On a Tiling Paradox

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Abstract

A cluster of $2n+1$ cubes comprising the central cube and reflections in all its faces is called the n-dimensional cube. If $2n+1$ is not a prime, then there are infinitely many tilings of $R^n$by crosses but it has been conjectured that there is a unique tiling of R^n by crosses otherwise. The conjecture has been proved for $n = 2, 3$, and in this paper we prove it also for $n = 5$. So there is a unique tiling of R^3 by crosses, there are infinitely many tilings of R^4, but for R^5 there is again only one tiling by crosses. We consider this result to be a paradox as our intuition suggests that "the higher the dimension of the space, the more freedom we get".

Tilings of R^n by unit cubes go back to 1907 when Minkowski conjectured [17] that each lattice tiling of R^n by unit cubes contains twins, a pair of cubes sharing a complete $n-1$ dimensional face. This conjecture was proved by Hajós [6] in 1942.

In 1930, when Minkowski's conjecture was still open, Keller [13] suggested that the lattice condition in the conjecture is redundant, that the nature of the problem is purely geometric, and not algebraic as assumed by Minkowski. Thus he conjectured that each tiling of R^n by unit cubes contains twins. It is trivial to see that each tiling of R^2 by unit cubes contains twins, and it is also easy to verify it for R^3. However, a proof that each tiling of R^n, $4 \leq n \leq 6$, contains twins takes in aggregate 80 pages, see [16]. There was no progress on Keller's conjecture for more than 50 years. Only in 1992 Lagarias and Shor [14] constructed a tiling of $R^n, n \geq 10$, by unit cubes with no twins. First they found such a tiling in R^{10}, which we consider a very surprising and remarkable result. However, once one has such a tiling in hand, it is relatively easy to find it for $R^n, n > 10$, as well. The second part supports our belief that "the higher the dimension of the space, the more freedom we get". Mackey [15] proved that the Keller's

conjecture is false for $n = 8, 9$ as well. As to the remaining value of $n = 7$, there are only some partial results, see [3].

Since late fifties tilings of R^n by different clusters of unit cubes have been considered, see e.g. [20] and [22], many of them related to perfect error-correction codes in Lee metric (also called Manhattan metric in Z^n). The Golomb-Welch conjecture [4] has been a main motivating power of the research in this area for the last forty years. A perfect e-error correcting Lee code over Z of block size n, denoted $PL(n, e)$, is a set $C \subset Z^n$ of codewords so that each word $A \in Z^n$ is at Lee distance at most e from exactly one codeword in C. Similarly, a perfect e-error correcting Lee code over Z_q of block size n, denoted $PL(n, q, e)$, is a set $C \subset Z_q^n$ of codewords so that each word $A \in Z_q^n$ is at Lee distance at most e from exactly one codeword in C.

Conjecture 1 *Golomb-Welch. For $n \geq 3$ and $e > 1$, there is no $PL(n, e)$ code.*

Clearly, the above conjecture, if true, implies that there is no $PL(n, q, e)$ code for $n \geq 3$, $e > 1$, and $q \geq 2e + 1$. For the state of the art on the conjecture we refer the reader to [10].

In this paper we focus on tilings by n-crosses. An n-dimensional cross comprises $2n + 1$ unit cubes: the "central" one and reflections in all its faces. A tiling $\mathcal{L}$ of R^n by crosses is called a Z-tiling if centers of all crosses in $\mathcal{L}$ have integer coordinates. Further, $\mathcal{L}$ is called a lattice tiling if centers of all crosses in $\mathcal{L}$ form a lattice. A regular (also called a face-to-face) tiling is a tiling that is congruent to a Z-tiling; otherwise the tiling is called non-regular. We recall that two tilings $\mathcal{T}$ and $\mathcal{S}$ of R^n are congruent if there exists a linear, distance preserving bijection of R^n which maps $\mathcal{T}$ on $\mathcal{S}$. It seems that Kárteszi [12] was the first to ask whether there exists a tiling of R^3 by crosses. Such a tiling was constructed by Freller in 1970; Korchmáros about the same time treated the case $n > 3$. Golomb and Welch showed the existence of these tilings in terms of error-correcting codes, see Section 3.5 in [20]. Immediately after the existence question has been answered, the enumeration of tilings has been studied. In [18] Molnár proved:

Theorem 2 *Molnar. The number of pair-wise non-congruent lattice Z-tilings of R^n by crosses equals the number of non-isomorphic Abelian groups of order $2n + 1$.*

Szabó [21] constructed a non-regular lattice tiling of R^n by crosses in the case when $2n + 1$ is not a prime. Using refinements of this construction it

was proved in [9] that in this case there are $2^{\aleph_0}$ non-congruent Z-tilings of R^n by crosses. In a strict contrast to this result it was proved there that, for $n = 2$, and $n = 3$, there is a unique, up to a congruence, tiling of R^n by crosses. It is conjectured in [9], see also [1]:

Conjecture 3 *If $2n + 1$ is a prime then there exists, up to a congruence, only one Z-tiling of R^n by crosses.*

It seems to us that the above conjecture, if true, would totally go against our intuition that suggests: the higher the dimension of the space, the more freedom we get; see also an above comment related to the Lagarias-Shor result on Keller's conjecture.

To provide supporting evidence for Conjecture 3 we prove in this paper:

Theorem 4 *There exists, up to a congruence, a unique Z-tiling of R^5 by crosses.*

We note that a sketch of a proof of the above statement has been given in [11]. However, the sketch is so short that it is impossible for the interested reader to reconstruct the whole proof from it. Therefore in this paper a complete version of the proof is provided. Although we proved Conjecture 3 only for $n = 5$, an essential part of the proof of Theorem 4 holds for all $n = 2 (\mathrm{mod}\, 3)$. We believe that this part will be helpful when proving this conjecture for some other values of n.

Clearly, if $\mathcal{L}$ is a Z-tiling of R^n by crosses, then centers of crosses in $\mathcal{L}$ form a $PL(n, 1)$ code. It is easy to check that the unique tiling of R^5 by crosses is 11-periodic. Thus, as an immediate consequence we get:

Corollary 5 *There is a $PL(5, q, 1)$ code if and only if $11|q$.*

As to the non-regular tilings of R^n by crosses, it was mentioned above that such a tiling exists if $2n + 1$ is not a prime. A result of Redei [19] implies that, if $2n+1$ is a prime, then there is no lattice non-regular tiling of R^n by crosses. It is easy to check that a non-regular tiling of R^2 by crosses does not exist. The same result for $n = 3$ has been proved in [5]. The following theorem has been proved in [11]:

Theorem 6 *Let $2n + 1$ be a prime. If there is a unique Z-tiling of R^n by crosses, then there is no non-regular tiling of R^n by crosses.*

Combining Theorem 4 with 6 we get:

Corollary 7 *There is a unique, up to a congruence, tiling of R^5 by crosses, and this tiling is a Z-tiling.*

Thus, there is a unique tiling of R^3 by crosses, there are $2^{\aleph_0}$ pair-wise non-congruent Z-tilings of R^4 by crosses, but for R^5 there is again a unique tiling by crosses.

Also, by means of Theorem 6, it is straightforward that Conjecture 3 is equivalent to

Conjecture 8 *If $2n+1$ is a prime then there exists, up to a congruence, a unique tiling of R^n by crosses, and this tiling is a lattice Z-tiling.*

In the next section we introduce needed notation, definitions and state some auxiliary results. Theorem 4 will be proved in Section 2.

1 Preliminaries

In this section we recall some notation, notions, and results which will turn out to be useful in proving the main result of the paper, Theorem 4.

Since the problem of tilings by crosses comes originally from the area of error-correcting codes we will stick to some of its terminology. Let $\mathcal{L}$ be a Z-tiling of R^n by crosses. We will denote by $\mathcal{T}_{\mathcal{L}} \subset Z^n$ the set of centers of crosses in $\mathcal{L}$. The elements of Z^n will be called words while the words in $\mathcal{T}_{\mathcal{L}}$ will be called codewords. We will also say that a codeword W covers a word V if $\rho_M(V,W) \leq 1$. As usual ρ_M stands for the Manhattan distance of $V = (v_1, ..., v_n)$ and $W = (w_1, ..., w_n)$ given by

$$\rho_M(V,W) = \sum_{i=1}^{n} |v_i - w_i| \,.$$

The weight $|V|_M$ of $V \in Z^n$ is given by $|V|_M := \sum_{i=1}^{n} v_i = \rho_M(V,O)$, where $O = (0, ..., 0)$. The following simple observation will be used several times:

Claim 9 *Let $\mathcal{L}$ be a tiling of R^n by crosses. Then permuting the order of coordinates of each codeword in $\mathcal{T}_{\mathcal{L}}$ and/or changing a sign of a coordinate*

for each codeword in $T_{\mathcal{L}}$ and/or adding a word $V \in R^n$ to each codeword results in a set T' which induces a tiling of R^n by crosses congruent to $\mathcal{L}$.

If $\mathcal{L}$ is a tiling of R^n by crosses then for each word V in Z^n there is a unique codeword W in $\mathcal{T}_{\mathcal{L}}$ so that $\rho_M(V, W) \leq 1$. Therefore $\mathcal{T}_{\mathcal{L}}$ can be also seen as a decomposition (tiling) of Z^n by Lee spheres $S_{n,1}$ of radius 1 centered at O, where $S_{n,1} = \{V \in Z^n, \rho_M(V, O) \leq 1\} = \{O\} \cup \{e_i, i = 1, ..., n\}$; and vice versa, each tiling of Z^n by spheres $S_{n,1}$ induces a tiling of R^n by crosses. As usual, $e_i = (0, ..., 0, 1, 0, ..., 0)$ where the i-th coordinate equal to 1.

In general, if S is a subset of R^n (Z^n), a tiling $\mathcal{L}$ of R^n (Z^n) by translations of S can be described in the form $\{S + \mathbf{u}, \mathbf{u} \in \mathcal{U}\}$, where $\mathbf{u}$ is a vector. Then $\mathcal{L}$ is a lattice tiling if $\mathcal{U}$ is a lattice. For the sake of simplicity we will abuse slightly the language and a subset $\mathcal{U}$ of R^n (Z^n) will be understood sometimes as a set of vectors with the obvious $U \in \mathcal{U}$ meaning that the vector $\mathbf{u} = U - O$ is in $\mathcal{U}$. The following theorem stated in [10] turns out to be useful when proving the main result of the paper.

Theorem 10 *Let S be a subset of Z^n. Then there is a lattice tiling of Z^n by translations of S if and only if there is an abelian group G of order $|S|$ and a homomorphism $\phi : Z^n \to G$ so that the restriction of ϕ to S is a bijection. In addition, if ϕ satisfies this condition, then the lattice tiling of Z^n by translations of S is given by $\{S + \mathbf{u}, \mathbf{u} \in \ker(\phi)\}$.*

As an immediate consequence we get the following.

Corollary 11 *Let $\phi : Z^n \to Z_{2n+1}$, the cyclic group of order $2n + 1$, be a homomorphism so that, for all $1 \leq i < j \leq n$, $\phi(e_i)$ is not an inverse element to $\phi(e_j)$, that is $\phi(e_i) \neq -\phi(e_j)$. Then $\{S_{n,1} + \mathbf{u}, \mathbf{u} \in \ker \phi\}$ is a lattice tiling of Z^n by $S_{n,1}$.*

We note that tiling of R^n by crosses given in [10] and other papers is a lattice tilling. Therefore these tilings can be seen as obtained by Corollary 11.

Let $\mathcal{L}$ be a collection of crosses that tile R^n. We will always assume wlog that the cross K_O centered at the origin belongs to $\mathcal{L}$. Then each cross $K \in \mathcal{L}$ can be seen as a translation of K_O by a vector $\mathbf{u}$. So $\mathcal{L} = \{K_O + \mathbf{u}, \mathbf{u} \in \mathcal{T}_{\mathcal{L}}\}$. For the sake of brevity we will use $K_{\mathbf{u}}$ for a cross centered at a point $U = O + \mathbf{u}$.

2 Proof of Theorem 4

Let $\mathcal{L}$ be a Z-tiling of R^n by crosses, and let $\mathcal{T}_{\mathcal{L}} \subset Z^n$ be the set of centers of crosses in $\mathcal{L}$. Since we will deal only with Z-tilings by crosses most of the time we will drop Z- and refer to $\mathcal{L}$ as a tiling of R^n by crosses. We use the terminology of coding theory; that is, the elements of Z^n will be called words and the elements of $\mathcal{T}_{\mathcal{L}}$ will be called codewords. In this section we provide a complete proof of Theorem 4.

As mentioned in the introduction Molnar [18] proved that the number of non-congruent lattice tilings of R^n by crosses equals the number of non-isomorphic abelian groups of order $2n+1$. As $2n+1$ is a prime for $n=5$, there is only one abelian group of order 11, and thus there is a unique, up to congruence, lattice tiling of R^5 by crosses. Thus, to prove the main result it suffices to show:

Theorem 12 *Let $\mathcal{L}$ be a tiling of R^5 by crosses. Then $\mathcal{L}$ is a lattice tiling.*

Let W be a codeword in $\mathcal{T}_{\mathcal{L}}$. Then $N_k(W)$, the k-neighborhood of W, will be the set of codewords V in $\mathcal{T}_{\mathcal{L}}$ at the distance at most k from W, that is, $N_k(W) = \{V \in \mathcal{T}_{\mathcal{L}},\ \rho_M(W,V) \leq k\}$. In the case of $W = O$, we will write N_k instead of $N_k(O)$. We will say that two k-neighborhoods $N_k(W)$ and $N_k(W')$ are equal if $\{V - W, V \in N_k(W)\} = \{V - W', V \in N_k(W')\}$; and we will say that $N_k(W)$ and $N_k(W')$ are congruent if there is a linear distance preserving transformation mapping $N_k(W)$ on $N_k(W')$. Clearly, for each codeword W, the neighborhoods $N_1(W)$, and $N_2(W)$ are empty sets.

The proof of Theorem 12 will be based on:

Theorem 13 *Let $\mathcal{L}$ be a tiling of R^5 by crosses. Then, for each codeword W in $\mathcal{T}_{\mathcal{L}}$, the neighborhood $N_3(W)$ and $N_3(O)$ are equal, and $N_3(O)$ is symmetric; that is, if $W \in N_3(O)$ then $-W \in N_3(O)$ as well.*

Now we show that the above theorem implies Theorem 12.

Proof. of Theorem 12. To show that $\mathcal{L}$ is a lattice tiling it suffices to prove that, for all codewords $W, Z \in \mathcal{T}_{\mathcal{L}}$, $W - Z \in \mathcal{T}_{\mathcal{L}}$ as well. As $\mathcal{L}$ is a tiling by crosses, it is not difficult to see that, for each codeword $Z \in \mathcal{T}_{\mathcal{L}}$, there is a sequence $Z_0 = O, Z_1, ..., Z_{m-1}, Z_m = Z$ of codewords in $\mathcal{T}_{\mathcal{L}}$ such that $\rho_M(Z_{i-1}, Z_i) = 3, i = 1, ..., m$. Then $Z_i \in N_3(Z_{i-1})$ and because, by

Theorem 13, the -neighborhoods $N_3(Z_{i-1})$ and $N_3(O)$ are equal, which in turn implies, again by Theorem 13, that $-U_i \in \mathcal{T}_{\mathcal{L}}$ as well for all $i = 1, ..., m$. Repeatedly applying Theorem 13 we get that $W - U_1, W - U_1 - U_2, ..., W - U_1 - U_2 - ... - U_m = W - (Z_1 - O) - (Z_2 - Z_1) - ... - (Z_{m-1} - Z_{m-2}) - (Z_m - Z_{m-1}) = W - Z_m = W - Z$ is a codeword. The proof of Theorem 12 is complete. ■

Hence, to prove the main result it suffices to prove Theorem 13. It turns out that in order to be able to do so one needs to look at "wider" neighbourhoods. In fact, to be able to prove Theorem 13 we will have to prove the same type of a theorem for 5-neighbourhoods. This recalls a situation when one wants to prove a statement P by using mathematical induction, but to be able to prove the inductive step a statement stronger than P has to be proved.

Theorem 14 *Let $\mathcal{L}$ be a tiling of R^5 by crosses. Then, for each W in $\mathcal{T}_{\mathcal{L}}$, the neighborhood $N_5(W)$ and $N_5(O)$ are equal, and $N_5(O)$ is symmetric.*

We will do it in four steps. To facilitate our discussion we introduce more notation and terminology. By a word of type $[m_1^{\alpha_1}, ..., m_s^{\alpha_s}]$ we mean a word having α_1 coordinates equal to $\pm m_1$, ..., α_s coordinates equal to $\pm m_s$, the other coordinates equal to 0. E.g., both words $(-2, -2, -1, -2, 0, 0)$ and $(1, 0, 2, 0, -2, 2)$ are of type $[2^3, 1^1]$. There are three types of words V with its weight$|V|_M = 3$; either V is of type $[3^1]$, or of type $[2^1, 1^1]$, or of type $[1^3]$. Let $Z \in N_k(W)$. Then Z will be called a codeword of a type with respect to W if $Z - W$ is of the given type; the number of codewords of type $[m_1^{\alpha_1}, ..., m_s^{\alpha_s}]$ in $N_k(W)$ will be denoted $|[m_1^{\alpha_1}, ..., m_s^{\alpha_s}]|_W$. If the codeword W will be clear from the context, we will drop the subscript W. Similarly, each word $V, |V|_M = 4$, is either of type $[4^1]$, or $[3^1, 1^1]$, or $[2^2]$, or $[2^1, 1^2]$, or $[1^4]$.

Now we are ready to describe the four phases of proving Theorem 13.

(A) Let $\mathcal{L}$ be a tiling of Z^n by crosses. First we prove a quantitative statement, which will be proved not only for $n = 5$ but for all $n = 2 (\mathrm{mod}\, 3)$. We believe that this statement might turn to be very useful when proving Conjecture 3 for other values of n, where $2n + 1$ is a prime. Let W be a codeword. The statement claims that the number of codewords of type $[m_1^{\alpha_1}, ..., m_s^{\alpha_s}]$, where $\sum_{i=1}^{s} \alpha_i m_i \leq 4$, with respect to W depends only on n and does not depend on $\mathcal{L}$.

Theorem 15 *Let $\mathcal{L}$ be a tiling of R^n by crosses where $n = 2(\bmod 3)$ and W be a codeword. Then the number of codewords of given type with respect to W is: $\left|[3^1]\right|_W = 0$, $\left|[2^1, 1^1]\right|_W = 2n$, and $\left|[1^3]\right|_W = \frac{2n(n-2)}{3}$. Further, $\left|[4^1]\right|_W = \left|[2^2]\right|_W = 0$, $\left|[3^1, 1^1]\right|_W = 2n$, $\left|[2^1, 1^2]\right|_W = 2n(n-2)$, and $\left|[1^4]\right|_W = \frac{n(n-2)(n-3)}{3}$.*

(B) We prove an analogue of Theorem 15 for the number of codewords of type $[m_1^{\alpha_1}, ..., m_s^{\alpha_s}]$, where $\sum_{i=1}^{s} \alpha_i m_i \leq 5$. However, we get the explicit values for the number of codewords of individual types only for $n = 5$, while for $n = 2(\bmod 3)$ we get those values only as a function of the number of codewords of type $\left[5^1\right]$. We point out, that this is not because the methods used are not satisfactory but for some values $n = 2(\bmod 3)$, say $n = 62$, there are two (lattice) tilings of Z^n by crosses with different number of codewords of type $\left[5^1\right]$. We stress that for $n = 62$, the number $2n+1 = 125$ is not a prime, hence it does not provide a counterexample to our conjecture.

(C) In this phase we prove that for any two codewords in $\mathcal{T}_{\mathcal{L}}$ their 5-neighborhoods are congruent.

(D) As the last step we show that for any two codewords in $\mathcal{T}_{\mathcal{L}}$ their 5-neighborhoods are not only congruent but the two 5-neighborhoods equal, and this joint neighborhood is symmetric, so we prove Theorem 14.

2.1 Phase A

In this subsection we prove Theorem 15. In fact we prove an extended version of the statement.

For any codeword W in $\mathcal{T}_{\mathcal{L}}$ there are $2n$ words V of type $\left[2^1\right]$ with respect to W. (We recall that this means that $V - W$ is of given type). Each of them is covered by a codeword of type $\left[3^1\right]$, or by a codeword of type $\left[2^1, 1^1\right]$, with respect to W. On the other hand, each codeword of type $\left[3^1\right]$ and of type $\left[2^1, 1^1\right]$, with respect to W, covers exactly one word of type $\left[2^1\right]$ with respect to W. Thus we get, for each codeword W,

$$\left|[3^1]\right| + \left|[2^1, 1^1]\right| = 2n \tag{1}$$

The above and the following equalities are valid for each codeword W, therefore in what follows we drop the index $_W$. Also we will not repeat any longer that all codewords of given type are meant with respect to W.

In Z^n there are $2^2\binom{n}{2}$ words V of type $[1^2]$. Each of them is covered either by a codeword of type $[1^3]$, or by a codeword of type $[2^1, 1^1]$. Further, each codeword of type $[1^3]$ covers three of them while a codeword of type $[2^1, 1^1]$ covers exactly one codeword of type $[1^2]$. Hence

$$\left|[2^1, 1^1]\right| + 3\left|[1^3]\right| = 4\binom{n}{2} \tag{2}$$

Equation (1) and (2) are "global" equations. To get their "local" form we need to introduce some more notation. Often we will need to express the number of words, or codewords, in a set $\mathcal{A}$ having their i-th coordinate positive, or their i-th coordinate negative. Therefore, to simplify the language, we will introduce the notion of the *signed coordinate* in Z^n. For the rest of the paper by the set of signed coordinates we will understand the set $I = \{+1, ..., +n, -1, ..., -n\}$. Let $V = (v_1, ..., v_n)$ be a word in Z^n. Then the signed coordinates V_i of V are given by: $V_i = |v_i|$ and $V_{-i} = 0$ for $v_i > 0$, $V_i = 0$ and $V_{-i} = |v_i|$ for $v_i < 0$, and $V_i = V_{-i} = 0$ for $v_i = 0$. E.g., if $V = (2, 0, -5)$ then $V_1 = 2, V_{-1} = 0, V_2 = V_{-2} = 0$, and$V_3 = 0$, $V_{-3} = 5$. For a signed coordinate $i \in I$, by $|\mathcal{A}_i|$ we will denote the number of words in $\mathcal{A}$ with a non-zero i-th coordinate. That is, $|\mathcal{A}_1|$ stands for the number of words in $\mathcal{A}$ with the first coordinate being a positive number, while $|\mathcal{A}_{-3}|$ represents the number of words in $\mathcal{A}$ with the third coordinate being a negative number. If we need to stress that the value of the i-th signed coordinate is m, we will use $\left|\mathcal{A}_i^{(m)}\right|$ for the number of words with the i-th coordinate equal m. Thus, for each $i \in I$, $\left|[2^1, 1^1]_i\right|$ is the number of words of type $[2^1, 1^1]$ with the i-th signed coordinate being non-zero, while $\left|[2^1, 1^1]_i^{(2)}\right|$ stands for the set of codewords of type $[2^1, 1^1]$ with the i-th signed coordinate equal to 2.

Now we are ready to state the local form of (1) and (2). As for each $i \in I$ there is in Z^n one word V of type $[2^1]$ with $V_i = 2$, and $2(n-1)$ words U of type $[1^2]$ with $U_i = 1$, we get:

$$\left|[3^1]_i\right| + \left|[2^1, 1^1]_i^{(2)}\right| = 1, \tag{3}$$

and

$$\left|[2^1, 1^1]_i\right| + 2\left|[1^3]_i\right| = 2(n-1). \tag{4}$$

Indeed, if A is a codeword of type $[3^1]$ with $A_i = 3$ (and then $A_j = 0$ for all $j \neq i, j \in I$) then A covers a word V of type $[2^1]$ with $V_i = 2$. However,

a codeword B of type $[2^1, 1^1]$ covers V only if $B_i = 2$, but does not cover it if $B_i = 1$. On the other hand, a codeword B with $B_i \neq 0$ covers one word D of type $[1^2]$ with $D_i = 1$ regardless whether $B_i = 2$ or $B_i = 1$. Clearly, a codeword C of type $[1^3]$ with $C_i = 1$ covers exactly two words D of type $[1^2]$ with $D_i = 1$.

Now we derive identities analogous to (1) - (4) for words of weight equal to 3. As (1) - (4) have been derived in great detail, and the same type of ideas are used to prove identities (5) - (11) we will leave a part of the proofs to the reader.

In Z^n there are $2n$ words of type $[3^1]$. Each of them is covered by a codeword of type $[3^1]$ or $[4^1]$ or $[3^1, 1^1]$, and each of those codewords covers exactly one word of type $[3^1]$. Therefore,

$$|[3^1]| + |[4^1]| + |[3^1, 1^1]| = 2n, \tag{5}$$

and, for each $i \in I$, we have

$$|[3^1]_i| + |[4^1]_i| + \left|[3^1, 1^1]_i^{(3)}\right| = 1. \tag{6}$$

Further, in Z^n there are $2^3\binom{n}{2}$ words of type $[2^1, 1^1]$. They are covered by codewords of type $[2^1, 1^1]$, or $[3^1, 1^1]$, or $[2^2]$, or $[2^1, 1^2]$. Each codeword of type $[2^2]$, or $[2^1, 1^2]$ covers two such words, while each codeword of type $[2^1, 1^1]$, or $[3^1, 1^1]$ covers one of them. Hence

$$|[2^1, 1^1]| + |[3^1, 1^1]| + 2\,|[2^2]| + 2\,|[2^1, 1^2]| = 2^3\binom{n}{2} \tag{7}$$

The above identity has two local forms. There are $2(n-1)$ words U of type $[2^1, 1^1]$ with $U_i = 2$, and $2(n-1)$ words U of type $[2^1, 1^1]$ with $U_i = 1$. For each $i \in I$ we get

$$\left|[2^1, 1^1]_i^{(2)}\right| + \left|[3^1, 1^1]_i^{(3)}\right| + |[2^2]_i| + 2\left|[2^1, 1^2]_i^{(2)}\right| = 2(n-1), \tag{8}$$

and

$$\left|[2^1, 1^1]_i^{(1)}\right| + \left|[3^1, 1^1]_i^{(1)}\right| + |[2^2]_i| + \left|[2^1, 1^2]_i^{(1)}\right| = 2(n-1). \tag{9}$$

Further, in Z^n there are $2^3\binom{n}{3}$ words of type $[1^3]$. They are covered by codewords of type $[1^3]$, or $[2^1, 1^2]$, or $[1^4]$. Each codeword of type $[1^4]$ covers four of them. Hence,

$$|[1^3]| + |[2^1, 1^2]| + 4\,|[1^4]| = 2^3\binom{n}{3} \tag{10}$$

The local form of (10) reads as follows:

$$\left|[1^3]_i\right| + \left|[2^1,1^2]_i\right| + 3\left|[1^4]_i\right| = 2^2\binom{n-1}{2} \tag{11}$$

as in Z^n there are $2^2\binom{n-1}{2}$ words U of type $[1^3]$ with $U_i = 1$, and each codeword V of type $[1^4]$ with $V_i = 1$ covers three of them.

Clearly, there are many solutions of (1),...,(11) in natural numbers. We will prove, that only one corresponds to a tiling of R^n by crosses.

We will split Theorem 15 into two statements but will determine also the local values for individual types. We start with the number of codewords of weight 3.

Theorem 16 *Let $n = 2(\mathrm{mod}\,3)$, $\mathcal{L}$ be a tiling of R^n by crosses, and W be a codeword. Then, the number of codewords of given type with respect to W is: $\left|[3^1]\right| = 0$, $\left|[2^1,1^1]\right| = 2n$, and $\left|[1^3]\right| = \frac{2n(n-2)}{3}$. As to the local values, for each $i \in I$, $\left|[2^1,1^1]_i^{(2)}\right| = \left|[2^1,1^1]_i^{(1)}\right| = 1$, that is, $\left|[2^1,1^1]_i\right| = 2$, and $\left|[1^3]_i\right| = n-2$.*

Proof. Let W be a codeword in $\mathcal{T}_{\mathcal{L}}$. Clearly, then also the set $\mathcal{T}' = \{U, U \in Z^n, U = V - W$ for some V in $\mathcal{T}_{\mathcal{L}}\}$ is a tiling of Z^n by Lee spheres. Therefore, wlog we assume $W = O$. From (3) we have $\left|[2^1,1^1]_i^{(2)}\right| \leq 1$, while from (4) we get $\left|[2^1,1^1]_i\right|$ is even, hence $\left|[2^1,1^1]_i^{(2)}\right| \leq \left|[2^1,1^1]_i\right|$. On the other hand, there is no $i \in I$ with $\left|[2^1,1^1]_i^{(2)}\right| < \left|[2^1,1^1]_i\right|$as $\sum_{i\in I}\left|[2^1,1^1]_i^{(2)}\right| = \sum_{i\in I}\left|[2^1,1^1]_i^{(1)}\right|$. Thus we proved:

Lemma A. For each $i \in I$, either $\left|[2^1,1^1]_i\right| = 0$ or $\left|[2^1,1^1]_i\right| = 2$. In the latter case $\left|[2^1,1^1]_i^{(2)}\right| = \left|[2^1,1^1]_i^{(1)}\right| = 1$.

Now we are ready to prove that $\left|[3^1]\right| = 0$. We consider two cases.

(i) Let $\left|[3^1]_i\right| = 1$. Then, by (6), $\left|[3^1,1^1]_i^{(3)}\right| = 0$, and by (3), $\left|[2^1,1^1]_i^{(2)}\right| = 0$, which implies, by Lemma A, that $\left|[2^1,1^1]_i\right| = 0$. This in turn implies, see (4), $\left|[1^3]_i\right| = n-1$. Substituting it into (11) gives $\left|[2^1,1^2]_i\right| +$

$3\left|\left[1^4\right]_i\right| = (n-1)(2n-5)$. As we deal with the case $n = 2(\bmod 3)$, then $(n-1)(2n-5) = 2(\bmod 3)$ as well, and therefore $\left|\left[2^1,1^2\right]_i\right| = 2(\bmod 3)$. Subtracting (9) from (8), and using $\left|[3^1,1^1]_i^{(3)}\right| = \left|\left[2^1,1^1\right]_i^{(2)}\right| = \left|\left[2^1,1^1\right]_i^{(1)}\right| = 0$, we get $2\left|\left[2^1,1^2\right]_i^{(2)}\right| = \left|\left[2^1,1^2\right]_i^{(1)}\right| + \left|[3^1,1^1]_i^{(1)}\right|$. As $\left|\left[2^1,1^2\right]_i\right| = \left|\left[2^1,1^2\right]_i^{(2)}\right| + \left|\left[2^1,1^2\right]_i^{(1)}\right|$, adding $\left|\left[2^1,1^2\right]_i^{(2)}\right|$ to both sides yields $3\left|\left[2^1,1^2\right]_i^{(2)}\right| = \left|\left[2^1,1^2\right]_i\right| + \left|[3^1,1^1]_i^{(1)}\right|$. We showed above that in this case of $\left|[3^1]_i\right| = 1$ it is $\left|\left[2^1,1^2\right]_i\right| = 2(\bmod 3)$. Therefore $\left|[3^1,1^1]_i^{(1)}\right| > 0$, that is, $\left|[3^1,1^1]_i^{(1)}\right| > \left|[3^1,1^1]_i^{(3)}\right|$.

(ii) Now let $\left|[3^1]_i\right| = 0$. By (3), we get $\left|\left[2^1,1^1\right]_i^{(2)}\right| = 1$, which implies, by Lemma A, that $\left|\left[2^1,1^1\right]_i\right| = 2$. This in turn implies, see (4), $\left|\left[1^3\right]_i\right| = n-2$. Substituting it into (11) gives $\left|\left[2^1,1^2\right]_i\right| + 3\left|\left[1^4\right]_i\right| = (n-2)(2n-3)$. As $n = 2(\bmod 3)$, it is $(n-2)(2n-3) = 0(\bmod 3)$, and therefore $\left|\left[2^1,1^2\right]_i\right| = 0(\bmod 3)$. Subtracting (9) from (8), and using $\left|\left[2^1,1^1\right]_i^{(2)}\right| = \left|\left[2^1,1^1\right]_i^{(1)}\right| = 1$, we get $2\left|\left[2^1,1^2\right]_i^{(2)}\right| + \left|[3^1,1^1]_i^{(3)}\right| - \left|[3^1,1^1]_i^{(1)}\right| = \left|\left[2^1,1^2\right]_i^{(1)}\right|$, and adding $\left|\left[2^1,1^2\right]_i^{(2)}\right|$ to both sides gives $3\left|\left[2^1,1^2\right]_i^{(2)}\right| + \left|[3^1,1^1]_i^{(3)}\right| - \left|[3^1,1^1]_i^{(1)}\right| = \left|\left[2^1,1^2\right]_i\right|$. As $\left|\left[2^1,1^2\right]_i\right| = 0(\bmod 3)$ in this case, we have $\left|[3^1,1^1]_i^{(3)}\right| - \left|[3^1,1^1]_i^{(1)}\right| = 0(\bmod 3)$, which yields $\left|[3^1,1^1]_i^{(1)}\right| \geq \left|[3^1,1^1]_i^{(3)}\right|$ as $\left|[3^1,1^1]_i^{(3)}\right| \leq 1$ for all $i \in I$, see (6).

So, $\left|[3^1]_i\right| = 1$ implies $\left|[3^1,1^1]_i^{(1)}\right| > \left|[3^1,1^1]_i^{(3)}\right|$, while $\left|[3^1]_i\right| = 0$ gives $\left|[3^1,1^1]_i^{(1)}\right| \geq \left|[3^1,1^1]_i^{(3)}\right|$. However, $\sum_{i\in I}\left|[3^1,1^1]_i^{(1)}\right| = \sum_{i\in I}\left|[3^1,1^1]_i^{(3)}\right|$, therefore there is no $i \in I$ with $\left|[3^1]_i\right| = 1$, that is $\left|[3^1]\right| = 0$, and, for all $i \in I$,

$$\left|[3^1,1^1]_i^{(1)}\right| = \left|[3^1,1^1]_i^{(3)}\right|, \text{ and } 3\left|\left[2^1,1^2\right]_i^{(2)}\right| = \left|\left[2^1,1^2\right]_i\right| \qquad (12)$$

Since $\left|[3^1]\right| = 0$, by (1) we get $\left|[2^1,1^1]\right| = 2n$, which in turn implies, by (2), that $\left|\left[1^3\right]\right| = \frac{2n(n-2)}{3}$. Further, from $\left|\left[2^1,1^1\right]_i\right| = 2$, we get $\left|\left[1^3\right]_i\right| = n-2$. The proof is complete. ■

Now we prove an analogue of Theorem 16 for the values of $\left|\left[2^1,1^2\right]\right|$ and $\left|\left[1^4\right]\right|$.

Theorem 17 *Let $\mathcal{L}$ be a tiling of R^n by crosses where $n = 2(\bmod 3)$. Then, for each $W \in \mathcal{T}_{\mathcal{L}}$, $\left|[2^1, 1^2]\right| = 2n(n-2)$, and $\left|[1^4]\right| = \frac{n(n-2)(n-3)}{3}$. In addition, for all $i \in I$, it is, $\left|[2^1, 1^2]_i\right| = 3(n-2)$, $\left|[2^1, 1^2]_i^{(2)}\right| = n-2$, and $\left|[1^4]_i\right| = \frac{2(n-2)(n-3)}{3}$.*

Proof. As with Theorem 16, w.l.o.g we assume that $W = O$. In order to determine the value of $\left|[2^1, 1^2]\right|$ we need the following lemma:

Lemma 18 *For each $i \in I$, it is $\left|[2^2]_i\right| \leq 1$; hence $\left|[2^2]\right| \leq n/2$.*

Proof of Lemma 18. Assume by contradiction that there is $i \in I$, say $i = 1$, such that $\left|[2^2]_1\right| \geq 2$. Let, w.l.o.g, $F = (2, 2, 0, ..., 0)$, $F' = (2, 0, 2, 0, ..., 0)$ be two codewords of type $[2^2]$ with $F_1 = F_1' = 2$. We proved that, for each $i \in I$, it is $\left|[2^1, 1^1]_i^{(2)}\right| = 1$. So there is a codeword B of type $[2^1, 1^1]$, with $B_1 = 2$. We may assume w.l.o.g. that $B = (2, ..., 0, \pm 1, 0, ...0)$. If $B = (2, -1, 0, ..., 0)$, then $F_1 - B = (0, 3, 0, ..., 0)$, that is, the codeword F_1 is with respect to the codeword B of type $[3^1]$, which is a contradiction as we proved that $\left|[3^1]\right| = 0$. So let $B = (2, 0, 0, 1, 0, ..., 0)$. Then $F_1 - B = (0, 2, 0, -1, 0, ..., 0)$ and $F_2 - B = (0, 0, 2, -1, 0, ..., 0)$. That is, with respect to the codeword B, we get $\left|[2^1, 1^1]_i^{(1)}\right| = 2$, which contradicts that $\left|[2^1, 1^1]_i^{(1)}\right| = 1$ for all $i \in I$. Therefore, $\left|[2^2]_i\right| \leq 1$ for all $i \in I$, which in turn implies $\left|[2^2]\right| = \frac{1}{2}\sum_{i \in I}\left|[2^2]_i\right| \leq n$. This proves Lemma 18.

With this in hand we find the values of $\left|[2^1, 1^2]\right|$ and $\left|[2^1, 1^2]_i\right|$. The equality (8) states that $\left|[2^1, 1^1]_i^{(2)}\right| + \left|[3^1, 1^1]_i^{(3)}\right| + \left|[2^2]_i\right| + 2\left|[2^1, 1^2]_i^{(2)}\right| = 2(n-1)$. In addition, by Lemma 16, it is $\left|[2^1, 1^1]_i^{(2)}\right| = 1$, by (6) $\left|[3^1, 1^1]_i^{(3)}\right| \leq 1$, and by the above lemma $\left|[2^2]_i\right| \leq 1$. As $\left|[2^1, 1^1]_i^{(2)}\right| + \left|[3^1, 1^1]_i^{(3)}\right| + \left|[2^2]_i\right|$ is an even number, we get

$$\left|[2^1, 1^1]_i^{(2)}\right| + \left|[3^1, 1^1]_i^{(3)}\right| + \left|[2^2]_i\right| = 2. \tag{13}$$

Therefore $\left|[2^1, 1^2]_i^{(2)}\right| = n-2$, thus $\left|[2^1, 1^2]\right| = \sum_{i \in I}\left|[2^1, 1^2]_i^{(2)}\right| = 2n(n-2)$. By (12), $\left|[2^1, 1^2]_i\right| = 3\left|[2^1, 1^2]_i^{(2)}\right| = 3(n-2)$. The values of $\left|[1^4]\right|$ and $\left|[1^4]_i\right|$ are easily obtained from (10) and (11), respectively. The proof is complete. ■

To be able to determine the values of $\left|\left[4^1\right]\right|$, $\left|\left[3^1,1^1\right]\right|$, and $\left|\left[2^2\right]\right|$, we need to consider codewords from the 5-neighbourhood. Each word $V, |V|_M = 5$, is either of type $\left[5^1\right]$, or $\left[4^1,1^1\right]$, or $\left[3^1,2^1\right]$, or $[3^1,1^2]$, or $\left[2^2,1^1\right]$, or $\left[2^1,1^3\right]$, or $\left[1^5\right]$. Let W be a codeword in $\mathcal{T}_{\mathcal{L}}$. Then the number of codewords Z in the 5-neighbourhood of W of the given type $\left[5^1\right]$ will be denoted by $\left|\left[5^1\right]\right|$, of type $\left[4^1,1^1\right]$ by $\left|\left[4^1,1^1\right]\right|$, etc.

We start with a series of auxiliary statements.

Lemma 19 *For each $i \in I$, $\left|\left[3^1,2^1\right]_i^{(3)}\right| \leq 1$, and $\left|\left[3^1,2^1\right]_i^{(2)}\right| \leq 2$.*

Proof. Assume by contradiction that there are two codewords C^1 and C^2 of type $[3^1,2^1]$ with $C_i^k = 3$ for $k = 1,2$. By (6) we have $\left|\left[4^1\right]_i\right| + \left|\left[3^1,1^1\right]_i^{(3)}\right| = 1$ as we know from Theorem 16 that $\left|\left[3^1\right]_i\right| = 0$.

So, assume first that $\left|\left[4^1\right]_i\right| = 1$. Then there is a codeword D, with $D_i = 4$. Say, w.l.o.g, $D = (4,0,...,0)$, and $C^1 = (3,2,0,...,0)$, $C^2 = (3,0,2,0,...,0)$. As $C^1 - D = (-1,2,0,...,0)$, and $C^2 - D = (-1,0,2,0,...,0)$, we arrived at a contradiction since with respect to D we have $\left|\left[2^1,1^1\right]_i^{(1)}\right| > 1$.

Suppose now that $\left|\left[3^1,1^1\right]_i^{(3)}\right| = 1$, i.e., there is codeword E of type $[3^1,1^1]$ so that $E_i = 3$; say $E = (3,1,0,...,0)$. If $C^1 = (3,-2,0,...,0)$ then $C^1 - E = (0,3,0,...,0)$ a contradiction as $\left|\left[3^1\right]\right| = 0$ with respect to all codewords. So we may assume that $C^1 = (3,0,2,0,..,0)$, and $C^2 = (3,0,0,2,0,...,0)$. Then $C^1 - E = (0,-1,2,0,...,0)$, and $C^2 - E = (0,-1,0,2,0,...,0)$; i.e., with respect to W, $\left|\left[2^1,1^1\right]_i^{(1)}\right| > 1$, a contradiction. The proof of the first part follows.

Now let B be a codeword of type $\left[2^1,1^1\right]$ with $B_i = 2$. Further, let $\left|\left[3^1,2^1\right]_i^{(2)}\right| \geq 3$, and C^1, C^2, and C^3 be codewords of type $[3^1,2^1]$ with $C_i^j = 2$, $j = 1,...,3$. We assume w.l.o.g. that $i = 1$, and $B = (2,1,0,..,0)$. Then there are at least two of the codewords C^j, say C^1 and C^2 having the second coordinate equal to 0, as otherwise the two codewords would be at distance less than 3. We assume w.l.o.g. $C^1 = (2,0,3,0,...,0)$, and $C^2 = (2,0,0,3,0,...,0)$. Hence $C^1 - B = (0,-1,3,0,...,0)$ and $C^2 - B = (0,-1,0,3,0,...,0)$; i.e., with respect to the codeword B we get

$\left|[3^1,1^1]^{(1)}_{-2}\right| > 1$. This contradicts (12) because $\left|[3^1,1^1]^{(3)}_i\right| \leq 1$ for all $i \in I$. The proof is complete. ■

Before we prove the next lemma we get equalities related to covering words of absolute value 4. In Z^n there are $2n$ words of type $[4^1]$. By Theorem 16, there is no codeword of type $[3^1]$. Hence we have

$$|[4^1]| + |[5^1]| + |[4^1,1^1]| = 2n, \tag{14}$$

and, for all $i \in I$, the local form reads as follows:

$$|[4^1]_i| + |[5^1]_i| + \left|[4^1,1^1]^{(4)}_i\right| = 1 \tag{15}$$

Further, in Z^n there are $2^3\binom{n}{2}$ words of type $[3^1,1^1]$. Each of them is covered by a codeword of type $[3^1]$, or $[2^1,1^1]$, or $[3^1,1^1]$, or $[4^1,1^1]$, or $[3^1,2^1]$, or $[3^1,1^2]$. Only codewords of type $[3^1,1^2]$ cover two words of type $[3^1,1^1]$. In addition we know that there is no codeword of type $[3^1]$. Thus,

$$|[2^1,1^1]| + |[3^1,1^1]| + |[4^1,1^1]| + |[3^1,2^1]| + 2\,|[3^1,1^2]| = 8\binom{n}{2} \tag{16}$$

For each $i \in I$, there are $2(n-1)$ words V of type $[3^1,1^1]$ with $V_i = 3$, and, at the same time, $2(n-1)$ words V of type $[3^1,1^1]$ with $V_i = 1$. Thus the local forms of (16) read as follows:

$$\begin{gathered}\left|[2^1,1^1]^{(2)}_i\right| + \left|[3^1,1^1]^{(3)}_i\right| + \left|[4^1,1^1]^{(4)}_i\right| + \left|[3^1,2^1]^{(3)}_i\right| + \\ 2\left|[3^1,1^2]^{(3)}_i\right| = 2(n-1),\end{gathered} \tag{17}$$

and

$$\begin{gathered}\left|[2^1,1^1]^{(1)}_i\right| + \left|[3^1,1^1]^{(1)}_i\right| + \left|[4^1,1^1]^{(1)}_i\right| + \left|[3^1,2^1]^{(2)}_i\right| + \\ \left|[3^1,1^2]^{(1)}_i\right| = 2(n-1).\end{gathered} \tag{18}$$

In Z^n there are $2^2\binom{n}{2}$ words of type $[2^2]$. Each word of this type is covered by a codeword of type $[2^1,1^1]$, or $[2^2]$, or $[3^1,2^1]$, or $[2^2,1^1]$. As each such codeword covers one word of type $[2^2]$ we have

$$|[2^1,1^1]| + |[2^2]| + |[3^1,2^1]| + |[2^2,1^1]| = 2^2\binom{n}{2} \tag{19}$$

and, for each $i \in I$, we get

$$\left|[2^1,1^1]_i\right| + \left|[2^2]_i\right| + \left|[3^1,2^1]_i\right| + \left|[2^2,1^1]_i^{(2)}\right| = 2(n-1) \tag{20}$$

In Z^n there are $3 \cdot 2^3 \binom{n}{3}$ words of type $[2^1,1^2]$. Each of these words is covered by a codeword of type $[2^1,1^1]$, or $[1^3]$, or $[2^1,1^2]$, or $[3^1,1^2]$, or $[2^1,1^3]$, or $[2^2,1^1]$. As each codeword of type $[2^1,1^1]$ covers $2(n-2)$ of them, each codeword of type $[1^3]$ and of type $[2^1,1^3]$ covers three of them, and each codeword of type $[2^2,1^1]$ covers two of them, we get

$$\begin{aligned} 2(n-2)\left|[2^1,1^1]\right| + 3\left|[1^3]\right| + \left|[2^1,1^2]\right| + \left|[3^1,1^2]\right| + \\ 3\left|[2^1,1^3]\right| + 2\left|[2^2,1^1]\right| = 24\binom{n}{3} \end{aligned} \tag{21}$$

For each $i \in I$, in Z^n there are $2^2\binom{n-1}{2}$ words V of type $[2^1,1^2]$ with $V_i = 2$. Hence, a local form of (21) is

$$\begin{aligned} 2(n-2)\left|[2^1,1^1]_i^{(2)}\right| + \left|[1^3]_i\right| + \left|[2^1,1^2]_i^{(2)}\right| + \left|[3^1,1^2]_i^{(3)}\right| + \\ 3\left|[2^1,1^3]_i^{(2)}\right| + \left|[2^2,1^1]_i^{(2)}\right| = 2^2\binom{n-1}{2}. \end{aligned} \tag{22}$$

In Z^n there are $8\binom{n-1}{2}$ words V of type $[2^1,1^2]$ with $V_i = 1$. It is not difficult to see that:

$$\begin{aligned} \left|[2^1,1^1]_i^{(0)}\right| + 2(n-2)\left|[2^1,1^1]_i^{(1)}\right| + 2\left|[1^3]_i\right| + \left|[2^1,1^2]_i^{(1)}\right| + \\ \left|[3^1,1^2]_i^{(1)}\right| + 2\left|[2^1,1^3]_i^{(1)}\right| + \left|[2^2,1^1]_i^{(2)}\right| + \\ 2\left|[2^2,1^1]_i^{(1)}\right| = 8\binom{n-1}{2}, \end{aligned} \tag{23}$$

where $\left|[2^1,1^1]_i^{(0)}\right|$ stands for codeword V of type $[2^1,1^1]$ with $V_i = V_{-i} = 0$. As $\left|[2^1,1^1]_i\right| = \left|[2^1,1^1]_{-i}\right| = 2$, we have $\left|[2^1,1^1]_i^{(0)}\right| = 2(n-2)$. Finally, in Z^n there are $2^4\binom{n}{4}$ words of type $[1^4]$. Each of them is covered by a codeword of type $[1^3]$, or $[1^4]$, or $[1^5]$, or $[2^1,1^3]$. Clearly each codeword of type $[1^3]$ covers $2(n-3)$ words of type $[1^4]$, while each codeword of type $[1^5]$ covers 5 words of type $[1^4]$. Therefore:

$$2(n-3)\left|[1^3]\right| + \left|[1^4]\right| + \left|[2^1,1^3]\right| + 5\left|[1^5]\right| = 2^4\binom{n}{4}. \tag{24}$$

The local form of (24) reads as follows

$$\left|[1^3]_i^{(0)}\right| + 2(n-3)\left|[1^3]_i\right| + \left|[1^4]_i\right| + \left|[2^1,1^3]_i\right| + 4\left|[1^5]_i\right| = 2^3\binom{n-1}{3}, \tag{25}$$

where $\left|[1^3]_i^{(0)}\right|$ stands for the number of codewords V of type $[1^3]$ with $V_i = V_{-i} = 0$.

Now we are ready to prove a lemma crucial for the proof of the next theorem. (15) states that $\left|[4^1]_i\right| + \left|[5^1]_i\right| + \left|[4^1,1^1]_i^{(4)}\right| = 1$. Thus, the lemma covers all possible cases.

Lemma 20 *Let $i \in I$. If $\left|[4^1]_i\right| = 1$, then $\left|[2^2]_i\right| = 1$, $\left|[3^1,1^1]_i^{(3)}\right| = 0$, and $\left|[3^1,2^1]_i^{(3)}\right| = 1$, while if $\left|[5^1]_i\right| = 1$, then $\left|[2^2]_i\right| = 0$, $\left|[3^1,1^1]_i^{(3)}\right| = 1$, and $\left|[3^1,2^1]_i^{(3)}\right| = 0$, but if $\left|[4^1,1^1]_i^{(4)}\right| = 1$ then $\left|[2^2]_i\right| = 0$, $\left|[3^1,1^1]_i^{(3)}\right| = 1$, and $\left|[3^1,2^1]_i^{(3)}\right| = 1$. In particular, $\left|[3^1,2^1]\right| = \left|[4^1]\right| + \left|[4^1,1^1]\right|$.*

Proof. Assume first that $\left|[4^1]_i\right| = 1$, and let D be a codeword in $\mathcal{T}_{\mathcal{L}}$ of type $[4^1]$ with $D_i = 4$. As $\left|[4^1]_i\right| = 1$, then taking into account $\left|[3^1]_i\right| = 0$ and (6), we get $\left|[3^1,1^1]_i^{(3)}\right| = 0$. This in turn implies, due to (13), that $\left|[2^2]_i\right| = 1$. Consider the 3-neighbourhood $\mathcal{N}$ of D. By Theorem 16, we have for $\mathcal{N}$ that $\left|[2^1,1^1]_i^{(1)}\right| = 1$ for all $i \in I$. That is there has to be in $\mathcal{T}_{\mathcal{L}}$ a codeword B of type $[2^1,1^1]$ with respect to D, with $B_i - D_i = -1$, that is $(B-D)_{-i}^{(1)} = 1$. Thus $B_i = 3$ and there is a $j \in J$ so that $B_j = 2$. Hence, B is a codeword of type $[3^1,2^1]$ with respect to the origin O, and therefore $\left|[3^1,2^1]_i^{(3)}\right| \geq 1$, while by Lemma 19 we get $\left|[3^1,2^1]_i^{(3)}\right| = 1$. The first part of the proof is complete.

Let now $\left|[5^1]_i\right| = 1$. Then there is a codeword W in $\mathcal{T}_{\mathcal{L}}$ of type $[5^1]$ with $W_i = 5$. Further, by (15), $\left|[4^1]_i\right| = 0$, which in turn implies, see (6), that $\left|[3^1,1^1]_i^{(3)}\right| = 1$, and by (13), $\left|[2^2]_i\right| = 0$. Now we prove that in this case $\left|[3^1,2^1]_i^{(3)}\right| = 0$ as well. Let B be a codeword of type $[2^1,1^1]$ with $B_i = 2$; w.l.o.g., let $W = (5,0,...,0)$, and $B = (2,1,0,...,0)$. Assume that there is a codeword C of type $[3^1,2^1]$ with $C_i = 3$. Then $B - W = (-3,1,0,...,0)$

and $C - W = (-2, 0, 2, 0, ..., 0)$. That is, with respect to the codeword W, we have $\left|[3^1, 1^1]_{-1}^{(3)}\right| = 1$ but also $\left|[2^2]_{-1}\right| = 1$, which contradicts (13) as $\left|[2^1, 1^1]_i^{(2)}\right| = 1$ for all $i \in I$. Thus $\left|[3^1, 2^1]_i^{(3)}\right| = 0$ in this case. The proof of the second part of the statement is complete.

Now, assume that $\left|[4^1, 1^1]_i^{(4)}\right| = 1$. We need to show that in this case $\left|[3^1, 2^1]_i^{(3)}\right| = 1$ as well. However, by (17), $\left|[2^1, 1^1]_i^{(2)}\right| + \left|[3^1, 1^1]_i^{(3)}\right| + \left|[4^1, 1^1]_i^{(4)}\right| + \left|[3^1, 2^1]_i^{(3)}\right|$ is an even number, and in this case we have $\left|[2^1, 1^1]_i^{(2)}\right| = \left|[3^1, 1^1]_i^{(3)}\right| = \left|[4^1, 1^1]_i^{(4)}\right| = 1$. Hence $\left|[3^1, 2^1]_i^{(3)}\right|$ is odd, and, by Lemma 19, $\left|[3^1, 2^1]_i^{(3)}\right| = 1$.

Finally, $\left|[3^1, 2^1]\right| = \sum_{i \in I} \left|[3^1, 2^1]_i^{(3)}\right| = \sum_{\left|[4^1]_i\right|=1} \left|[3^1, 2^1]_i^{(3)}\right| + \sum_{\left|[4^1, 1^1]_i^{(4)}\right|=1} \left|[3^1, 2^1]_i^{(3)}\right| + \sum_{\left|[5^1]_i\right|=1} \left|[3^1, 2^1]_i^{(3)}\right| = \sum_{\left|[4^1]_i\right|=1} \left|[3^1, 2^1]_i^{(3)}\right| + \sum_{\left|[4^1, 1^1]_i^{(4)}\right|=1} \left|[3^1, 2^1]_i^{(3)}\right| = \left|[4^1]\right| + \left|[4^1, 1^1]\right|$. The proof is complete. ■

We are ready to determine the remaining values of $\left|[4^1]\right|, \left|[3^1, 1^1]\right|$, and $\left|[2^2]\right|$.

Theorem 21 *For each codeword W in $\mathcal{T}_{\mathcal{L}}$, we have $\left|[3^1, 1^1]\right| = 2n$, and $\left|[4^1]\right| = \left|[2^2]\right| = 0$. In addition, for all $i \in I$, $\left|[3^1, 1^1]_i^{(3)}\right| = \left|[3^1, 1^1]_i^{(1)}\right| = 1$.*

Proof. As above, we assume w.l.o.g. that $W = O$. We recall that, by (12), $\left|[3^1, 1^1]_i^{(3)}\right| = \left|[3^1, 1^1]_i^{(1)}\right|$ for all $i \in I$. Since $\left|[3^1, 2^1]_i\right| = \left|[3^1, 2^1]_i^{(3)}\right| + \left|[3^1, 2^1]_i^{(2)}\right|$, we get from (20)

$$\left|[2^1, 1^1]_i\right| + \left|[2^2]_i\right| + \left|[3^1, 2^1]_i^{(3)}\right| + \left|[3^1, 2^1]_i^{(2)}\right| + \left|[2^2, 1^1]_i^{(2)}\right| = 2(n-1) \tag{26}$$

Combining Lemma 20 with (17), (18), and (26) yields:

If $\left|\left[4^1\right]_i\right| = 1$, then

$$\begin{gathered}\left|\left[3^1,1^2\right]_i^{(3)}\right| = n-2,\\ \left|\left[3^1,1^2\right]_i^{(1)}\right| = 2n-3-\left|\left[4^1,1^1\right]_i^{(1)}\right| - \left|\left[3^1,2^1\right]_i^{(2)}\right|,\\ \text{and } \left|\left[2^2,1^1\right]_i^{(2)}\right| = 2n-6-\left|\left[3^1,2^1\right]_i^{(2)}\right|\end{gathered} \tag{27}$$

For $\left|\left[5^1\right]_1\right| = 1$, it is

$$\begin{gathered}\left|\left[3^1,1^2\right]_i^{(3)}\right| = n-2,\\ \left|\left[3^1,1^2\right]_i^{(1)}\right| = 2n-4-\left|\left[4^1,1^1\right]_i^{(1)}\right| - \left|\left[3^1,2^1\right]_i^{(2)}\right| =,\\ \text{and } \left|\left[2^2,1^1\right]_i^{(2)}\right| = 2n-4-\left|\left[3^1,2^1\right]_i^{(2)}\right|,\end{gathered} \tag{28}$$

and for $\left|\left[4^1,1^1\right]_i\right| = 1$ we get

$$\begin{gathered}\left|\left[3^1,1^2\right]_i^{(3)}\right| = n-3,\\ \left|\left[3^1,1^2\right]_i^{(1)}\right| = 2n-4-\left|\left[4^1,1^1\right]_i^{(1)}\right| - \left|\left[3^1,2^1\right]_i^{(2)}\right| =,\\ \text{and } \left|\left[2^2,1^1\right]_i^{(2)}\right| = 2n-5-\left|\left[3^1,2^1\right]_i^{(2)}\right|.\end{gathered} \tag{29}$$

Summing (27), (28), and (29) yields:

$$\left|\left[3^1,1^2\right]\right| = \sum_{i\in I}\left|\left[3^1,1^2\right]_i^{(3)}\right| = \sum_{\left|\left[4^1\right]_i\right|=1}\left|\left[3^1,1^2\right]_i^{(3)}\right| + \sum_{\left|\left[4^1,1^1\right]_i^{(4)}\right|=1}\left|\left[3^1,1^2\right]_i^{(3)}\right| +$$

$\sum_{\left|\left[5^1\right]_i\right|=1}\left|\left[3^1,1^2\right]_i^{(3)}\right| = \left|\left[4^1\right]\right|(n-2) + \left|\left[4^1,1^1\right]\right|(n-3) + \left|\left[5^1\right]\right|(n-2)$. Hence, using $\left|\left[4^1\right]\right| + \left|\left[4^1,1^1\right]\right| + \left|\left[5^1\right]\right| = 2n$, see (14), we get

$$\left|\left[3^1,1^2\right]\right| = 2n(n-2) - \left|\left[4^1,1^1\right]\right|; \tag{30}$$

and

$$2\left|\left[2^2,1^1\right]\right| = \sum_{i\in I}\left|\left[2^2,1^1\right]_i^{(2)}\right| = \sum_{\left|\left[4^1\right]_i\right|=1}\left|\left[2^2,1^1\right]_i^{(2)}\right| + \sum_{\left|\left[4^1,1^1\right]_i^{(4)}\right|=1}\left|\left[2^2,1^1\right]_i^{(2)}\right|$$

$+ \sum_{\left|\left[5^1\right]_i\right|=1}\left|\left[2^2,1^1\right]_i^{(2)}\right| = \left|\left[4^1\right]\right|(2n-6) + \left|\left[4^1,1^1\right]\right|(2n-5) + \left|\left[5^1\right]\right|(2n-$

4) $-\sum_{i\in I}\left|[3^1,2^1]_i^{(2)}\right| = 2n(2n-6)+\left|[4^1,1^1]\right|+2\left|[5^1]\right|-\left|[3^1,2^1]\right|$ as, see Lemma 20, $\left|[3^1,2^1]\right| = \left|[4^1]\right|+\left|[4^1,1^1]\right|$. Thus

$$\left|[2^2,1^1]\right| = n(2n-6)+\left|[5^1]\right|-\frac{\left|[4^1]\right|}{2}. \tag{31}$$

Substituting into (21) for $\left|[2^1,1^1]\right|, \left|[1^3]\right|, \left|[2^1,1^2]\right|$ from Theorem 17 and for $\left|[3^1,1^2]\right|$ and $2\left|[2^2,1^1]\right|$ from above we get

$$4n(n-2)+2n(n-2)+2n(n-2)+2n(n-2)-\left|[4^1,1^1]\right|+$$
$$3\left|[2^1,1^3]\right|+2n(2n-7)+\left|[4^1,1^1]\right|+3\left|[5^1]\right| = 24\binom{n}{3},$$

i.e.,

$$3\left|[2^1,1^3]\right|+3\left|[5^1]\right| = 2n(n-3)(2n-7) \tag{32}$$

Substituting for $\left|[2^1,1^1]_i^{(2)}\right|, \left|[1^3]_i\right|, \left|[2^1,1^2]_i^{(2)}\right|$ from Theorem 17, to (22) yields $\left|[3^1,1^2]_i^{(3)}\right|+3\left|[2^1,1^3]_i^{(2)}\right|+\left|[2^2,1^1]_i^{(2)}\right| = 4\binom{n-1}{2}-4(n-2) = 2(n-2)(n-3)$. Applying (27), (29), and (28) in turn implies, if $\left|[4^1]_i\right| = 1$ or $\left|[4^1,1^1]_i^{(4)}\right| = 1$, then

$$3\left|[2^1,1^3]_i^{(2)}\right| = 2(n-2)(n-3)-3n+8+\left|[3^1,2^1]_i^{(2)}\right|, \tag{33}$$

and if $\left|[5^1]_i\right| = 1$, then

$$3\left|[2^1,1^3]_i^{(2)}\right| = 2(n-2)(n-3)-3n+6+\left|[3^1,2^1]_i^{(2)}\right|. \tag{34}$$

As $n = 2(\bmod 3)$, also $2(n-2)(n-3) = 0(\bmod 3)$. Thus, for $\left|[4^1]_i\right| = 1$ and $\left|[2^1,1^1]_i^{(4)}\right| = 1$, we have $\left|[3^1,2^1]_i^{(2)}\right| = 1(\bmod 3)$, while for $\left|[5^1]_i\right| = 1$ we get $\left|[3^1,2^1]_i^{(2)}\right| = 0(\bmod 3)$. By Lemma 19, $\left|[3^1,2^1]_i^{(2)}\right| \le 2$ for all $i \in I$. Hence, for $\left|[4^1]_i\right| = 1$ and $\left|[4^1,1^1]_i^{(4)}\right| = 1$, we have $\left|[3^1,2^1]_i^{(2)}\right| = 1$, and for $\left|[5^1]_i\right| = 1$ we get $\left|[3^1,2^1]_i^{(2)}\right| = 0$. Again, by Lemma 19, we have the same conclusion for $\left|[3^1,2^1]_i^{(3)}\right|$. Thus,

$$\left|[3^1,2^1]_i^{(3)}\right| = \left|[3^1,2^1]_i^{(2)}\right| \tag{35}$$

for all $i \in I$.

Substituting into (23) for $\left|[2^1,1^1]_i^{(1)}\right|, \left|[1^3]_i\right|$, and $\left|[2^1,1^2]_i^{(1)}\right|$ from Theorem 17 yields $\left|[3^1,1^2]_i^{(1)}\right| + 2\left|[2^1,1^3]_i^{(1)}\right| + \left|[2^2,1^1]_i^{(2)}\right| + 2\left|[2^2,1^1]_i^{(1)}\right| = 4(n-1)(n-2)-8(n-2) = 4(n-2)(n-3)$. If $\left|[4^1]_i\right| = 1$ or $\left|[4^1,1^1]_i^{(4)}\right| = 1$, then in this case $\left|[3^1,2^1]_i^{(2)}\right| = 1$, and by (27), (29), we get

$$2\left|[2^1,1^3]_i^{(1)}\right| + 2\left|[2^2,1^1]_i^{(1)}\right| = 4(n-2)(n-3) - 4n + 11 + \left|[4^1,1^1]_i^{(1)}\right|.$$

Thus, $\left|[4^1,1^1]_i^{(1)}\right|$ is odd for $\left|[4^1]_i\right| = 1$ and for $\left|[4^1,1^1]_i^{(4)}\right| = 1$. However, $\sum_{i\in I}\left|[4^1,1^1]_i^{(1)}\right| = \sum_{i\in I}\left|[4^1,1^1]_i^{(4)}\right|$ and $\left|[4^1,1^1]_i^{(4)}\right| \leq 1$. Therefore, for all $i \in I$,

$$\left|[4^1,1^1]_i^{(4)}\right| = \left|[4^1,1^1]_i^{(1)}\right|, \tag{36}$$

which in turn implies $\left|[4^1]_i\right| = 0$ for all i, hence $\left|[4^1]\right| = 0$. Combining $\left|[3^1]\right| = 0$ with (5) yields $\left|[3^1,1^1]\right| = 2n$, which in turn implies $\left|[2^2]\right| = 0$, see (13). The proof is complete. ■

2.2 Phase B

In this subsection we deal with the number of codewords of individual types of weight equal to 5. First we will summarize results for all $n = 2(\bmod 3)$, then we concentrate on the case $n = 5$. For $n = 2(\bmod 3)$, all these values are expressed as a function of the number of codewords of type $[5^1]$. We point out that for some $n = 2(\bmod 3)$, there are two tilings of R^n by crosses with different number of codewords of type $[5^1]$. Hence, unlike with codewords of weight equal to 3 or 4, the values of $\left|[5^1]\right|, \left|[4^1,1^1]\right|, \left|[3^1,2^1]\right|, \left|[3^1,1^2]\right|, \left|[2^1,1^3]\right|, \left|[2^2,1^1]\right|$, and $\left|[1^5]\right|$ do not depend only on the value of n but also on a given tiling of R^n by crosses.

Theorem 22 *Let $n = 2(\bmod 3)$, and W in $\mathcal{T}_{\mathcal{L}}$. Then the number of codewords of a given type with respect to W is: $\left|[4^1,1^1]\right| = 2n - \left|[5^1]\right|, \left|[3^1,2^1]\right| = 2n - \left|[5^1]\right|, \left|[3^1,1^2]\right| = 2n(n-3) + \left|[5^1]\right|, \left|[2^1,1^3]\right| = n(2n-6) + \left|[5^1]\right|, \left|[2^1,1^3]\right| = \frac{1}{3}2n(n-3)(2n-7) - \left|[5^1]\right|$, and $\left|[1^5]\right| = \frac{1}{5}(2^4\binom{n}{4} - n(n-3)(3n-8) + \left|[5^1]\right|)$.*

Proof. We have proved in the previous theorem that $\left|[4^1]\right| = 0$. Therefore, by (14), we have $\left|[5^1]\right| + \left|[4^1,1^1]\right| = 2n$. In addition, by Lemma 20, it is

$\left|[3^1,2^1]\right| = \left|[4^1]\right| + \left|[4^1,1^1]\right|$, that is $\left|[3^1,2^1]\right| = \left|[4^1,1^1]\right| = 2n - \left|[5^1]\right|$. It is $\left|[3^1,1^2]\right| = 2n(n-2) - \left|[4^1,1^1]\right|$, see(30), hence $\left|[5^1]\right| + \left|[4^1,1^1]\right| = 2n$ implies $\left|[3^1,1^2]\right| = 2n(n-3) + \left|[5^1]\right|$, while (31) implies $\left|[2^2,1^1]\right| = n(2n-6) + \left|[5^1]\right|$. The value of $\left|[2^1,1^3]\right|$ is given by (32). Finally, the value of $\left|[1^5]\right|$ follows from (24) after substituting for $\left|[1^3]\right|$ and $\left|[1^4]\right|$ from Theorem 15. The proof is complete. ■

The next theorem determines the local values of $\left|[3^1,2^1]\right|, \ldots, \left|[1^5]\right|$ with respect to $\left|[5^1]_1\right|$.

Theorem 23 *Let $n = 2 (\bmod 3)$. Then for each codeword in $\mathcal{T}_{\mathcal{L}}$ we have: If $\left|[5^1]_i\right| = 1$, then $\left|[4^1,1^1]_i\right| = 0, \left|[3^1,2^1]_i\right| = 0, \left|[3^1,1^2]_i^{(1)}\right| = 2\left|[3^1,1^2]_i^{(3)}\right| = 2(n-2), \left|[2^2,1^1]_i^{(2)}\right| = 2\left|[2^2,1^1]_i^{(1)}\right| = 2(n-2), \left|[2^1,1^3]_i^{(1)}\right| = 3\left|[2^1,1^3]_i^{(2)}\right| = (n-2)(2n-9)$. If $\left|[5^1]_i\right| = 0$, then $\left|[4^1,1^1]_i^{(4)}\right| = \left|[4^1,1^1]_i^{(1)}\right| = 1, \left|[3^1,2^1]_i^{(3)}\right| = \left|[3^1,2^1]_i^{(2)}\right| = 1, \left|[3^1,1^2]_i^{(1)}\right| = 2\left|[3^1,1^2]_i^{(3)}\right| = 2(n-3), \left|[2^2,1^1]_i^{(2)}\right| = 2\left|[2^2,1^1]_i^{(1)}\right| = 2(n-3), \left|[2^1,1^3]_i^{(1)}\right| = 3\left|[2^1,1^3]_i^{(2)}\right| = (n-3)(2n-7)$. In both cases $\left|[1^5]_i\right| = \frac{1}{4}(8\binom{n-1}{3} - \left|[2^1,1^3]_i\right| - \frac{10}{3}(n-2)(n-3))$.*

Proof. Let $\left|[5^1]_i\right| = 1$. Then by, Lemma 20, $\left|[3^1,2^1]_i^{(3)}\right| = 0$, and by (35) $\left|[3^1,2^1]_i^{(2)}\right| = 0$; also $\left|[4^1,1^1]_i^{(4)}\right| = 0$, and (35) implies $\left|[4^1,1^1]_i^{(i)}\right| = 0$. Hence $\left|[3^1,2^1]\right| = \left|[4^1,1^1]\right| = 0$. Using the same arguments in the case $\left|[5^1]_i\right| = 0$ yields $\left|[4^1,1^1]_i^{(4)}\right| = \left|[4^1,1^1]_i^{(1)}\right| = 1, \left|[3^1,2^1]_i^{(3)}\right| = \left|[3^1,2^1]_i^{(2)}\right| = 1$. With this in hand, the values of $\left|[3^1,1^2]_i\right|$ and $\left|[2^2,1^1]_i\right|$ follow from (28) and (29), while the values of $\left|[2^1,1^3]_i\right|$ are obtained from (33) and (34). Finally, to determine $\left|[1^5]_i\right|$ it suffices to substitute into (25). The proof is complete. ■

We showed above, that the number of codewords in $\mathcal{T}_{\mathcal{L}}$ of weight 3 and 4 does depends only on n, while the number of codewords of weight 5 depends also on the tiling $\mathcal{L}$. However, for $n = 5$ also all these values are constant.

Theorem 24 *If $n = 5$, then $\left|[3^1]\right| = \left|[4^1]\right| = \left|[2^2]\right| = \left|[5^1]\right| = 0, \left|[2^1,1^1]\right| = \left|[1^3]\right| = \left|[3^1,1^1]\right| = \left|[1^4]\right| = \left|[4^1,1^1]\right| = \left|[3^1,2^1]\right| = 10, \left|[3^1,1^2]\right| = \left|[2^2,1^1]\right| = \left|[2^1,1^3]\right| = 20$, while $\left|[2^1,1^2]\right| = 30$, and $\left|[1^5]\right| = 2$, and $\left|[1^5]_i\right| = 1$ for all $i \in I$.*

Proof. The values of $\left|[3^1]\right|, \left|[2^1,1^1]\right|, \left|[1^3]\right|, \left|[4^1]\right|, \left|[3^1,1^1]\right|, \left|[2^2]\right|$, $\left|[2^1,1^2]\right|$, and $\left|[1^4]\right|$ are obtained from Theorem 16, 17, and 21 by substituting $n=5$. The other values depend on the value of $\left|[5^1]\right|$, see Theorem 22. We will prove that there is no codeword of type $[5^1]$, i.e., that $\left|[5^1]\right| = 0$. In order to do it we have to consider not only local equalities but also so-called double-local equalities for the individual type of codewords. To be able to introduce these we need one more piece of notation. Let $\mathcal{K}$ be a set of codewords. Then, for $i,j \in I$, we denote by $|\mathcal{K}_{ij}|$ the number of codewords K in $\mathcal{K}$ so that $K_i \neq 0 \neq K_j$. For an ordered pair (i,j) we denote by $\left|\mathcal{K}_{ij}^{(a,b)}\right|$ the number of codewords K in $\mathcal{K}$ with $K_i = a$ and $K_j = b$.

Substituting $n=5$ into Theorem 22 and into Theorem 23 yields $\left|[1^5]\right| = 2+\frac{\left|[5^1]\right|}{5}$, and $\left|[1^5]_i\right| > 0$ for all $i \in I$, respectively. Assume by contradiction that $\left|[5^1]\right| > 0$. Then we have $\left|[1^5]\right| \geq 3$, and at least two codewords of type $[1^5]$ have to coincide in at least two coordinates. Thus, there have to be signed coordinates $i,j \in I$ such that $\left|[1^5]_{ij}\right| \geq 2$. To reject the assumption of $\left|[5^1]\right| > 0$ we prove that $\left|[1^5]_{ij}\right| \leq 1$ for all $i,j \in I$. We will start by setting several double-local equalities.

For each $i,j \in I$ there is a unique word V in Z^5 of type $[1^2]$ with $V_i = V_j = 1$. Therefore, see also the explanation to (2),

$$\left|[2^1,1^1]_{ij}\right| + \left|[1^3]_{ij}\right| = 1 \tag{37}$$

For each $i,j \in I$ there are six words V in Z^5 of type $[1^3]$ with $V_i = V_j = 1$. Therefore, see also (10),

$$\left|[1^3]_{ij}\right| + \left|[2^1,1^2]_{ij}\right| + 2\left|[1^4]_{ij}\right| = 6 \tag{38}$$

Clearly, it is not difficult to see that $\left|[2^1,1^2]_{ij}\right| = \left|[2^1,1^2]_{ij}^{(2,1)}\right| + \left|[2^1,1^2]_{ij}^{(1,2)}\right| + \left|[2^1,1^2]_{ij}^{(1,1)}\right| \leq 3$. Indeed, $\left|[2^1,1^2]_{ij}^{(2,1)}\right| > 1$ ($\left|[2^1,1^2]_{ij}^{(1,2)}\right| > 1$) would imply that there are in $\mathcal{T}_{\mathcal{L}}$ two codewords of type $[2^1,1^2]$ of distance 2, a contradiction. Finally, $\left|[2^1,1^2]_{ij}^{(1,1)}\right| > 1$ would imply, that there are in $\mathcal{T}_{\mathcal{L}}$ two codewords of type $[2^1,1^2]$ coinciding in two coordinates where their value is 1, say $A = (1,1,2,0,0)$, and $B = (1,1,-2,0,0)$ or $B = (1,1,0,2,0)$. However, in the former case $B - A = (0,0,4,0,0)$, and in the latter $B - A = (0,0,-2,2,0)$, thus B would be with respect to A a codeword of type $[4^1]$,

and of type $[2^2]$, respectively, which contradicts $\left|[4^1]\right| = \left|[2^2]\right| = 0$, see Theorem 21. Combining (37) with (38) we get:

Claim B. For all $i, j \in I$, if $\left|[1^3]_{ij}\right| = 1$, then $\left|[1^4]_{ij}\right| \geq 1$, and for $\left|[1^3]_{ij}\right| = 0$, we get $\left|[1^4]_{ij}\right| \geq 2$.

Now we state a double-local equality for codewords of type $[1^4]$. For each $i, j \in I$, there are twelve words V in Z^5 of type $[1^4]$ with $V_i = V_j = 1$, using arguments similar to proving (24) yields:

$$\left|[1^3]_{ij}^{(0,1)}\right| + \left|[1^3]_{ij}^{(1,0)}\right| + 4\left|[1^3]_{ij}\right| + \left|[1^4]_{ij}\right| + \left|[2^1, 1^3]_{ij}\right| + 3\left|[1^5]_{ij}\right| = 12, \tag{39}$$

where $\left|[1^3]_{ij}^{(0,1)}\right|$ is the number of codewords C of type $[1^3]$ such that $C_i = C_{-i} = 0$, and $C_j = 1$. For each $i, j \in I$, we have $\left|[1^3]_{ij}\right| \leq 1$, otherwise there would be two codewords of type $[1^3]$ at distance less than 3. Therefore, for all $i, j \in I$, there is at most one codeword V of type $[1^3]$ with $V_i = V_j = 1$, or $V_{-i} = V_j = 1$, or $V_i = V_{-j} = 1$. As $\left|[1^3]_i\right| = 3$, we get that both $\left|[1^3]_{ij}^{(0,1)}\right| \geq 2$ and $\left|[1^3]_{ij}^{(1,0)}\right| \geq 2$ for $\left|[1^3]_{ij}\right| = 0$, and both $\left|[1^3]_{ij}^{(0,1)}\right| \geq 1$ and $\left|[1^3]_{ij}^{(1,0)}\right| \geq 1$ if $\left|[1^3]_{ij}\right| = 1$.

In aggregate, if $\left|[1^3]_{ij}\right| = 1$, we get $\left|[1^3]_{ij}^{(0,1)}\right| + \left|[1^3]_{ij}^{(1,0)}\right| + 4\left|[1^3]_{ij}\right| \geq 6$, and $\left|[1^3]_{ij}^{(0,1)}\right| + \left|[1^3]_{ij}^{(1,0)}\right| + 4\left|[1^3]_{ij}\right| \geq 4$ for $\left|[1^3]_{ij}\right| = 0$.

Substituting to (39) for $\left|[1^4]_{ij}\right|$ from Claim B implies

$$\left|[2^1, 1^3]_{ij}\right| + 3\left|[1^5]_{ij}\right| \leq 5 \text{ for } \left|[1^3]_{ij}\right| = 1, \text{ and}$$
$$\left|[2^1, 1^3]_{ij}\right| + 3\left|[1^5]_{ij}\right| \leq 6 \text{ for } \left|[1^3]_{ij}\right| = 0.$$

Thus, $\left|[1^5]_{ij}\right| \leq 1$ for $\left|[1^3]_{ij}\right| = 1$. To prove that $\left|[1^5]_{ij}\right| \leq 1$ also when $\left|[1^3]_{ij}\right| = 0$, we will show that in this case $\left|[2^1, 1^3]_{ij}\right| > 0$. If $\left|[1^3]_{ij}\right| = 0$, then $\left|[2^1, 1^1]_{ij}\right| = 1$, see (3). Let B be the codeword of type $[2^1, 1^1]$ having

non-zero the i-th and the j-th sign coordinate. Then either $B_i = 2$ and $B_j = 1$, or $B_i = 1$, and $B_j = 2$. Assume that the former is the case. In Z^5 there are six words V of type $[2^1, 1^2]$ with $V_i = 2$, and $V_j = 1$. Each of these is covered either (i) by a codeword W of type $[2^1, 1^1]$ with $W_i = 2, W_j = 0$; or (ii) by a codeword W of type $[2^1, 1^1]$ with $W_i = 2, W_j = 1$; or (iii) by a codeword W of type $[1^3]$ with $W_i = 1, W_j = 1$; or (iv) by a codeword W of type $[2^1, 1^2]$ with $W_i = 2, W_j = 1$; or (v) by a codeword W of type $[3^1, 1^2]$ with $W_i = 3, W_j = 1$; or (vi) by a codeword W of type $[2^2, 1^1]$ with $W_i = 2, W_j = 2$; or (vii) by a codeword W of type $[2^2, 1^1]$ with $W_i = 2, W_j = 1$; or (viii) by a codeword W of type $[2^1, 1^3]$ with $W_i = 2, W_j = 1$. Using our notation we can write

$$\left|[2^1,1^1]_{ij}^{(2,0)}\right| + 6\left|[2^1,1^1]_{ij}^{(2,1)}\right| + \left|[1^3]_{ij}\right| + \left|[2^1,1^2]_{ij}^{(2,1)}\right| + \left|[3^1,1^2]_{ij}^{(3,1)}\right| + \left|[2^2,1^1]_{ij}^{(2,2)}\right| + \left|[2^2,1^1]_{ij}^{(2,1)}\right| + \left|[2^1,1^3]_{ij}^{(2,1)}\right| = 6.$$

We have chosen i, j so that $\left|[2^1,1^1]_{ij}^{(2,1)}\right| = \left|[1^3]_{ij}\right| = 0$. Further, it is easy to see that in each of the cases (i) and (iv)-(vi) there is at most one codeword of each type, as otherwise we would have two codewords at distance less than 3. As to (vii), we have $\left|[2^2,1^1]_{ij}^{(2,1)}\right| \leq 1$ as well, because if there were two codewords W of type $[2^2, 1^1]$ with $W_i = 2$, and $W_j = 1$, then their difference would be a codeword either of type $[4^1]$ or $[2^2]$, a contradiction. So we get $\left|[2^1,1^3]_{ij}^{(2,1)}\right| \geq 1$ in this case, thus $\left|[2^1,1^3]_{ij}\right| \geq 1$, and in turn $\left|[1^5]_{ij}\right| \leq 1$ also in the case $\left|[1^3]_{ij}\right| = 0$. The proof is complete. ■

2.3 Phase C

In the previous subsection we proved that the 5-neighborhood of each codeword has the same quantitative properties. Now we prove that it has also the same structure.

Let V be a word in Z^5. Then by $< V >$ we denote the collection of words comprising V, and the words obtained by cyclic shifts of coordinates of V. Hence, e.g., $< (2,1,0,0,0) >= \{(2,1,0,0,0), (0,2,1,0,0), (0,0,2,1,0), (0,0,0,2,1), (1,0,0,0,2)\}$. We note that $< V >$ contains five words except for the case when V has all coordinates equal to the same number. Finally, we set $\pm < V > \triangleq < V > \cup < -V >$. By the canonical 5- neighborhood, or simply a canonical neighborhood, we mean the set of

words
$\{\pm < (2,1,0,0,0) >, \pm < (1,0,1,0,-1) >, \pm < (3,0,0,-1,0) >,$
$\pm < (2,0,1,0,1) >, \pm < (2,0,0,1,-1) >, \pm < (2,-1,-1,0,0) >,$
$\pm < (1,1,1,-1,0) >, \pm < (4,0,-1,0,0) >, \pm < (3,0,2,0,0) >,$
$\pm < (3,0,0,1,1) >, \pm < 3,-1,0,0,-1) >, \pm < (2,-2,0,0,1) >,$
$\pm < (2,0,-2,0,-1) >, \pm < (2,-1,1,1,0) >, \pm < (2,0,-1,-1,1) >,$
$\pm < (1,1,1,1,1) >\}$. A simple inspection shows that the number of words of individual types in the canonical neighborhood coincides with the values given by Theorem 24. E.g., $\pm < (2,1,0,0,0) >$ is the set of ten words of type $[2^1,1^1]$, ($|[2^1,1^1]| = 10$), while $\pm < (1,0,1,0,-1) >$ comprises ten words of type $[1^3]$ ($|[1^3]| = 10$).

Theorem 25 *Let $\mathcal{L}$ be a tiling of R^5 by crosses. Then, for each codeword W in $\mathcal{T}_{\mathcal{L}}$, the 5-neighbourhood of W is congruent to the canonical one. Moreover, the 5-neighborhood of W is uniquely determined by the set of codewords of type $[2^1,1^1]$.*

Proof. As in other proofs in this paper we assume w.l.o.g. that $W = O$. Two words $U = (u_1,...,u_5), V = (v_1,..,v_5)$ will be called sign equivalent in the j-th coordinate if $u_j v_j > 0$; that is, they are sign equivalent if $u_j \neq 0 \neq v_j$, and the two non-zero values have the same sign.

Codewords of type $[1^5]$. It was proved in Theorem 24 that $|[1^5]| = 2$, and $|[1^5]_i| = 1$ for each $i \in I$; i.e., the two codewords of type $[1^5]$ differ in each coordinate (= are not sign equivalent in any coordinate). That is, if M is a codeword of type $[1^5]$, then $-M$ is the other codeword of type $[1^5]$.

Codewords of type $[2^1,1^1]$. Let $\mathcal{B}$ be a set of codewords of type $[2^1,1^1]$, and $\mathcal{C}$ be the set of codewords of type $[1^3]$. We know by Theorem 16 that $|\mathcal{B}| = |\mathcal{C}| = 10$. There are in total 10 words of type $[1^2]$ that are sign equivalent in two coordinates with M and another 10 words of type $[1^2]$ that are sign equivalent in two coordinates with $-M$. Each word of type $[1^2]$ is covered by a codeword in $\mathcal{B} \cup \mathcal{C}$, thus each of these 20 words is covered by a codeword in $\mathcal{B} \cup \mathcal{C}$. If a codeword C in $\mathcal{C}$ covered two of these 20 words, then C would be sign equivalent in three coordinates with M or $-M$, and the distance of C to one of M or $-M$ would be less than 3. Therefore, each codeword in $\mathcal{C}$ covers at most one of these 20 words of type $[1^2]$. Hence, each codeword in $\mathcal{B}$ has to cover one of these 20 words, that is, each codeword of type $[2^1,1^1]$ is sign equivalent in both non-zero coordinates either with the codeword M or with the codeword $-M$. We know by Theorem 16 that, for each $i \in I$, $\left|[2^1,1^1]_i^{(2)}\right| = \left|[2^1,1^1]_i^{(1)}\right|$. Thus, five codewords in $\mathcal{B}$ are sign equivalent in two coordinates with M, the other five with $-M$.

It turns out that graph theory has a very suitable language to describe the structure of the set $\mathcal{B}$. Let G be a graph with the vertex set I, the set of signed coordinates, and the edges of G be all pairs of vertices in I except for $\{i, -i\}$, $i = 1, ..., 5$. Thus G is a complete graph on 10 vertices, K_{10}, with a 1-factor (=perfect matching) removed. We denote this one factor by M. So $G = K_{10} - M$. In Z^5 there are forty words of type $[1^2]$. In a natural way each word of type $[1^2]$ is associated with an edge of G. If V is a word of type $[1^2]$ with $V_i = V_j = 1$, (and then $V_k = 0$ for all $k \in I - \{i, j\}$) then we assign to V the edge $\{i, j\}$ of G. So there is a one-to-one correspondence between words of type $[1^2]$ and the edges of G. In addition, each codeword $W \in \mathcal{B}$ is associated with the edge (word of type $[1^2]$) covered by W. The set of words of type $[1^2]$ covered by codewords in $\mathcal{B}$ will be denoted by $\mathcal{B}^*$. The condition $\left|[2^1, 1^1]_i^{(2)}\right| = \left|[2^1, 1^1]_i^{(1)}\right| = 1$ implies that the words in $\mathcal{B}^*$ form a 2-factor, say F. It follows from the above discussion that F consists of two cycles of length 5 such that both cycles contain exactly one vertex of each edge in the matching M. Let C be the set of five words in $\mathcal{B}^*$ constituting one of the two 5-cycles in F. Clearly, by suitably permuting the order of coordinates of each codeword in $\mathcal{T}_\mathcal{L}$ and/or changing a sign of a coordinate for each codeword in $\mathcal{T}_\mathcal{L}$ maps C onto $< (1, 1, 0, 0, 0) >$, and the codewords covering words in C onto $< (2, 1, 0, 0, 0) >$. By Claim 9 the above transformation is a congruence mapping. Therefore we will assume that $< (2, 1, 0, 0, 0) >$ are codewords in $\mathcal{T}_\mathcal{L}$.

We will show that by choosing $< (2, 1, 0, 0, 0) > \in \mathcal{T}_\mathcal{L}$ all the other codewords will be uniquely determined. First of all $< (2, 1, 0, 0, 0) > \in \mathcal{T}_\mathcal{L}$ implies that the codewords of type $[1^5]$ are $M = (1, 1, 1, 1, 1)$ and $-M$.

If one of the two 5-cycles of $\mathcal{B}^*$ is $< (1, 1, 0, 0, 0) >$, then the other one is formed by codewords having both non-zero coordinates negative. There are four non-isomorphic ways how to choose it. It is either the 5-cycle comprising the edges corresponding to $< (-1, -1, 0, 0, 0) >$, or to $< (-1, 0, 0, -1, 0) >$, or
$\{(-1, -1, 0, 0, 0), (0, -1, 0, 0, -1), (0, 0, -1, 0, -1), (0, 0, -1, -1, 0), (-1, 0, 0, -1, 0)\}$, or
$\{(-1, -1, 0, 0, 0), (0, -1, -1, 0, 0), (0, 0, 0, -1, -1), (0, 0, -1, 0, -1), (-1, 0, 0, -1, 0)\}$.

The graph consisting of the edges of the 2-factor F and the matching M is a cubic graph. The four cubic graph corresponding to the cases described above are depicted in Figure 1. The first one is the prism on 10 vertices, the second is the Petersen graph, the labels for the other two are taken from [2].

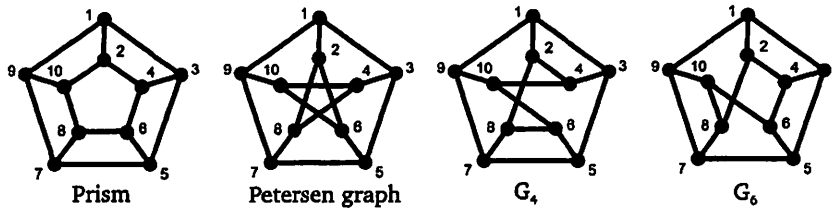

Figure 1: Corresponding cubic graphs

Codewords of type $[1^3]$. Each codeword of type $[1^3]$ covers three words of type $[1^2]$, so we associate with each codeword of type $[1^3]$ a triangle (cycle of length 3) in the graph G. As mentioned many times, all words of type $[1^2]$ are covered by codewords in $\mathcal{B} \cup \mathcal{C}$. Therefore, the triangles corresponding to codewords in $\mathcal{C}$ have to form an edge decomposition of the complement of the cubic graph consisting of the edges of the 2-factor F and the matching M.

It is known, see [2], that the complement of the two cubic graphs G_4 and G_6 is not decomposable into triangles. Therefore, in the case of G_4 and G_6 it is impossible to choose the codewords of type $[1^3]$. Thus, we are left with the prism and the Petersen graph.

(i) The prism. Then $\mathcal{B}^* = \pm < (1,1,0,0,0) >$. There are two different ways how to choose codewords of type $[1^3]$ (how to decompose the complement of the prism into triangles). Either

(ia) $\mathcal{C} = \pm < (1,0,1,0,-1) >$, or

(ib) $\mathcal{C} = \pm < (1,0,1,-1,0) >$.

These two decomposition are isomorphic but we will we need to consider both of them, as the automorphism of the graph G, which maps one decomposition on the other, maps the set $\pm < (2,1,0,0,0) >$ of codewords of type $[2^1, 1^1]$ on the set $\pm < (1,2,0,0,0) >$.

(ii) The Petersen graph.

Then $\mathcal{B}^* =< (1,1,0,0,0) > \cup < (-1,0,-1,0,0) >$. There are six different ways how to decompose the complement of the Petersen graph into triangles. One of them corresponds to the following codewords of type $[1^3]$:

(iia) $\mathcal{C} =< (1,-1,1,0,0) > \cup < -1,-1,0,1,0) >$, while the other five

are isomorphic to:

(iib) $\mathcal{C} = \{(1,-1,1,0,0),(1,0,-1,1,0),(-1,1,0,0,1),$ $(0,1,0,1,0,-1),(0,0,1,-1,1),(1,0,0,-1,-1),(-1,-1,0,1,0),$ $(-1,0,1,0,-1),(0,1,-1,-1,0),(0,-1,-1,0,1)\}$. Here we do not need to consider all 5 decompositions, as it is possible to prove that this case is not a viable one whether $<(2,1,0,0,0)>\in \mathcal{T}_{\mathcal{L}}$ or $<(1,2,0,0,0)>\in \mathcal{T}_{\mathcal{L}}$.

Codewords of type $[1^4]$. Let $\mathcal{H}$ be the set of ten codewords in $\mathcal{T}_{\mathcal{L}}$ of type $[1^4]$. Clearly, $\mathcal{H} \subset \mathcal{H}^*$ where $\mathcal{H}^*$ is constructed as follows: First let $\mathcal{H}^*$ be the set of all eighty words in Z^5 of type $[1^4]$. As any codeword in $\mathcal{H}$ has to have a distance at least 3 from both codewords of type $[1^5]$, and all ten codewords of type $[1^3]$, we delete from $\mathcal{H}^*$ all codewords that coincide with a codeword in all four non-zero coordinates, or with a codeword in $\mathcal{C}$ in three non-zero coordinates. After these two procedures there are exactly thirty words left in $\mathcal{H}^*$.

There are twenty words of type $[1^2]$ with both non-zero coordinates of the same sign. Ten of them are covered by codewords in $\mathcal{B}$, the other ten by the codewords in $\mathcal{C}$. Further, by Claim B stated in the proof of Theorem 24, it is $\left|[1^4]_{ij}\right| \geq 2$ for ij such that $\left|[2^1,1^1]_{ij}\right| = 1$, and $\left|[1^4]_{ij}\right| \geq 1$ for $\left|[1^3]_{ij}\right| = 1$. Hence, to each codeword W in $\mathcal{B}$, there are in $\mathcal{H}$ at least two codewords sign equivalent with W in two coordinates. Since no codeword in $\mathcal{H}$ has all non-zero coordinates of the same sign (otherwise its distance to a codeword of type $[1^5]$ would be less than 3), and the codewords in $\mathcal{B}$ form two 5-cycles, no codeword in $\mathcal{H}$ can be sign equivalent in two coordinates with three codewords in $\mathcal{B}$. This in turn implies, because $|\mathcal{H}| = 10$, that

Claim C. Each codeword in $\mathcal{H}$ has to be sign equivalent in two coordinates with two codewords of type $[2^1,1^1]$ and with one word of type $[1^2]$ with both non-zero coordinates of the same sign. In particular, each codeword in $\mathcal{H}$ has three coordinates of the same sign.

As in all cases $<(1,1,0,0,0)>\subset \mathcal{B}^*$, $\mathcal{H}$ has to contain a codeword $W = (1,1,1,a,b)$ where exactly one of a,b equals 0 and the other equals -1, and all cyclic shifts of coordinates of W.

(ia) First we describe the set $\mathcal{H}^*$. At the beginning of the process $\mathcal{H}^* = \{\pm<(1,1,1,1,0)>,\pm<(1,1,1,-1,0)>,\pm<(1,1,-1,1,0)>,$ $\pm<(1,-1,1,1,0)>,\pm<(-1,1,1,1,0)>,\pm<(1,1,-1,-1,0)>,$ $\pm<(1,-1,1,-1,0)>,\pm<(-1,1,1,-1,0)>\}$. We have to remove from

$\mathcal{H}^*$ words $\pm < (1,1,1,1,0) >$ that have the distance from the codewords $\pm(1,1,1,1,1)$ of type $[1^5]$ less than 3. Next, in this case, the set $\mathcal{C}$ of codewords of type $[1^3]$ is $\pm < (1,0,1,0,-1) >$. Therefore we need to remove from $\mathcal{H}^*$ forty words at distance less than 3 from any codeword in $\mathcal{C}$. These words are $\pm < (1,\pm1,1,0,-1) >$ and $\pm < (1,0,1,\pm1,-1) >$. Thus, at the end of the process $\mathcal{H}^* = \{\pm < (1,1,1,-1,0) >, \pm < (1,-1,1,1,0) >, \pm < (1,1,-1,-1,0,) >\}$. Clearly, the only way how to choose a set of ten codewords satisfying Claim C is to set $\mathcal{H} = \pm < (1,1,1,-1,0) >$.

(ib) We have $\mathcal{H}^* = \{\pm < (-1,1,1,1,0) >, \pm < (1,1,-1,1,0) >, \pm < (1,1,-1,-1,0) >\}$. Then a unique way how to fulfill Claim C is to set $\mathcal{H} = \pm < (-1,1,1,1,0) >$.

(iia) Let $H_1 =< (1,1,1,-1,0) >$, $H_2 =< (-1,1,1,1,0) >$, $H_3 =< (-1,1,-1,-1,0) >, H_4 =< -1,-1,1,-1,0) >$. Then $\mathcal{H}^*$ can be expressed as

$$\mathcal{H}^* = \bigcup_{i=1}^{4} H_i \cup < (-1,1,1,-1,0) > \cup < (-1,-1,1,1,0) > .$$

There are four options how to choose $\mathcal{H}$ in this case. Either $\mathcal{H}=H_1 \cup H_3$, or $H_1 \cup H_4$, or $H_2 \cup H_3$, or $H_2 \cup H_4$.

(iib) In this case all words in $\mathcal{H}^*$ that belong to $< (1,1,1,a,b) >$ are: $(1,1,1,-1,0), (1,1,1,0,-1)$, $(-1,1,1,1,0), (-1,0,1,1,1), (0,-1,1,1,1)$, $(1,-1,0,1,1), (1,1,0,-1,1)$. It is impossible to choose from this set three words of type $(a,1,1,1,b), (a,b,1,1,1)$, and $(1,a,b,1,1)$ that would be pairwise at distance at least 3. Therefore in this case it is impossible to choose a required set of codewords of type $[1^4]$, and we do not need to consider this case any longer.

<u>Codewords of type $[2^1,1^2]$ and $[2^1,1^3]$.</u> In Z^5 there are eighty words of type $[1^3]$. By (10), ten of them are covered by codewords in $\mathcal{C}$ of type $[1^3]$, forty of them by codewords in $\mathcal{H}$ of type $[1^4]$, and the set $\mathcal{G}^*$ of the remaining thirty words covered by the set $\mathcal{G}$ of codewords of type $[2^1,1^2]$. Clearly, if a codeword W of type $[2^1,1^2]$ covers a word V in $\mathcal{G}^*$, then we can see the codeword W as obtained from V by multiplying one of the non-zero coordinates of V by two.

Further, in Z^5 there are eighty words of type $[1^4]$. By (24), forty of them are covered by codewords in $\mathcal{C}$, ten of them by codewords in $\mathcal{H}$, ten by codewords of type $[1^5]$, and the remaining twenty, belonging to the set

$\mathcal{H}^* - \mathcal{H}$, by the set Λ of codewords of type $[2^1, 1^3]$. As above, if a codeword W in Λ covers a word V in $\mathcal{H}^* - \mathcal{H}$, then we can see W as obtained from V by multiplying one of the non-zero coordinates of V by two.

(ia) Since $\mathcal{H} = \pm < (1,1,1,-1,0) >$, and $\mathcal{C} = \pm < (1,0,1,0,-1) >$, it is $\mathcal{G}^* = \{\pm < (1,0,1,1,0) >, \pm < (1,-1,1,0,0) >, \pm < (0,-1,1,1,0) >\}$, and $\mathcal{H}^* - \mathcal{H} = \{\pm < (1,-1,1,1,0) >, \pm < (1,1,-1,-1,0) >\}$. We need to consider two possible choices of codewords of type $[2^1, 1^1]$ covering the other 5-cycle $< (-1,-1,0,0,0) >$ of the 2-factor F. It is either $< (-2,-1,0,0,0) > \in \mathcal{T}_{\mathcal{L}}$, or $< (-1,-2,0,0,0) > \in \mathcal{T}_{\mathcal{L}}$.

In the former case consider the set of words/codewords.

$$
\begin{array}{lrrrrr}
B_1 = (& -2, & -1, & 0, & 0, & 0) \\
B_2 = (& 0, & 0, & 2, & 1, & 0) \\
V_1 = (& 1, & -1, & 1, & 1, & 0) \\
V_2 = (& -1, & -1, & 1, & 1, & 0) \\
V_3 = (& 1, & 0, & 1, & 1, & 0) \\
V_4 = (& 0 & -1, & 1, & 1, & 0) \\
V_5 = (& 1, & -1, & 1, & 0, & 0)
\end{array}
$$

Clearly, B_1, B_2 are codewords of type $[2^1, 1^1]$, while $V_1, V_2 \in \mathcal{H}^* - \mathcal{H}$, and $V_3, V_4, V_5 \in \mathcal{G}^*$. To get a codeword W_j covering the word V_j, $j = 1, ..., 5$, we need to multiply a non-zero coordinate of V_j by 2. As there are five words V_j, and all of them have the fifth coordinate equal to zero, two of them have to have the same non-zero coordinate multiplied by 2. It is possible only for V_2 and V_3 if their forth coordinate is chosen, as otherwise the two resulting codewords would be at distance less than 3 (note that multiplying the first coordinate of V_2 by 2 results in a codeword at distance less than 3 from B_1). Because of the distance to the codeword B_2, only the word V_5 can have its third coordinate multiplied by 2. This implies that V_4 has to have its second coordinate multiplied by 2 while for $V_1$the first one is the only choice.

The same type of argument can be applied to a set of words/codewords obtained by the above one by cyclically shifting coordinates of each word/codeword and/or multiplying all words/codewords by -1. Therefore setting
$\Lambda = \{\pm < (2,-1,1,1,0) >, \pm < (-1,-1,1,2,0) >\}$, and
$\mathcal{G} = \{\pm < (1,0,1,2,0) >, \pm < (0,-2,1,1,0) >, \pm < (1,-1,2,0,0) >\}$,
which is identical to
$\Lambda = \{\pm < (2,-1,1,1,0) >, \pm < (2,0,-1,-1,1) >\}$,and
$\mathcal{G} = \{\pm < (2,0,1,0,1) >, \pm < (2,0,0,1,-1) >, \pm < (2,-1,-1,0,0) >\}$

is the unique choice so that all codewords in $\mathcal{B}$, $\mathcal{G}$, and Λ are pair-wise at distance at least 3.

In the latter case $< (-1,-2,0,0,0) > \in \mathcal{B}$. We will demonstrate that this is not a viable option. Let

$$\begin{array}{lrrrrr} B = (& 0, & 0, & -1, & -2, & 0) \\ V_1 = (& -1, & 1, & -1, & -1, & 0) \\ V_2 = (& -1, & 0, & -1, & -1, & 0) \\ V_3 = (& -1, & 1, & -1, & 0, & 0) \\ V_4 = (& 0, & 1, & -1, & -1, & 0) \end{array}$$

It is easy to check that $V_1 \in \pm < (1,-1,1,1,0) > \in \mathcal{H}^* - \mathcal{H}$, and $V_2 \in \pm < (1,0,1,1,0) > \in \mathcal{G}^*, V_3 \in \pm < (1,-1,1,0,0) > \in \mathcal{G}^*$, $V_4 \in \pm < (0,-1,1,1,0) > \in \mathcal{G}^*$, while B is a codeword. Let W_j be a codeword covering the word $V_j, j = 1,...,4$. As mentioned above, W_j can be viewed as obtained from V_j by multiplying one non-zero coordinate of V_j by 2. It is easy to see that, for $1 \leq j < k \leq 4$, if W_j and W_k were obtained by multiplying the same coordinate of V_j and V_k by 2 than their distance would be less than 3. On the other hand, if, for some $j, 1 \leq j \leq 4$, the fourth coordinate of W_j equaled to -2, then the distance of W_j to B would be less than three. Thus in this case, codewords covering $V_j, j = 1,...,4$, do not exist.

(ib) In this case $\mathcal{H} = \pm < (-1,1,1,1,0) >$, which in turn implies $\mathcal{H}^* - \mathcal{H} = \{\pm < (1,1,-1,1,0) >, \pm < (1,1,-1,-1,0) >\}$, and $\mathcal{G}^* = \{\pm <(1,1,0,1,0) >, \pm <(1,1,-1,0,0) >, \pm <(1,-1,1,0,0) >\}$. Let

$$\begin{array}{lrrrrr} B = (& 2, & 1, & 0, & 0, & 0) \\ V_1 = (& 1, & 1, & -1, & 1, & 0) \\ V_2 = (& 1, & 1, & 0, & 1, & 0) \\ V_3 = (& 1, & 1, & -1, & 0, & 0) \\ V_4 = (& 0, & 1, & -1, & 1, & 0) \end{array}$$

Then $V_1 \in \mathcal{H}^* - \mathcal{H}$, $V_2, V_3, V_4 \in \mathcal{G}^*$. By the same type of an argument as in (ia) it is easy to see that the required codewords $W_j, j = 1,...,4$, do not exist, as multiplying the first coordinate of V_j by 2 leads to a codeword at distance less than 3 from B. Moreover, our example shows that in this case (ib) it is impossible to choose the sets of codewords of type $[2^1, 1^2]$, and $[2^1, 1^3]$, regardless whether $< (-2,-1,0,0,0) >$, or $< (-1,-2,0,0,0) >$ is in $\mathcal{B}$.

(iia) We show that also in this case it is impossible to choose the sets of

codewords of type $[2^1, 1^2]$, and $[2^1, 1^3]$. There are two ways how to choose codewords in $\mathcal{B}$ of type $[2^1, 1^1]$ covering words $< (-1, 0, -1, 0, 0) >$; either $< (-2, 0, -1, 0, 0,) >$, or $< (-1, 0, -2, 0, 0,) >$. There are four ways how to choose the set $\mathcal{H}$ of codewords of type $[1^4]$. We treat here only one of them as the other three are nearly identical to this one. We deal with the option $\mathcal{H} = H_2 \cup H_3$. Then $\mathcal{H}^* - \mathcal{H} = \{ < (1, 1, 1, -1, 0) >,$ $< -1, -1, 1, -1, 0) > , < (1, 1, -1, 0, -1) >, < (1, 1, 0, -1, -1) >\}$, while $\mathcal{G}^* = \{< (1, 0, 1, 0, 1) >, < (1, 0, -1, 1, 0) >, < (1, 0, 0, -1, -1) >,$ $< (1, 1, 0, 0, -1) >, < (-1, -1, 0, 0, -1) >, < (-1, 0, -1, 0, 1) >\}$. Of the two cases when $< (-2, 0, -1, 0, 0,) > \in \mathcal{T}_{\mathcal{L}}$ or $< (-1, 0, -2, 0, 0,) > \in \mathcal{T}_{\mathcal{L}}$ we treat here the first one. Let

$$\begin{array}{lllllll} B_1 = (& 2, & 1, & 0, & 0, & 0) \\ B_2 = (& 0, & 0, & -2, & 0, & -1) \\ V_1 = (& 1, & 1, & -1, & 0, & -1) \\ V_2 = (& 0, & 1, & -1, & 0, & -1) \\ V_3 = (& 1, & 1, & 0, & 0, & -1) \end{array} ,$$

where B_1 and B_2 are codewords and $V_1 \in \mathcal{H}^* - \mathcal{H}$, $V_2 \in < (-1, 0, -1, 0, 1) > \in \mathcal{G}^*$, $V_3 \in < (1, 1, 0, 0, -1) > \in \mathcal{G}^*$. Because of B_1 none of V_j's can have the first coordinate multiplied by 2, while because of B_2 none of V_j's can have its third coordinate multiplied by 2. Thus, codewords covering V_1, V_2, V_3 do not exist.

Thus, in what follows, it suffices to consider the case (ia) with $\mathcal{B} = \pm < (2, 1, 0, 0, 0) >$.

Now it is relatively simple to show the uniqueness of the codewords of the remaining types.

Codewords of type $[3^1, 1^1]$. In Z^5 there are eighty words of type $[2^1, 1^1]$. By (7), ten of them are covered by the codewords of type $[2^1, 1^1]$, and sixty by the codewords of type $[2^1, 1^2]$. The remaining ten words $\pm < (2, 0, 0, -1, 0) >$ are to be covered by codewords of type $[3^1, 1^1]$. Thus, $\pm < (3, 0, 0, -1, 0) >$ are in $\mathcal{T}_{\mathcal{L}}$.

Codewords of type $[3^1, 1^2]$ and $[2^2, 1^1]$. In Z^5 there are 240 words of type $[2^1, 1^2]$. By (21) sixty of them are covered by codewords of type $[2^1, 1^1]$, thirty by codewords of type $[1^3]$, another thirty by codewords of type $[2^1, 1^2]$, and sixty by codewords of type $[2^1, 1^3]$. The remaining sixty words of type $[2^1, 1^2]$ are $\pm < (2, 0, 0, 1, 1) >$, $\pm < (2, -1, 0, 0, -1) >$, then $\pm < (2, -1, 0, 0, 1) >$, $\pm < (1, -2, 0, 0, 1) >$, and $\pm < (2, 0, -1, 0, -1) >$, $\pm < (1, 0, -2, 0, -1) >$. As each codeword of

type $[2^2, 1^1]$ covers two words of type $[2^1, 1^2]$ there have to be in $\mathcal{T}_{\mathcal{L}}$ codewords $\pm < (2,-2,0,0,1) >$ and $\pm < (2,0,-2,0,-1) >$ of type $[2^2, 1^1]$, and the remaining twenty words of type $[2^1, 1^2]$ are covered by codewords $\pm < (3,0,0,1,1) >, \pm < (3,-1,0,0,-1) >$ of type $[3^1, 1^2]$.

Codewords of type $[3^1, 2^1]$ and $[4^1, 1^1]$. Finally, the ten remaining words $\pm < (3,0,-1,0,0) >$ of type $[3^1, 1^1]$ have to be covered by codewords $\pm < (4,0,-1,0,0) >$ of type $[4^1, 1^1]$, and the ten remaining words $\pm < (2,0,2,0,0) >$ of type $[2^2]$ by codewords $\pm < (3,0,2,0,0) >$ of type $[3^1, 2^1]$.

So we have proved that the 5-neighborhood of each point is congruent to the canonical neighborhood, and that the neighborhood is uniquely determined by the codewords of type $[2^1, 1^1]$. ■

At the end of this subsection we describe an important attribute of the canonical 5-neighborhood.

Theorem 26 *Let $U, Z \in \mathcal{T}_{\mathcal{L}}$ be two words from the 5-neighborhood of a codeword W. Then $2W - U$ is in this neighborhood, that is, the 5-neighborhood is symmetric, and if $|U + Z - 2W| \leq 5$, then $U + Z - W$ belongs to this 5-neighborhood as well. In particular, if U, Z are from the 5-neighborhood of the origin then $-U$ and $U + Z$, if $|U + Z|_M \leq 5$, belong to this neighborhood as well.*

Proof. Again it suffices to prove the statement for $W = O$. From the previous theorem we know, that the 5-neighborhood of each codeword in congruent to the canonical one. Clearly, a congruence mapping retains the properties described in this theorem. Therefore, it suffices to prove the statement for the canonical neighborhood. To show that the canonical neighborhood satisfies these properties we prove that this neighborhood is a part of (the unique) lattice tiling of R^5 by crosses. Then the proof will follow from the fact that if words U, V belong to a lattice $\mathcal{L}$ then also $-U$ and $U + V$ are in $\mathcal{L}$.

Consider a homomorphism $\phi : Z^5 \to Z_{11}$, the cyclic group of order 11, given by $\phi(e_1) = 1, \phi(e_2) = 9, \phi(e_3) = 4, \phi(e_4) = 3$, and $\phi(e_5) = 5$. Then ϕ satisfies the assumptions of Corollary 11. Thus, ϕ induces a lattice tiling $\mathcal{L}$ of R^5 by crosses, where the set $\ker \phi$ is the set $\mathcal{T}_{\mathcal{L}}$ of the centers of crosses in this tilling. It is easy, although time consuming, to check, that all codewords from the canonical neighborhood belong to $\ker \phi$, that is, if $U = (u_1, ..., u_5)$ belongs to the 5-neighborhood then $u_1 + 9u_2 + 4u_3 + 3u_4 + 5u_5 = 0 (\mathrm{mod}\, 11)$. The proof is complete. ■

2.4 Phase D

As the closing part of the proof of Theorem 13 we show that for any two codewords in $\mathcal{T}_{\mathcal{L}}$ their 5-neighborhoods are not only congruent but that they are identical.

Theorem 27 *The 5-neighborhood of each codeword in the tiling $\mathcal{L}$ is equal to the 5-neighborhood of the origin.*

Proof. Let $\mathcal{L}$ be a tiling of R^5 by crosses. We proved that the 5-neighborhood of any codeword W in $\mathcal{T}_{\mathcal{L}}$ is congruent to the 5-neighborhood of the origin. By Claim 9, we may assume that the 5-neighborhood of the origin is the canonical one.

We have proved, see Theorem 16, that for each codeword W the 3-neighborhood of W comprises twenty codewords; i.e., there are in $\mathcal{T}_{\mathcal{L}}$ twenty codewords at distance 3 from W. We will call these codewords at distance 3 from W the codewords adjacent to W. Ten of the adjacent codewords are of type $[2^1, 1^1]$, and ten of them are of type $[1^3]$. The proof of the theorem is based on the following claim, which states that all codewords adjacent to W have "the same" set of codewords of type $[2^1, 1^1]$ as W has.

Claim D. Let W be a codeword in $\mathcal{T}_{\mathcal{L}}$, and let U be a codeword adjacent to W. Further, let S_W, and S_U be the set of codewords of type $[2^1, 1^1]$ with respect to W and U, respectively. Then $\{Z - W;$ where $Z \in S_W\} = \{Z - U,$ where $Z \in S_U\}$.

Theorem 25 states that the 5-neighbourhood of each codeword W is uniquely determined by the codewords of type $[2^1, 1^1]$. Thus, with Claim D in hands, we know that any two adjacent codewords have the same 5-neighborhood. The rest of the proof of the theorem follows easily by induction because to each codeword W there is a sequence of codewords $O = Z_0, Z_1, ..., Z_{m-1}, Z_m = W$ such that the codeword Z_j is adjacent to the codeword Z_{j-1} for all $j = 1, ..., m$.

W.l.o.g. we prove Claim D only for $W = O$, the origin. Consider a codeword U that is adjacent to O. To prove Claim D for U we need to show that $U + V$, where $V \in \pm < (2, 1, 0, 0, 0) >$ is a codeword in $\mathcal{T}_{\mathcal{L}}$. To do so, it suffices either

(a) to show that $U - V$ is a codeword, or

(b) to choose X, Y, Z so that

(i) $X, Y, Z, Y - X, Z - X$, and $Y + Z - 2X$ are in the canonical neighborhood; and

(ii) $Y + Z - X = U + V$.

Indeed, in the case (a), we know that for each codeword W its 5-neighborhood is symmetric with respect W, hence if $U - Z$ is a codeword then also $U + Z$ is a codeword because $|Z|_M \leq 5$. In the case (b) consider a codeword X. By Theorem 26, if there are codewords Y, Z, so that $\rho_M(Y, X) \leq 5, \rho_M(Z, X) \leq 5$, and $\rho_M(Y+Z-X, X) \leq 5$, then $Y+Z-X$ is a codeword as well. However, (i) guarantees that all assumptions of Theorem 26 are satisfied, therefore (ii) guarantees that $U + V$ is in the 5-neighborhood of the codeword U.

First we choose the codeword adjacent to the origin to be of type $[2^1, 1^1]$. Let $U = (2, 1, 0, 0, 0)$. If $V = (2, 1, 0, 0, 0)$ then $U - V = (0, ..., 0)$ is a codeword, and hence by (a) $U + V = (4, 2, 0, 0, 0)$ is a codeword as well. The following table provides a suitable choice for the other four codewords in $< (2, 1, 0, 0, 0) >$.

V	X	Y	Z	$Y + Z - X = U + V$
$(0, 2, 1, 0, 0)$	$(1, 0, 1, 0, -1)$	$(0, 3, 0, 0, -1)$	$(3, 0, 2, 0, 0)$	$(2, 3, 1, 0, 0)$
$(0, 0, 2, 1, 0)$	$(2, 1, 0, 0, 0)$	$(1, 2, 0, 1, 0)$	$(3, 0, 2, 0, 0)$	$(2, 1, 2, 1, 0)$
$(0, 0, 0, 2, 1)$	$(2, 1, 0, 0, 0)$	$(1, 2, 0, 1, 0)$	$(3, 0, 0, 1, 1)$	$(2, 1, 0, 2, 1)$
$(1, 0, 0, 0, 2)$	$(0, 1, 0, -1, 1)$	$(3, 0, 0, -1, 0)$	$(0, 2, 0, 0, 3)$	$(3, 1, 0, 0, 2)$

Theorem 26 guarantees that the 5-th neighborhood of each codeword is symmetric. Therefore $U + V$ is a codeword also for all $V \in< (-2, -1, 0, 0, 0) >$. Let $U' \in< (2, 1, 0, 0, 0) >$. Then we apply the same cyclic shift to X,Y, Z given in the table to obtain the required codewords. The same applies to $-U$ and its cyclic shifts. Finally, let U be a codeword of type $[1^3]$ adjacent to the origin. say $U = (1, 0, 1, 0, -1)$. The proper choice of X, Y, Z for each $V \in< (2, 1, 0, 0, 0) >$ is given in the table below:

V	X	Y	Z	$Y + Z - X = U + V$
$(2, 1, 0, 0, 0)$	$(1, 0, 1, 0, -1)$	$(1, 1, 0, 0, -2)$	$(3, 0, 2, 0, 0)$	$(3, 1, 1, 0, -1)$
$(0, 2, 1, 0, 0)$	$(0, 0, 2, 1, 0)$	$(0, 1, 1, 1, -1)$	$(1, 1, 3, 0, 0)$	$(1, 2, 2, 0, -1)$
$(0, 0, 2, 1, 0)$	$(0, 0, 2, 1, 0)$	$(1, 0, 1, 2, 0)$	$(0, 0, 4, 0, -1)$	$(1, 0, 3, 1, -1)$
$(0, 0, 0, 2, 1)$	$(2, 1, 0, 0, 0)$	$(2, 0, 1, 0, 1)$	$(1, 1, 0, 2, -1)$	$(1, 0, 1, 2, 0)$
$(1, 0, 0, 0, 2)$	$(2, 1, 0, 0, 0)$	$(3, 0, 0, -1, 0)$	$(1, 1, 1, 1, 1)$	$(2, 0, 1, 0, 1)$

while for $V \in< (-2, -1, 0, 0, 0) >$, $U + V$ is a codeword as the 5-neighbourhood is symmetric. As in the case of U being an adjacent codeword of type

$[2^1, 1^1]$, a suitable choice of X, Y, Z for other cases can be obtained by a cyclic shift. The proof is complete. ■

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