On an Extension of an Observation of Hamilton

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Abstract

For a Hamiltonian graph G, the Hamiltonian cycle extension number of G is the maximum positive integer k for which every path of order k or less is a subpath of some Hamiltonian cycle of G. The Hamiltonian cycle extension numbers of all Hamiltonian complete multipartite graphs are determined. Sharp lower bounds for the Hamiltonian cycle extension number of a Hamiltonian graph are presented in terms of its minimum degree and order, its size and the sum of the degrees of every two nonadjacent vertices. Hamiltonian cycle extension numbers are also determined for powers of cycles.

Key Words: Hamiltonian graph, Hamiltonian cycle extension number.
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1 Introduction

The concepts of Hamiltonian cycles, Hamiltonian paths and Hamiltonian graphs are, of course, named for the famous Irish physicist and mathematician Sir William Rowan Hamilton. In 1856 Hamilton introduced a game he called the *Icosian Game* from a non-commutative algebraic system he developed. This two-person game could be played on the vertices and edges of a dodecahedron (a polyhedron with twenty vertices). Hamilton later sold the rights of this game to the well-known game company John Jacques & Son, which is still in existence. The preface to the instruction pamphlet for the Icosian Game was written by Hamilton himself. Below are excerpts from this preface:

> *In this new Game ... a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board ... in such a manner as always to proceed along the lines of the figure, and also to fulfill certain other conditions, which may in various ways be assigned by another player. ... For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may be cyclical, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind.*

In other words, beginning with any path P of order 5 (or less) on the graph G of the dodecahedron, P may be extended to a Hamiltonian cycle of G. That is, for every path P of order 5 in G, there exists a Hamiltonian cycle C of G such that P is a path on C. What Hamilton observed for paths of order 5 on the graph of the dodecahedron does not hold for all paths of order 6 as is illustrated in Figure 1 since this path of order 6 (drawn in bold edges) cannot be extended to a Hamiltonian cycle on the graph of the dodecahedron. This leads to a concept that is defined for every Hamiltonian graph. We refer to the book [4] for graph theory notation and terminology not described in this paper.

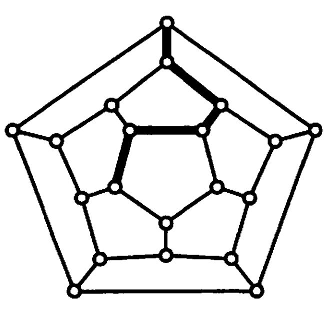

Figure 1: The graph G of the dodecahedron

2 The Hamiltonian Cycle Extension Number of a Hamiltonian Graph

A Hamiltonian graph G of order $n \geq 3$ is said to be a *k-path Hamiltonian graph* for some integer k with $1 \leq k \leq n$ if for every path P of order k, there exists a Hamiltonian cycle C of G such that P is a path on C. Certainly, every Hamiltonian graph is 1-path Hamiltonian. The largest integer k for which a Hamiltonian graph G is j-path Hamiltonian for every integer j with $1 \leq j \leq k$ is the *Hamiltonian cycle extension number* $\text{hce}(G)$ of G. Therefore, $1 \leq \text{hce}(G) \leq n$. Furthermore, $\text{hce}(G) = 1$ if and only if G contains an edge that lies on no Hamiltonian cycle of G. All graphs G of order n were determined for which $\text{hce}(G) = n$ (see [3]). In particular, for $n \geq 3$, $\text{hce}(G) = n$ if and only if G is the complete graph K_n, the n-cycle C_n or, when n is even, the regular complete bipartite graph $K_{n/2,n/2}$. Thus, for an integer $n \geq 3$, there are three graphs G of order n such that $\text{hce}(G) = n$ if n is even and two such graphs if n is odd. There are no Hamiltonian graphs G of order n, however, for which $\text{hce}(G) = n - 1$ or $\text{hce}(G) = n - 2$.

Proposition 2.1 *If G is a Hamiltonian graph of order n, then either* $1 \leq \text{hce}(G) \leq n - 3$ *or* $\text{hce}(G) = n$.

Proof. It suffices to show that if $\text{hce}(G) \geq n - 2$, then $\text{hce}(G) = n$. Let G be an $(n-2)$-path Hamiltonian graph and consider a Hamiltonian path $P = (v_1, v_2, \ldots, v_n)$ in G. Since G is $(n-2)$-path Hamiltonian, the subpath $(v_2, v_3, \ldots, v_{n-1})$ of P must lie on a Hamiltonian cycle, which implies that $v_1v_n \in E(G)$. Thus, P can be extended to a Hamiltonian cycle. ■

As Hamilton observed, the graph of the dodecahedron is 5-path Hamiltonian but not 6-path Hamiltonian. The dodecahedron is one of the five Platonic solids, the other four being the tetrahedron, octahedron, cube and icosahedron. For $n \in \{4, 6, 8, 12, 20\}$, let G_n be the graph of the Platonic solid having order n. Thus, $\text{hce}(G_{20}) = 5$. Note also that $\text{hce}(G_4) = \text{hce}(K_4) = 4$.

The graph G_6 of the octahedron is the complete 3-partite graph $K_{2,2,2}$. Let $V(G_6) = \{v_1, v_2, \ldots, v_6\}$ with the partite sets V_1, V_2, V_3, where $V_i = \{v_i, v_{i+3}\}$ for $i = 1, 2, 3$. Then the paths (v_1, v_2, v_3) and (v_1, v_2, v_4) can both be extended to a Hamiltonian cycle but the path (v_1, v_2, v_4, v_5) cannot be extended to a Hamiltonian path. Thus, $\text{hce}(G_6) = 3$.

In the example above, deleting the vertices in the path (v_1, v_2, v_4, v_5) from G_6 results in a disconnected graph. This illustrates the following observation.

Observation 2.2 *If a graph G contains a $u - v$ path P of order k such that the graph $G - (V(P) - \{u, v\})$ does not contain a Hamiltonian $u - v$*

path, then hce(G) $< k$. *In particular, if* $G - V(P)$ *is disconnected, then* hce(G) $< k$.

For the cube G_8, let $V(G_8) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ and $E(G_8) = \{u_i v_j : 1 \le i, j \le 4 \text{ and } i \ne j\}$. Since both paths (u_1, v_2, u_3, v_1) and (u_1, v_2, u_3, v_4) can be extended to Hamiltonian cycles, G_8 is 4-path Hamiltonian. However, Observation 2.2 with the path $(u_1, v_2, u_3, v_1, u_2)$ shows that hce$(G_8) < 5$. Hence, hce$(G_8) = 4$.

We next determine the value of hce(G_{12}). The graph G_{12} is given in Figure 2 with bold edges forming a Hamiltonian cycle. With the path $(v_1, v_2, v_{11}, v_7, v_6)$, for example, it is immediate that G_{12} is not 5-path Hamiltonian by Observation 2.2. Furthermore, for every path P of order 4 in G_{12}, observe that there is an automorphism ϕ of G_{12} such that the image of P under ϕ is one of the following five paths:

$$(v_1, v_2, v_3, v_4), \quad (v_4, v_5, v_6, v_7), \quad (v_7, v_8, v_9, v_{10}),$$
$$(v_{10}, v_{11}, v_{12}, v_1), \quad (v_2, v_3, v_4, v_5).$$

Since these five paths all lie on the Hamiltonian cycle $(v_1, v_2, \ldots, v_{12}, v_1)$ shown in Figure 2, we conclude that hce$(G_{12}) = 4$.

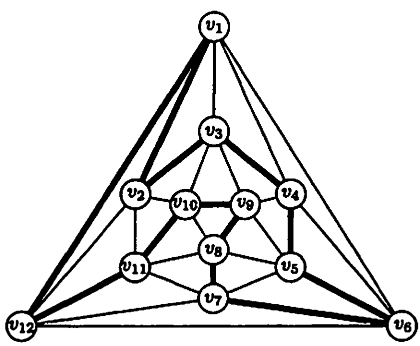

Figure 2: The graph G_{12} with a Hamiltonian cycle

These facts are summarized in Table 1.

3 Hamiltonian Cycle Extension Numbers of Hamiltonian Complete Multipartite Graphs

We have observed for $n \ge 3$ that hce$(K_n) = n$ and for $n = 2r \ge 4$ that hce$(K_{r,r}) = n$. We next determine hce(G) for all other Hamiltonian complete multipartite graphs. The following observation concerning complete multipartite graphs will be useful to us.

The graph G	The order of G	hce(G)
Octahedron	6 (4-regular)	3
Tetrahedron	4 (3-regular)	4
Cube	8 (3-regular)	4
Icosahedron	12 (5-regular)	4
Dodecahedron	20 (3-regular)	5

Table 1: The Hamiltonian cycle extension numbers of the graphs of the Platonic solids

Observation 3.1 *Let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph where $\ell \geq 2$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$.*

(a) *The graph G is Hamiltonian if and only if $n_\ell \leq \sum_{i=1}^{\ell-1} n_i$.*

(b) *The graph G contains a Hamiltonian path if and only if $n_\ell \leq 1 + \sum_{i=1}^{\ell-1} n_i$.*

(c) *If $n_\ell = 1 + \sum_{i=1}^{\ell-1} n_i$, then each Hamiltonian path of G must begin and end in the partite set whose cardinality equals n_ℓ.*

The following lemma is known (see [4, p. 42], for example).

Lemma 3.2 *Let G be a connected graph of order $n \geq 3$ and let d be an integer with $2 \leq d \leq n-1$. If $\deg u + \deg v \geq d$ for every two nonadjacent vertices u and v of G, then G contains a path of order $d+1$.*

We are now in a position to determine the Hamiltonian cycle extension numbers of all Hamiltonian complete multipartite graphs that are distinct from complete graphs and regular complete bipartite graphs. That the expression for the Hamiltonian cycle extension number of such a graph G given in the following result is a lower bound for hce(G) can be derived from a theorem of Kronk [8] dealing with linear forests. However, we present a complete and independent proof of this fact.

Theorem 3.3 *If G is a Hamiltonian complete ℓ-partite graph of order n for some integer $\ell \in \{3, 4, \ldots, n-1\}$, then $\text{hce}(G) = n + 1 - 2\alpha(G)$.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_\ell}$, where then $n = n_1 + n_2 + \cdots + n_\ell$ and $3 \leq \ell \leq n-1$. We may assume that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell$ and so $\alpha(G) = n_\ell$. Since G is Hamiltonian and not complete, it follows that $2 \leq n_\ell \leq \sum_{i=1}^{\ell-1} n_i$. Suppose that the partite sets of G are $V_1, V_2, \ldots, V_\ell$ with $|V_i| = n_i$ for $1 \leq i \leq \ell$. To verify that $\text{hce}(G) \leq n + 1 - 2n_\ell$, we show G contains a path of order $n + 2 - 2n_\ell$ that lie on no Hamiltonian cycle in

G. Let $H = G - V_\ell = K_{n_1,n_2,\ldots,n_{\ell-1}}$. First, we claim that H contains a path of order $n+2-2n_\ell$. If $n_{\ell-1} = 1$, then H is a complete graph of order $n - n_\ell$ and so there is certainly a path of order $n - n_\ell \geq n+2-2n_\ell$ (since $n_\ell \geq 2$). Thus, we may now assume that $n_{\ell-1} \geq 2$. Let u and v be two nonadjacent vertices of H. Since $\ell \geq 3$ and $1 \leq n_1 \leq n_{\ell-1} \leq n_\ell$, it follows that $n \geq n_1 + n_{\ell-1} + n_\ell \geq 2n_{\ell-1} + 1$ and so

$$\deg u + \deg v \geq 2\delta(H) = n + (n - 2n_{\ell-1}) - 2n_\ell \geq n + 1 - 2n_\ell.$$

By Lemma 3.2, H contains a path P of order $n+2-2n_\ell$, as claimed. Now if P lies on a Hamiltonian cycle in G, then the graph $F = G - V(P)$ must contain a Hamiltonian path. However, this is not the case since F contains the independent set V_ℓ of n_ℓ vertices while the order of F is $2n_\ell - 2$. Consequently, $\text{hce}(G) \leq n+1-2n_\ell$.

Since every path of order less than $n+1-2n_\ell$ lies on a path of order $n+1-2n_\ell$, to show that $\text{hce}(G) \geq n+1-2n_\ell$, it suffices to show that an arbitrary $u-v$ path P of order $n+1-2n_\ell$ lie on a Hamiltonian cycle in G. Let $F = G - V(P)$. Then F is a complete ℓ'-partite graph of order $2n_\ell - 1$ for some integer ℓ' with $2 \leq \ell' \leq \ell$. Let the partite sets of F be $U_1, U_2, \ldots, U_{\ell'}$ where $1 \leq |U_1| \leq |U_2| \leq \cdots \leq |U_{\ell'}| \leq n_\ell$. If $|U_{\ell'}| = n_\ell$, then $|U_{\ell'}| = 1 + \sum_{i=1}^{\ell'-1} |U_i|$ and so there exists a Hamiltonian $x-y$ path Q in H where $x, y \in U_{\ell'}$ by Observation 3.1(c). Since $ux, vy \in E(G)$, the path P can be extended to a Hamiltonian cycle in G containing Q as a subpath. Thus, we may now assume that $|U_{\ell'}| \leq n_\ell - 1$, which implies that $\ell' \geq 3$. Since $|U_{\ell'}| < \sum_{i=1}^{\ell'-1} |U_i|$, it follows that F has a Hamiltonian cycle C' by Observation 3.1(a). We now consider two cases, according to whether u and v belong to distinct partite sets of G or to the same partite set of G.

Case 1. *u and v belong to distinct partite sets of G, say $u \in V_a$ and $v \in V_b$, where $1 \leq a, b \leq \ell$ and $a \neq b$.* Let xy be an edge of C' and let P' be the $x-y$ path of C' not containing xy. Suppose first that one of x and y belongs to V_a or to V_b, say $x \in V_a$. Hence $u, x \in V_a$ and $v, y \notin V_a$, which implies that $uy, vx \in E(G)$. Then the two edges uy, vx together with P and P' produce a Hamiltonian cycle in G that is an extension of P. If neither x nor y belongs to V_a or to V_b, then $uy, vx \in E(G)$ here also and once again, P can be extended to a Hamiltonian cycle of G.

Case 2. *u and v belong to the same partite set of G, say $u, v \in V_a$, where $1 \leq a \leq \ell$.* Since $|V_a - \{u,v\}| \leq n_\ell - 2$ and C' is a $(2n_\ell - 1)$-cycle, it follows that C' contains an edge xy such that $x \notin V_a$ and $y \notin V_a$. Hence $ux, vy \in E(G)$ and so P can be extended to a Hamiltonian cycle of G.

As a result, we conclude that G is $(n+1-2n_\ell)$-path Hamiltonian and so $\text{hce}(G) = n+1-2n_\ell = n+1-2\alpha(G)$. ∎

We now know the value of the Hamiltonian cycle extension number of each Hamiltonian complete multipartite graph.

Corollary 3.4 *If G is a Hamiltonian complete multipartite graph of order n, then*

$$\text{hce}(G) = \begin{cases} n & \text{if } G \text{ is complete or bipartite} \\ n+1-2\alpha(G) & \text{otherwise.} \end{cases}$$

4 A Minimum Degree Condition for Hamiltonian Cycle Extension Numbers

The first result of a theoretical nature dealing with Hamiltonian graphs occurred in 1952 and is due to Dirac [5].

Theorem 4.1 *If G is a graph of order $n \geq 3$ with the minimum degree $\delta(G) \geq n/2$, then G is Hamiltonian.*

A graph G is *Hamiltonian-connected* if G contains a Hamiltonian $u - v$ path for every pair u, v of distinct vertices of G. The following theorem, due to Ore [11] in 1963, provides a sufficient condition for a graph to be Hamiltonian-connected.

Theorem 4.2 *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq (n+1)/2$, then G is Hamiltonian-connected.*

If G is a Hamiltonian graph, then $\delta(G) \geq 2$. Now, suppose that G is a Hamiltonian graph of order n and $\delta(G) = 2$. If $G \neq C_n$, then $\Delta(G) \geq 3$ and so there is an edge $uv \in E(G)$ such that $\deg u = 2$ and $\deg v \geq 3$. Thus, there exists a path (x, v, y) where $u \notin \{x, y\}$, which cannot be extended to a Hamiltonian cycle. Hence $\text{hce}(G) \leq 2$. Therefore, if G is a Hamiltonian graph of order n and $\delta(G) = 2$, then

$$\text{hce}(G) = \begin{cases} n & \text{if } G = C_n \\ 1 & \text{if } G \text{ contains an edge not belonging to a Hamiltonian cycle} \\ 2 & \text{otherwise.} \end{cases}$$

We now present a lower bound for the Hamiltonian cycle extension number of a graph G in terms of the minimum degree and order of G.

Theorem 4.3 *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then*

$$\text{hce}(G) \geq 2\delta(G) - n + 1.$$

Proof. By Dirac's Theorem, G is Hamiltonian and so $1 \leq \text{hce}(G) \leq n$. Since G is Hamiltonian and $1 \leq 2\delta(G) - n + 1 \leq n - 1$, it follows that G contains a path of order $2\delta(G) - n + 1$. Here, every path of order less than $2\delta(G) - n + 1$ lies on a path of order $2\delta(G) - n + 1$. Thus, it suffices to show that $\text{hce}(G) \geq 2\delta(G) - n + 1$ by verifying that every path of order $2\delta(G) - n + 1$ can be extended to a Hamiltonian cycle in G. Since the result is immediate when $\delta(G) \in \{n/2, n-1\}$, we may assume that $n/2 < \delta(G) \leq n - 2$. Thus, $n \geq 5$ and $2\delta(G) - n + 1 \geq 2$. Let P be a $u - v$ path of order $2\delta(G) - n + 1$ and let H be the subgraph of G induced by $(V(G) - V(P)) \cup \{u, v\}$. Thus, the order of H is $n' = 2(n - \delta(G)) + 1 \geq 5$. Furthermore,

$$\delta(H) \geq \delta(G) - |V(P)| + 2 = n - \delta(G) + 1 = (n' + 1)/2.$$

It then follows that H is Hamiltonian-connected and so there exists a Hamiltonian $u - v$ path in H. Thus, P can be extended to a Hamiltonian cycle in G. ■

The lower bound in Theorem 4.3 is sharp for if G is a Hamiltonian complete multipartite graph of order n that is neither complete nor bipartite, then $\text{hce}(G) = 2\delta(G) - n + 1$ since $\delta(G) = n - \alpha(G)$. The following is a consequence of Theorem 4.3.

Corollary 4.4 *If G is a graph of order $n \geq 4$ such that $\delta(G) \geq rn$ for some rational number r with $1/2 \leq r < 1$, then* $\text{hce}(G) \geq (2r - 1)n + 1$.

The lower bound presented in Corollary 4.4 for the Hamiltonian cycle extension number of a graph is sharp for each rational number $r \in [1/2, 1)$. To see this, write $r = p/q$, where p, q are integers satisfying $2 \leq p < q \leq 2p$, and consider the graph $G = K_{n_1, n_2, \ldots, n_{p+1}}$ be the complete $(p+1)$-partite graph of order $n = \sum_{i=1}^{p+1} n_i = pq$, where

$$n_i = \begin{cases} p & \text{if } 1 \leq i \leq p \\ p(q-p) & \text{if } i = p+1. \end{cases}$$

Since $p < n_{p+1} \leq p^2 = \sum_{i=1}^{p} n_i$, the graph G is Hamiltonian and $\delta(G) = p^2$. Furthermore,

$$\begin{aligned} \text{hce}(G) &= 2\delta(G) - n + 1 = 2p^2 - pq + 1 \\ &= (2p/q - 1)pq + 1 = (2r - 1)n + 1. \end{aligned}$$

If G is a Hamiltonian graph of order n, then certainly $C_n \subseteq G \subseteq K_n$. While $\text{hce}(C_n) = \text{hce}(K_n) = n$, there are graphs for which $1 \leq \text{hce}(G) < n$.

Proposition 4.5 *If G is a Hamiltonian graph of order $n \geq 4$ and clique number $n - 1$, then* $\text{hce}(G) = \delta(G) - 1$.

Proof. Since G is Hamiltonian but not complete, $2 \leq \delta(G) \leq n-2$. Let v^* be the vertex with $\deg v^* = \delta(G)$. Since a path whose vertex set equals $N(v^*)$ cannot be extended to a Hamiltonian cycle in G, it is follows that $\text{hce}(G) \leq \delta(G) - 1$.

To show that $\text{hce}(G) \geq \delta(G) - 1$, let P be a $u-v$ path of order $\delta(G)-1$. Let $H = G - V(P)$. We show that H contains a Hamiltonian $x-y$ path Q such that $ux, vy \in E(G)$. If v^* belongs to P, then H is complete and at least two vertices in H are adjacent to v^* in G, say $x, y \in V(H)$ and $xv^*, yv^* \in E(G)$. Thus, both x and y are adjacent to each of u and v in G. Let Q be any Hamiltonian $x-y$ path in H. If v^* does not belong to P, then both u and v are adjacent to every vertex in $V(H) - \{v^*\}$. If at least one of u and v is adjacent to v^*, say $uv^* \in E(G)$, then $\deg_H v^* \geq 1$ and H contains a Hamiltonian $x-y$ path Q with v^* as its initial vertex. Otherwise, $\deg_H v^* \geq 3$ and H contains a Hamiltonian $x-y$ path Q with v^* as one of its internal vertices. Thus, P and Q together with the edges ux and vy form Hamiltonian cycle of G. The desired result now follows. ■

Corollary 4.6 *For a pair n, a of positive integers, there exists a Hamiltonian graph G of order n with $\text{hce}(G) = a$ if and only if $1 \leq a \leq n-3$ or $a = n$.*

5 Sufficient Conditions for the Hamiltonian Cycle Extension Number of a Graph

It is well known that if G is a graph of order $n \geq 3$ and size $m \geq \binom{n-1}{2} + 2$, then G is Hamiltonian; while if G is a graph of order $n \geq 4$ and size $m \geq \binom{n-1}{2} + 3$, then G is Hamiltonian-connected (see [11]). Consequently, every graph G of order $n \geq k+2$ where $k = 1, 2$ and size $m \geq \binom{n-1}{2} + k + 1$ is k-path Hamiltonian. The following result is an extension of these two statements.

Theorem 5.1 *Let k and n be positive integers such that $n \geq k+2$. If G is a graph of order n and size $m \geq \binom{n-1}{2} + k + 1$, then G is k-path Hamiltonian.*

Proof. Since it is known that this theorem holds for $k = 1, 2$, we may assume that $k \geq 3$. For a graph G of order $n \geq k+2$ and size $m \geq \binom{n-1}{2} + k + 1$, let P be a $u-v$ path of order k. Consider the graph $H = G - S$, where $S = V(P) - \{u, v\}$. If n' and m' are the order and size of H, respectively, then $n' = n - k + 2$ and

$$m' \geq \binom{n'+k-3}{2} + k + 1 - \binom{k-2}{2} - n'(k-2) = \binom{n'-1}{2} + 3.$$

Therefore, H is Hamiltonian-connected and so H contains a Hamiltonian $u - v$ path Q. Since the paths P and Q produce a Hamiltonian cycle in G, it follows that G is k-path Hamiltonian. ∎

The bound for the size m of a graph in Theorem 5.1 cannot be improved. For example, let G be a connected graph of order $n \geq 3$ with clique number $n-1$ and $k = \delta(G)$. Hence, $3 \leq k+2 \leq n$ and the size of G equals $\binom{n-1}{2}+k$. The graph G is not Hamiltonian when $k = 1$. If G is Hamiltonian, then $k \geq 2$ and $\text{hce}(G) = k - 1$ by Proposition 4.5.

Two of the best known sufficient conditions for a graph G of order n to be Hamiltonian or Hamiltonian-connected are generalizations of Theorems 4.1 and 4.2 and are both due to Ore (see [10, 11]).

Theorem 5.2 (Ore) *If G is a graph of order $n \geq 3$ such that* $\deg u + \deg v \geq n$ *for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.*

Theorem 5.3 (Ore) *If G is a graph of order $n \geq 4$ such that* $\deg u + \deg v \geq n + 1$ *for every pair u, v of nonadjacent vertices of G, then G is Hamiltonian-connected.*

As a consequence of Theorems 5.2 and 5.3, it follows for $k = 1, 2$ that if G is a graph of order $n \geq k + 2$ such that $\deg u + \deg v \geq n + k - 1$ for every pair u, v of nonadjacent vertices of G, then G is k-path Hamiltonian. The following is an extension of this statement.

Theorem 5.4 *Let k and n be positive integers such that $n \geq k + 2$. If G is a graph of order n such that* $\deg u + \deg v \geq n + k - 1$ *for every pair u, v of nonadjacent vertices of G, then* $\text{hce}(G) \geq k$.

Proof. Since this theorem holds for $k = 1, 2$, we may assume that $k \geq 3$. Because every path of order less than k lies on a path of order k here, we consider a path $P = (u = u_1, u_2, \ldots, u_k = v)$ of order k in G. Let $S = \{u_2, u_3, \ldots, u_{k-1}\}$. Furthermore, let $H = G - S$ be the graph of order $n' = n - k + 2$. Then for every two nonadjacent vertices u and v of H, it follows that

$$\deg_H u + \deg_H v \geq n + k - 1 - 2(k - 2) = n - k + 3 = n' + 1.$$

By Theorem 5.3, H is Hamiltonian-connected and so H contains a Hamiltonian $u - v$ path Q. Hence the cycle formed from P and Q is a Hamiltonian cycle in G containing the path P. Consequently, G is k-path Hamiltonian and so $\text{hce}(G) \geq k$. ∎

Note that now Theorem 5.1 can be seen as a corollary of Theorem 5.4. The bound on the degree sum of nonadjacent vertices in Theorem 5.4 cannot be improved. To see this, let a be an integer satisfying $1 \le a \le n - k - 1$. Consider a graph G of order n consisting of complete subgraphs G_1 and G_2 of order $k+a$ and $n-a$, respectively, such that $E(G) = E(G_1) \cup E(G_2)$ and $V(G) = V(G_1) \cup V(G_2)$ where $V(G_1) \cap V(G_2) = \{u_1, u_2, \ldots, u_k\}$. Then $\deg u + \deg v = n + k - 2$ whenever u and v are two distinct nonadjacent vertices in G. When $k = 1$, the graph G is not Hamiltonian. For $k \ge 2$, Ore's Theorem guarantees that G is Hamiltonian. However, the path $(u_1, u_2, \ldots, u_k)$ lies on no Hamiltonian cycle in G. Consequently, G is not k-path Hamiltonian.

6 Hamiltonian Cycle Extension Numbers of Powers of Cycles

For a connected graph G and a positive integer k, the *kth power* G^k of G is that graph whose vertex set is $V(G)$ such that uv is an edge of G^k if $1 \le d_G(u, v) \le k$. The graph G^2 is called the *square* of G and G^3 is the *cube* of G.

In 1960, Sekanina [12] proved that the cube of every connected graph G is Hamiltonian-connected and, consequently, G^3 is Hamiltonian if its order is at least 3. In the 1960s, it was conjectured independently by Nash-Williams [9] and Plummer (see [4, p.139]) that the square of every 2-connected graph is Hamiltonian. In 1974, Fleischner [6] verified this conjecture. Also, in 1974 and using Fleischner's result, Chartrand, Hobbs, Jung, Kapoor and Nash-Williams [1] proved that the square of every 2-connected graph is Hamiltonian-connected.

For a connected graph G of order $n \ge 4$ and an integer k with $1 \le k \le n - 3$, the graph G is *k-Hamiltonian* if $G - S$ is Hamiltonian for every set S of k vertices of G and *k-Hamiltonian-connected* if $G - S$ is Hamiltonian-connected for every set S of k vertices of G. If the order of a graph G is at least 4, then Chartrand and Kapoor [2] showed that G^3 is 1-Hamiltonian. Since the square of every 2-connected graph is Hamiltonian-connected, the square of every Hamiltonian graph is Hamiltonian-connected.

In 1973 Hobbs [7] made the following conjecture.

Conjecture 6.1 (Hobbs) *If G is a 2-connected graph of order n and k is an integer with $3 \le k \le (n+1)/2$, then G^k is $(2k-3)$-Hamiltonian-connected.*

In this section, we verify Conjecture 6.1 when $G = C_n$ for $k = 2$ as well as for $k \ge 3$ and use the resulting theorem to obtain a formula for $\text{hce}(C_n^k)$. We begin by stating the following two lemmas, the first of which is straightforward to verify.

Lemma 6.2 *If $P = (v_1, v_2, \ldots, v_n)$ is a path of order $n \geq 2$, then P^2 contains a Hamiltonian $v_1 - v_2$ path.*

Lemma 6.3 *Let P be a path of order $n \geq 2$. For distinct vertices u and v in P, there exists a Hamiltonian $u - v$ path in P^2 unless neither u nor v is an end-vertex of P and uv is an edge of P.*

Proof. Since P^2 is complete when $n = 2, 3$, we may assume that $n \geq 4$. Suppose that neither u nor v is an end-vertex of P and $uv \in E(P)$. Since deleting u and v from P^2 results in a disconnected graph, there is no Hamiltonian $u - v$ path in P^2.

Next, suppose that either at least one of u and v is an end-vertex of P or $uv \notin E(P)$. We show that P^2 contains a Hamiltonian $u - v$ path. Let $P = (v_1, v_2, \ldots, v_n)$. We may assume that $u = v_i$ and $v = v_j$ where $1 \leq i < j \leq n$. Let

$$Q_1 = (v_1, v_2, \ldots, v_i = u) \text{ and } Q_2 = (v = v_j, v_{j+1}, \ldots, v_n)$$

be two subpaths of P. With the aid of Lemma 6.2, we define

$$Q_1' = \begin{cases} Q_1 & \text{if } i = 1 \\ \text{a Hamiltonian } v_i - v_{i-1} \text{ path in } Q_1^2 & \text{if } i \geq 2 \end{cases}$$

$$Q_2' = \begin{cases} Q_2 & \text{if } j = n \\ \text{a Hamiltonian } v_{j+1} - v_j \text{ path in } Q_2^2 & \text{if } j \leq n-1. \end{cases}$$

Then

$$Q = \begin{cases} (Q_1', Q_2') & \text{if } j - i = 1 \\ (Q_1', v_{i+1}, v_{i+2}, \ldots, v_{j-1}, Q_2') & \text{if } j - i \geq 2 \end{cases}$$

is a Hamiltonian $u - v$ path in P^2. ∎

Corollary 6.4 *If P is a $u - v$ path of order at least 4, then $P^2 + uv$ is Hamiltonian-connected.*

We are now prepared to prove the following result.

Theorem 6.5 *For every two integers k and n for which $6 \leq 2k + 2 \leq n$, the graph C_n^k is $(2k-3)$-Hamiltonian-connected.*

Proof. Let $G = C_n^k$ be the graph constructed by taking the kth power of the n-cycle $C = (v_1, v_2, \ldots, v_n, v_1)$. Since $6 \leq 2k + 2 \leq n$, the graph G is neither C_n nor K_n. For each vertex v_i $(1 \leq i \leq n)$, the neighborhood of v_i in G is $N(v_i) = \{v_{i\pm1}, v_{i\pm2}, \ldots, v_{i\pm k}\}$, where each subscript is expressed as one of the integers $1, 2, \ldots, n$ modulo n. The right (or "clockwise") neighborhood of v_i in G is the set

$$N_R(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+k}\}.$$

Similarly, the left (or "counter-clockwise") neighborhood of v_i in G is the set

$$N_L(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\}.$$

Hence

$$\text{if } x, y \in \{v_i\} \cup N_R(v_i), \text{ then } d_C(x,y) \leq k \text{ and so } xy \in E(G). \qquad (1)$$

Let S be a $(2k-3)$-element subset of $V(G)$ and let $G' = G - S$ where the order of G' is $n' = n - (2k-3) = n - 2k + 3$. We show that G' is Hamiltonian-connected.

The subgraph $C[S]$ of the cycle C induced by S is a linear forest (and so each component of $C[S]$ is a path). Let S_1 be a subset of S such that $C[S_1]$ is a component of maximum order in $C[S]$, say $|S_1| = p$. Thus $1 \leq p \leq 2k-3$. By relabeling the vertices of C, if necessary, we may assume that $C[S_1] = (v_{n-p+1}, v_{n-p+2}, \ldots, v_n)$. Consequently, $v_1, v_{n-p} \in V(G')$. Express the vertex set of G' by

$$V(G') = \{v_{j_i} = u_i : 1 \leq i \leq n'\} = \{u_1, u_2, \ldots, u_{n'}\},$$

where $1 = j_1 < j_2 < \cdots < j_{n'} = n - p$. For each $u_i \in V(G')$, let

$$N'_R(u_i) = N_R(u_i) \cap V(G') \subseteq N_R(u_i).$$

Since $\{u_i\} \cup N'_R(u_i) \subseteq \{u_i\} \cup N_R(u_i)$, it follows by (1) that

$$\text{if } x, y \in \{u_i\} \cup N'_R(u_i), \text{ then } xy \in E(G) \text{ and so } xy \in E(G'). \qquad (2)$$

Since $p \leq 2k-3$, there is a $u_i - u_{i+1}$ path on C of length at most $k-1$ for every i $(1 \leq i \leq n'-1)$ and so $d_C(u_i, u_{i+1}) \leq k-1$. Therefore, $u_i u_{i+1} \in E(G')$ for each i for $1 \leq i \leq n'-1$. Hence $P = (u_1, u_2, \ldots, u_{n'})$ is a spanning path in G'. Next, we show that either G' contains P^3 as a subgraph or there exists a $u - v$ Hamiltonian path Q in G' such that $uv \in E(G')$ and G' contains $Q^2 + uv$ as a subgraph. We consider three cases, according to the value of p.

Case 1. $k \leq p \leq 2k-3$. Since $|S - S_1| = 2k - 3 - p$, it follows that $0 \leq |S - S_1| \leq k - 3$. In this case, we show that each vertex u_i is adjacent to each of $u_{i+1}, u_{i+2}, u_{i+3}$ in G' for $1 \leq i \leq n'-3$ and $u_{n'-2}$ is adjacent to $u_{n'-1}$ and $u_{n'}$ in G', which implies that P^3 is a subgraph of G. Since for each $u_i \in V(G')$, $1 \leq i \leq n'-2$, at most $k-3$ vertices of

$N_R(u_i)$ belong to $S - S_1$, it follows that $u_{i+1}, u_{i+2}, u_{i+3} \in N_R(u_i)$ when $1 \le i \le n' - 3$ and $u_{n'-1}, u_{n'} \in N_R(u_i)$ when $i = n' - 2$. This implies that P^3 is a spanning subgraph of G'. Since P^3 is Hamiltonian-connected, G' is Hamiltonian-connected.

Case 2. $p = k-1$. Thus $|S - S_1| = k-2$. Since $u_{n'} \in N_L(u_i)$, it follows that $d_C(u_1, u_{n'}) \le k - 1$. Therefore, $u_1 u_{n'} \in E(G)$ and so $P + u_1 u_{n'}$ is a Hamiltonian cycle in G'. For $1 \le i \le n' - 2$, at most $k - 2$ vertices of $N_R(u_i)$ belong to $S - S_1$ and so $u_{i+1}, u_{i+2} \in N'_R(u_i)$. Hence P^2 as well as $P^2 + u_1 u_{n'}$ are spanning subgraphs of G'. Since $P^2 + u_1 u_{n'}$ is Hamiltonian-connected by Corollary 6.4, it follows that G' is Hamiltonian-connected. This case then establishes the theorem when $k = 2$.

Case 3. $1 \le p \le k - 2$. Then $k \ge 3$. Since G is $(2k)$-regular, it follows that $\delta(G') \ge 2k - (2k - 3) = 3$. Furthermore, because $u_{n'} \in N_L(u_1)$, we have $u_1 u_{n'} \in E(G)$ and so $P + u_1 u_{n'}$ is a Hamiltonian cycle in G'. If $P^2 + u_1 u_{n'}$ is a subgraph of G', then G' is Hamiltonian-connected by Corollary 6.4. We claim that if $P^2 + u_1 u_{n'}$ is not a subgraph of G', then G' contains adjacent vertices u and v for which there is a $u - v$ Hamiltonian path Q such that $Q^2 + uv$ is a subgraph of G'. Suppose that $P^2 + u_1 u_{n'}$ is not a subgraph of G'. Since P is a subgraph of G' and $u_1 u_{n'} \in E(G)$, it follows that there is $i \in \{1, 2, \ldots, n' - 2\}$ such that $u_i u_{i+2} \notin E(G)$. Let A be the $u_i - u_{i+2}$ subpath of C containing u_{i+1} and B the $u_i - u_{i+2}$ subpath of C not containing u_{i+1}. Since $d_C(u_i, u_{i+2}) \ge k+1$, at least $k-1$ vertices of A belong to S. Consequently, at most $k - 2$ vertices of B belong to S. Let $T = (u_{i+2}, u_{i+3}, \ldots, u_{n'}, u_1, u_2, \ldots, u_i)$. If we express $u_1, u_2, \ldots, u_i$ as $u_{n'+1}, u_{n'+2}, \ldots, u_{n'+i}$ in T, then for each j with $i + 2 \le j \le n' + i - 2$, it follows that $d_C(u_j, u_{j+2}) \le k$ and so $u_j u_{j+2} \in E(G')$. Hence T^2 is a subgraph of $G' - u_{i+1}$. Since $\deg_{G'} u_{i+1} \ge 3$ and u_{i+1} is adjacent to u_i and u_{i+2} in G', either $u_{i+1} u_{i+3} \in E(G')$ or $u_{i+1} u_{i-1} \in E(G')$, say the former. Now $Q = (u_{i+1}, T) = (u_{i+1}, u_{i+2}, u_{i+3}, \ldots, u_{n'}, u_1, u_2, \ldots, u_i)$ is a Hamiltonian $u_{i+1} - u_i$ path in G'. Since T^2 is a subgraph of $G' - u_{i+1}$ and u_{i+1} is adjacent to u_i and u_{i+3} in G', it follows that $Q^2 + u_{i+1} u_i$ is a subgraph of G'. Hence G' is Hamiltonian-connected by Corollary 6.4. ■

Theorem 6.5 cannot be strengthened. That is, for integers k and n with $6 \le 2k + 2 \le n$, the graph C_n^k is not $(2k - 2)$-Hamiltonian-connected. To see this, let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. For the set $S = \{v_2, v_3, \ldots, v_k\} \cup \{v_{k+2}, v_{k+3}, \ldots, v_{2k}\}$ of $2k - 2$ vertices of C_n, there is no Hamiltonian $v_1 - v_{2k+1}$ path in $C_n^k - S$.

We are now in a position to establish a formula for the Hamiltonian cycle extension numbers of the graphs C_n^k for positive integers k and $n \ge 3$.

Theorem 6.6 *For positive integers k and $n \geq 3$,*

$$\text{hce}(C_n^k) = \begin{cases} n & \text{if } k = 1 \text{ or } k \geq \lfloor n/2 \rfloor \\ 2k-1 & \text{otherwise.} \end{cases}$$

Proof. If $k = 1$, then $C_n^k = C_n$ and if $k \geq \lfloor n/2 \rfloor$, then $C_n^k = K_n$. We have seen in these cases that the Hamiltonian cycle extension number is n.

It remains to show that $\text{hce}(C_n^k) = 2k - 1$ when $2 \leq k \leq \lfloor n/2 \rfloor - 1$ or, equivalently, when $6 \leq 2k + 2 \leq n$. Let $C_n = (v_1, v_2, \ldots, v_{n-1}, v_n, v_1)$. Then $P = (v_1, v_2, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{2k+1})$ is a path of of order $2k$ in $G = C_n^k$. Since v_{k+1} is an isolated vertex in $G - V(P)$, it follows by Observation 2.2 that $\text{hce}(C_n^k) \leq 2k - 1$.

Next, we show that $\text{hce}(C_n^k) \geq 2k-1$. Here, every path of order less than $2k - 1$ lies on a path of order $2k - 1$. Therefore, let $P = (u_1, u_2, \ldots, u_{2k-1})$ be a path of order $2k - 1$ in G and let $Q = (u_2, u_3, \ldots, u_{2k-2})$ be a subpath of P of order $2k - 3$. By Theorem 6.5, the graph $G - V(Q)$ contains a Hamiltonian $u_1 - u_{2k-1}$ path Q'. Then the two paths P and Q' produce a Hamiltonian cycle in G. Thus, $\text{hce}(G) \geq 2k - 1$ and we conclude that $\text{hce}(G) = 2k - 1$. ■

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