining H to $(G/\pi)-H$. Since $|V(H)| = 4$, at least one vertex in $V(H)$ is not incident with any edges in $E(v_0)$.

Let $V(H) = \{u_0, u_1, u_2, x\}$ where x is incident with e_π and u_0 is not incident with any edges in $E(v_0)$. Since $\delta(G/\pi) \geq 3$, either $\{u_1, u_2, x\} \subseteq N_H(u_0)$, or u_0 is adjacent to only one of the vertices in $\{u_1, u_2\}$ (say u_2) and two parallel edges joining u_0 and x (see Figure 4.7 (II)).

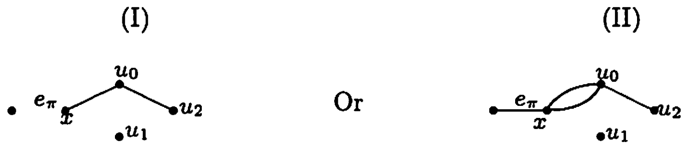


Figure 4.7

Subcase 1D(ii)(I). $N_H(u_0) = \{u_1, u_2, x\}$ (see Figure 4.7 (I)).

Since G is K_3-free, $u_1u_2 \notin E(G)$. Therefore, since G is essentially 4-edge-connected and $\delta(G/\pi) \geq 3$, u_i $(i = 1, 2)$ must be adjacent to x and incident with an edge in $E(v_0)$ as shown in Figure 4.8 below.

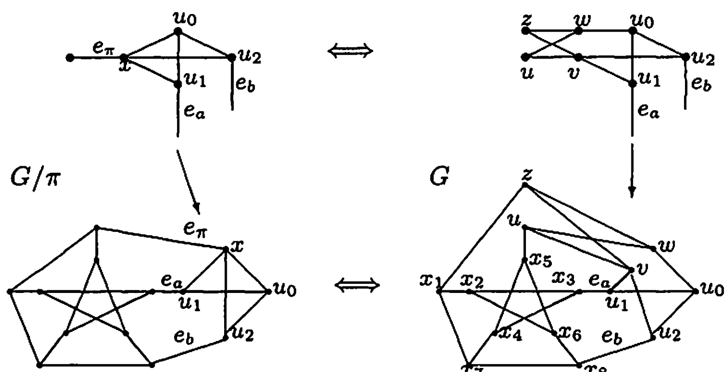


Figure 4.8

Thus G is the graph in Figure 4.8 which has a spanning closed trail $H_1 = x_1x_2x_6x_5uvu_1x_3x_4x_7x_8u_2u_0wzx_1$, contrary to (5).

Subcase 1D(ii)(II). $|N_H(u_0) \cap \{u_1, u_2\}| = 1$, and u_0 and x are joined by a pair of edges.

Since $|N_H(u_0) \cap \{u_1, u_2\}| = 1$, u_0 is adjacent to only one vertex in $\{u_1, u_2\}$, say u_2 as shown on Figure 4.7 (II). Since G is K_3-free, $u_2x \notin E(H)$. Since $d(u_i) \geq 3$ and u_1 cannot be adjacent to both u_0 and x, u_1 must be adjacent to x and u_2 (see Figure 4.9). But G has a Hamiltonian cycle: $x_1x_2x_6x_5uvu_1x_3x_4x_7x_8u_2u_0wzx_1$, a contradiction again.

This completes the proof of Claim 3 for the case $\kappa'(G/\pi) \geq 3$.

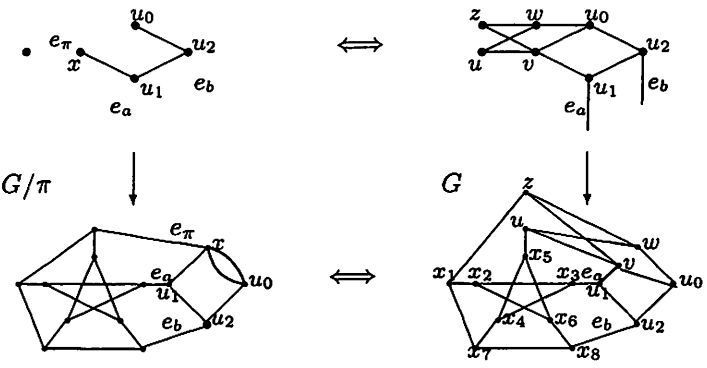


Figure 4.9

Case 2. $\kappa'(G/\pi) = 2$.

Since $\kappa'(G) \geq 3$, $e_\pi = xy$ must be in any edge-cuts of size 2 in G/π, where x is the vertex obtained from G by identifying u and z and y is the vertex obtained by identifying v and w. Let $X = \{e_\pi, e_0\}$ be an edge cut of size 2 in G/π where $e_0 = ab$. Let $X_1 = E(H_0) \cup \{e_0\}$. Therefore, $G - X_1$ has two components, say G_1 and G_2. We may assume that $u, z, a \in V(G_1)$ and $v, w, b \in V(G_2)$ and $|V(G_1)| \leq |V(G_2)|$.

Claim 2A. $d_{G/\pi}(y) \geq 4$.

By way of contradiction, suppose that $d_{G/\pi}(y) \leq 3$. Then e_0 with the edges other than e_π incident with y forms an essential edge-cut with size at most 3, contrary to Claim 2 that G is essentially 4-edge-connected. Thus, $d_{G/\pi}(y) \geq 4$. Claim 2A is proved.

Subcase 1. Every edge in G_1 is incident with a vertex in $\{u, z\} \cup \{a\}$.

Since G is K_3-free and $\delta(G) \geq 3$, either G_1 has a $K_{2,3}$ subgraph, or $G_1 = K_{1,2}$ with $V(G_1) = \{u, z, a\}$ where $d_{G_1}(a) = 2$.

If G_1 has a $K_{2,3}$ subgraph, then G has a $K_{3,3} - e$ subgraph, contrary to the fact that G is reduced.

If $G_1 = K_{1,2}$, then G/π has a C_2 with $V(C_2) = \{x, a\}$. Let $G_0 = (G/\pi)/C_2$. Let v_c be the contraction image of C_2 in G_0. Then $d_{G_0}(v_c) = 2$ with $v_c y$, $v_c b \in E(G_0)$. Note that $d_{G_0}(y) = d_{G/\pi}(y) \geq 4$. Furthermore, G_0 is essentially 3-edge-connected. Otherwise, if G_0 has an essential edge-cut X_0 with $|X_0| \leq 2$, then either X_0 is an essential edge-cut of G if $e_\pi \notin X_0$, or $(X_0 - e_\pi) \cup \{au, az\}$ if $e_\pi \in X_0$ is an essential edge-cut of G, contrary to the fact that G is essentially 4-edge-connected. Since G_0 is essentially 3-edge-connected with $|V(G_0)| = 12$ and $|D_2(G_0)| = 1$, by Corollary 2.3(b), either $G_0 \in \mathcal{SL}$ or $G_0' = P$.

If $G_0 \in \mathcal{SL}$, then by Theorem A, $G/\pi \in \mathcal{SL}$. By Theorem D, $G \in \mathcal{SL}$, a contradiction.

If $G'_0 = P$, since $|D_2(G_0)| = 1$ and $d_{G_0}(y) \geq 4$, G_0 has a collapsible subgraph H_0 that contains y and has $|V(H_0)| \geq 4$ such that $G'_0 = (G_0/H_0)' = P$. Then $10 = |V(G'_0)| \leq |V(G_0/H_0)| \leq 9$, a contradiction.

Subcase 2. G_1 has an edge x_1x_2 that is not incident with any vertices of $\{u, z\} \cup \{a\}$.

Since G is K_3-free, $N_G(x_1) \cap N_G(x_2) = \emptyset$ and $N_G(x_1) \cup N_G(x_2) \subseteq V(G_1)$. We have

$$|V(G_1)| \geq |N_G(x_1)| + |N_G(x_2)| \geq d(x_1) + d(x_2) \geq 3 + 3 = 6. \qquad (6)$$

Since $|V(G)| \leq 15$ and $|V(G_1)| \leq |V(G_2)|$, by (6),

$$6 \leq |V(G_1)| \leq 7 \text{ and } |V(G_2)| \leq 9.$$

Let H_1 and H_2 be the two components of $G/\pi - X$. Then $|V(H_1)| = |V(G_1)| - 1 = 6$ and $|V(H_2)| = |V(G_2)| - 1 \leq 8$, and $D_1(H_i) \cup D_2(H_i) \subseteq \{x, y, a, b\}$ $(i = 1, 2)$.

If G/π is simple, then since $\kappa'(G) \geq 3$, $D_1(H_i) = \emptyset$, and $\kappa'(H_i) \geq 2$ $(i = 1, 2)$ and $|D_2(H_i)| \leq 2$. By Corollary 2.1 and $\kappa'(H_i) \geq 2$, $H_i \in \mathcal{CL}$. Therefore, $(G/\pi)/(H_1 \cup H_2) = C_2$ and so $G/\pi \in \mathcal{CL}$. By Theorem D, $G \in \mathcal{CL}$, a contradiction.

If G/π is not simple, then G/π contains C_2 cycles formed by the π-reduction operation on G. Since reduced G has no $K_{3,3} - e$, G/π has no Θ as a subgraph. Thus, all the C_2 cycles are incident with only one end of the vertices of $e_\pi = xy$.

We may assume that all the C_2 cycles incident with x. (The case that all the C_2's incident with y can be proved in the same way and so omitted). Let H_c be the maximal collapsible subgraph in H_1 containing all the C_2 cycles. Let $H_1^c = H_1/H_c$. Let v_c be the contraction image of H_c. We regard $v_c = x$ and $v_c y = xy$. Then $|V(H_1^c)| \leq |V(H_1)| - 1 \leq 5$ and $d_{H_1^c}(v_c) \geq 2$. Thus, $D_1(H_1^c) \cup D_2(H_1^c) \subseteq \{v_c, a\}$. If v_c is a vertex of degree 2 in H_1^c, then let $N(v_c) = \{v_0, y\}$. Then H_1^c has a nontrivial 2-edge-connected subgraph H_1^* with at most two vertices of degree 2 and with $\{v_0, a\} \subseteq V(H_1^*)$. By Corollary 2.1, H_1^c and H_2 are collapsible. Therefore, $(G/\pi)/(H_1^c \cup H_2) = K_3$, and so $G/\pi \in \mathcal{CL}$. By Theorem D, $G \in \mathcal{CL}$, a contradiction. This proved the case $\kappa'(G/\pi) = 2$.

Case 3. $\kappa'(G/\pi) = 1$. Since $\delta(G/\pi) \geq 3$, by Lemma 2.2, G is not reduced, a contradiction.

This completes the proof of Claim 3, and so Theorem 4.1(c)(ii) holds.

Claim 4. $D_4^*(G)$ is an independent set.

By way of contradiction, suppose that G has an edge $e = ab$ with

$d(a) \geq 4$ and $d(b) \geq 4$. By Claim 2, G is essentially 4-edge-connected. Then since G is reduced, $G_e = G - e$ is a 3-edge-connected reduced graph with at most 15 vertices. Since G is the minimum counterexample, either $G_e \in \mathcal{SL}$ or $G_e \in \{P, P_{14}\}$.

If $G_e \in \mathcal{SL}$, then $G \in \mathcal{SL}$, a contradiction.

If $G_e = P_{14}$, then $G = P_{14} + e$ has girth at most 4, contrary to Claim 3.

If $G_e = P$, then $G = P + e \neq P$. By Theorem B, $G \in \mathcal{CL}$, a contradiction again. Claim 4 and Theorem 4.1(c)(iii) are proved.

Claim 5. $\Delta(G) \leq \lfloor \frac{|V(G)|-1}{3} \rfloor$.

Let $\Delta(G) = t$. Let v be a vertex with degree $d(v) = t$. Let $N(v) = \{x_1, x_2, x_3, \cdots, x_t\}$. Since G has no 3- and 4-cycles, $(N(x_i) - v) \cap (N(x_j) - v) = \emptyset$. Since $\delta(G) \geq 3$, $|N(x_i)| = d(x_i) \geq 3$ and so

$$|V(G)| \geq 1 + t + \sum_{i=1}^{t}(|N(x_i)| - 1) \geq 1 + t + 2t = 1 + 3t.$$

Since $\Delta(G) = t$ is an integer, $\Delta(G) \leq \lfloor \frac{|V(G)|-1}{3} \rfloor$. Claim 5 is proved. □

5. Proof of Theorem 1.1

Lemma 5.1. Let G be a 2-connected simple graph with $V(G) = D_3(G) \cup D_4(G)$. Let $D_4(G) = \{v_1, \cdots, v_s\}$. Let G_1 be a graph obtained from G by splitting each vertex v_i in $D_4(G)$ into two vertices v_i^1 and v_i^2 joint by an edge e_i (see Figure 5.1) such that G_1 is a 3-regular graph with $V(G_1) = (V(G) - D_4(G)) \cup_{i=1}^{s} \{v_i^1, v_i^2\}$ and $E(G_1) = E(G) \cup \{e_1, e_2, \cdots, e_s\}$. Then

(a) G_1 is 2-connected with order $|V(G)| + |D_4(G)|$ and has the girth greater or equal to the girth of G;

(b) if G_1 is hamiltonian, then G is supereulerian.

Splitting operation

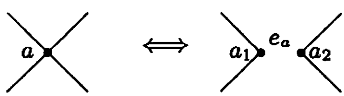


Figure 5.1

Proof. Lemma 5.1(a) follows from the definitions.

For (b), suppose that G_1 is hamiltonian. Let H_0 be a hamiltonian cycle in G_1. Let $E_0 = E(H_0) \cap \{e_1, e_2, \cdots, e_s\}$, then $G = G_1/\{e_1, e_2, \cdots, e_s\}$ has a spanning trail $H = H_0/E_0$. Lemma 5.1(b) is proved. □

Proof of Theorem 1.1(a). Let G' be the reduction of G. By way of contradiction, suppose that G is a smallest counterexample. Then $G = G'$,

$$G \notin \mathcal{SL} \text{ and } G \neq P. \tag{7}$$

Therefore, by Theorem 4.1(c), G is a reduced, 2-connected, 3-edge-connected and essentially 4-edge-connected graph with girth at least 5. By Theorem B, we only need to consider graph G with $12 \le |V(G)| \le 13$.

Case 1. $|V(G)| = 12$. By Theorem 4.1(c)(iv), $3 \le \Delta(G) \le \lfloor \frac{|V(G)|-1}{3} \rfloor$. Then $\Delta(G) = 3$. Thus, G is a cubic 2-edge-connected Non-Hamiltonian graph of order 12 with girth at least 5, contrary to Theorem F(a).

Case 2. $|V(G)| = 13$. Since $\delta(G) \ge 3$, $\Delta(G) = 4$. Thus, $V(G) = D_3(G) \cup D_4(G)$ and $|D_4(G)|$ must be an odd number. If $|D_4(G)| \ge 5$, then

$$|E(G)| = \frac{4|D_4(G)| + 3|D_3(G)|}{2} \ge \frac{5 \times 4 + 8 \times 3}{2} = 22$$

and so $F(G) = 2|V(G)| - |E(G)| - 2 \le 26 - 24 = 2$. By Theorem C(c) and $G \ne K_1$, $G \in \{K_2, K_{2,t}\}$, contrary to $\kappa'(G) \ge 3$. Thus, $|D_4(G)| = 1$ or 3.

Let G_1 be the graph obtained from G by splitting the vertices in $D_4(G)$ as defined in Lemma 5.1. G_1 is a cubic 2-connected graph of order 14 or 16. Since G has girth at least 5, G_1 has girth at least 5. By Theorem F(a), there is no 2-edge-connected cubic Non-Hamiltonian graph of order 14 or 16 with girth great than 4, G_1 must be hamiltonian. By Lemma 5.1, G is supereulerian, contrary to (7).

This completes the proof of Theorem 1,1(a).□

Let T_3 be a path of length 3.

Corollary 5.2. Let G be a connected simple graph with $|V(G)| \le 13$ and $\delta(G) \ge 3$. Then $G' \in \{K_1,\ K_2,\ K_{1,2}, K_{1,3},\ T_3,\ P\}$.

Proof. By Theorem 1.1(a), if $\kappa'(G) \ge 3$, then Corollary 5.2 follows. Thus we may assume that $\kappa'(G) \le 2$. Let $X \subseteq E(G)$ be an edge cut of G with $|X| \le 2$. Let G_1 and G_2 be the two components of $G - X$ with $|V(G_1)| \le |V(G_2)|$. Since $\delta(G) \ge 3$, it follows that $\delta(G_i) \ge 2$ $(i = 1, 2)$ and

$$4 \le |V(G_1)| \le 6 \text{ and } |V(G_2)| = 13 - |V(G_1)|. \tag{8}$$

Case 1. $\kappa'(G) = 1$. Then $|D_2(G_i)| \le 1$ $(i = 1, 2)$. If $|V(G_1)| = 6$, then by (8) $|V(G_2)| \le 7$. By Corollary 2.1(a) and (b), $G_1 \in \mathcal{CL}$ and the reduction of G_2 is in $\{K_1,\ K_2\}$. Hence the reduction of G is in $\{K_1,\ K_2,\ K_{1,2}\}$.

If $4 \le |V(G_1)| \le 5$, then by (8), $8 \le |V(G_2)| \le 9$. By Corollary 2.1(a), $G_1 \in \mathcal{CL}$. By Corollary 2.1(d) the reduction of G_2 is in $\{K_1,\ K_2,\ K_{1,2}\}$ and so the reduction of G is in $\{K_1,\ K_2,\ K_{1,2},\ K_{1,3},\ T_3\}$.

Case 2. $\kappa'(G) = 2$. Then $|D_2(G_i)| \le 2$ and $\kappa'(G_i) \ge 2$ $(i = 1, 2)$. Since $|V(G_1)| \le 6$ with $|D_2(G_1)| \le 2$, by Corollary 2.1(a), $G_1 \in \mathcal{CL}$. Since $|V(G_2)| \le 9$ with $|D_2(G_2)| \le 2$, by Corollary 2.1(e), $G_2' \in \{K_1, K_{2,3}\}$.

If $G_2' = K_1$, then $G_2 \in \mathcal{CL}$. $G/(G_1 \cup G_2) = C_2 \in \mathcal{CL}$. Thus, $G' = K_1$.

If $G_2' = K_{2,3}$, then let $D_2(K_{2,3}) = \{u_1, u_2, u_3\}$. If $H(u_i)$ is a non-trivial preimage of u_i, then since $\delta(G) \geq 3$, $|V(H(u_i))| \geq 4$. Since $|V(G_2)| \leq 9$ and $G_2' = K_{2,3}$, G_2 has at most one nontrivial collapsible subgraph with at least 4 vertices. Thus, at least two vertices of degree 2 in $G_2' = K_{2,3}$ are trivial contractions. Let u_1 and u_2 be the two trivial contractions of G_2'. Since $|X| = 2$ and $\delta(G) \geq 3$, u_i must be incident with an edge in X ($i = 1, 2$), and u_3 has a non-trivial preimage $H(u_3)$. Therefore, $G/(G_1 \cup H(u_3)) = K_{3,3} - e$. By Corollary 2.1(a), $K_{3,3} - e$ is collapsible and so $G' = K_1$. □

Remark. Theorem 1.1(a) and Corollary 5.2 were first proved in [17] (without using the computer search results [23]). The proof outlined in [17] was long and complicated which involved checking on many cases and was never submitted for publication in journals. However, the result has been used by several authors [8, 9, 12]. Using that result (i.e., Theorem 1.1(a)), Catlin and Lai obtained the following:

Theorem I (Catlin and Lai [12]). If G is a 3-edge-connected graph with at most 10 edge-cuts of size 3, then either $G \in \mathcal{SL}$ or $G' = P$.

We will make use of Theorem I in the proof of Theorem 1.1(b) and (c).
Proof of Theorem 1.1(b) and (c). By Theorem 1.1(a), we only need to consider graphs G with $14 \leq |V(G)| \leq 15$. By way of contradiction, suppose that G is a counterexample with $|E(G)|$ minimized. Then G is reduced. By Theorem 4.1(c), G is a 2-connected and essentially 4-edge-connected reduced graph with girth at least 5, $\Delta(G) \leq 4$ and $D_4(G)$ is an independent set. Thus, $V(G) = D_3(G) \cup D_4(G)$.

Case 1. $|V(G)| = 14$ (Theorem 1.1(b)). Then $|D_4(G)|$ must be even.

If $|D_4(G)| = 0$, then by Theorem F, $G \in \{P, P_{14}\}$, a contradiction.

If $|D_4(G)| \geq 4$, then $|D_3(G)| \leq 10$. Since G is essentially 4-edge-connected, G is 3-edge-connected with at most 10 edge-cuts of size 3. By Theorem I, either $G \in \mathcal{SL}$ or $G' = P$, a contradiction. Thus, $|D_4(G)| = 2$.

Let G_1 be the graph obtained from G by splitting the two vertices in $D_4(G)$ as stated in Lemma 5.1. Then G_1 is a 2-connected cubic graph of order 16 with girth at least 5. By Theorem F, since $G_1 \neq P$ with girth at least 5, G_1 is hamiltonian. By Lemma 5.1, $G \in \mathcal{SL}$, a contradiction.

Case 2. $|V(G)| = 15$ (Theorem 1.1(c)). Then $|D_4(G)|$ must be odd. If $|D_4(G)| = 1$, let G_1 be the graph obtained from G by splitting the vertex in $D_4(G)$ as defined in Lemma 5.1. Then G_1 is a 2-edge-connected cubic graph of order 16 with girth at least 5. By Theorem F, all the cubic 2-edge-connected Non-Hamiltonian graphs of order 16 have girth at most 4. Thus, G_1 is a hamiltonian. By Lemma 5.1, $G \in \mathcal{SL}$, a contradiction.

If $|D_4(G)| \geq 5$, then $|D_3(G)| = |V(G)| - |D_4(G)| \leq 10$. Since G is essentially 4-edge-connected, G is 3-edge-connected with at most 10 edge-cuts of size 3. By Theorem I, either $G \in \mathcal{SL}$ or $G' = P$, a contradiction. Thus, $|D_4(G)| = 3$ and so G is the graph defined in Theorem 1.1(c). □

We conclude this paper with a conjecture that is a refinement of Conjecture 1:

Conjecture 3. Any 3-edge-connected simple graph of order at most 17 is either supereulerian or its reduction is in $\{P, P_{14}, P_{16}\}$.

References

[1] D. Bauer, A note on degree conditions for Hamiltonian cycles in line graphs, Congressus Numeratium 49 (1985) 11-18.

[2] A. Benhocine, L. Clark, N. Köhler and H. J. Veldman, On circuits and pancyclic line graphs, J. Graph Theory 10 (1986), 411 - 425.

[3] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications", American Elsevier, New York (1976).

[4] P. A. Catlin, A reduction method to find spanning eulerian subgraphs, J. Graph Theory 12 (1988) 29-44.

[5] P. A. Catlin, Supereulerian graphs, collapsible graphs, and four-cycles, Congressus Numerantium 58 (1987) 233-246.

[6] P. A. Catlin, Contractions of graphs with non spanning Eulerian subgraphs, Combinatorica 8 (4) 1988 313-321.

[7] P. A. Catlin, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989) 465–483.

[8] P. A. Catlin, Supereulerian Graphs: A Survey, J. Graph Theory 16 (1992) 177-196.

[9] P. A. Catlin, Double cycle covers and the Petersen graph II, Proceedings 21st Southeastern Conference. Boca Raton, 1990, Congr. Numer. 76 (1990), 173-181.

[10] P. A. Catlin and Z.-H. Chen, Nonsupereulerian graphs with large size, Proc. 2nd Intl. Conf. Graph Theory. San Francisco, (July 1989).

[11] P. A. Catlin, Z. Han, and H.-J. Lai, Graphs without spanning eulerian trails, Discrete Math. 160 (1996) 81-91.

[12] P. A. Catlin and H.-J. Lai, On supereulerian graphs and the Petersen Graph, J. of Combinatorial Theorey, Series B 66, 123–139(1996).

[13] P. A. Catlin, Z.-H. Chen, and H.-J. Lai, Personal communication, (1991).

[14] Z.-H. Chen, Supereulerian graphs and the Petersen graph, J. of Combinatorial Math. and Combinatorial Computing. Vol 9 (1991) 79-89.

[15] Z.-H. Chen, A degree condition for spanning Eulerian Subgraphs, J. of Graph Theory, 17 (1993), 5-21.

[16] Z.H. Chen, Supereulerian graphs, independent sets, and degree-sum conditions, Discrete Math.179 (1998) 73-87.

[17] Z.-H. Chen, Reductions of graphs and spanning eulerian subgraphs, Ph.D Dissertation, Wayne State University 1991.

[18] Z.-H. Chen and H.-J. Lai, Supereulerian graphs and the Petersen graph II, Ars Comtinatoria 48 (1998) PP. 271-282.

[19] Z.-H. Chen and H.-J. Lai, Collapsible graphs and Matchings, J. of Graph Theorey, Vol 17, No. 5, 597-605 (1993).

[20] H.-J. Lai, Contractions and Hamiltonian line graphs, J. Graph Theory, 12 (1988) 11-15.

[21] L. Lesniak-Foster and J. E. Williamson, On Spanning and dominating circuits in graphs, Canad. Math. Bull. 20 (1977), 215-220.

[22] X.M. Li, D.X. Li, H.-J. Lai, On 3-edge-connected supereulerian graphs in graph family $C(l, k)$, Discrete Mathematics, 310(2010) 2455-2459.

[23] David A, Pike, Snarks and Non-Hamiltonian Cubic 2-edge-connected graphs of small order, Journal of Combinatorial Mathematics and Combinational Computing, 23 (1997) 129–141.

[24] H. J. Veldman, On dominating and spanning circuits in graphs, Discrete Math., 124 (1994) 229-239.

[25] H.Y. Yan, H.-J. Lai, Supereulerian graphs and matchings, Applied Mathematics Letters 24(2011) 1867-1869.